

FUKAYA CATEGORIES OF HIGHER-GENUS SURFACES AND PANTS DECOMPOSITIONS

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ABSTRACT. In this paper we prove a local-to-global principle for the Fukaya category of a closed Riemann surface Σ of genus $g \geq 2$. We show that $\text{Fuk}(\Sigma)$ can be glued from the Fukaya category of the pair-of-pants making up a pants decomposition of Σ . This extends our earlier results for the case of punctured Riemann surfaces. Our result has several interesting consequences: we obtain simple proofs of old and new HMS statements for Riemann surfaces, and establish a geometrization theorem for the objects of $\text{Fuk}(\Sigma)$.

CONTENTS

1. Introduction	1
1.1. Main results	3
2. Preliminaries	6
2.1. Notations and conventions	6
3. Graphs and categories	9
3.1. Local model for the restriction functors	10
3.2. Dependence of the local model on the weights	11
3.3. Relationship of the local model to the Fukaya category	13
3.4. Sheaves of categories on graphs	15
3.5. Dependence of $\mathcal{B}(G)$ on the weights and framings	17
4. Topological Fukaya category and pants decomposition	18
4.1. Riemann surfaces and maximal tropical covers	19
4.2. The topological Fukaya category of $\tilde{\Sigma}$	21
4.3. The locally finite topological Fukaya category	23
5. Singularity categories and graphs	24
5.1. Singularity categories of normal crossing surfaces	24
6. Covering spaces and equivariance	29
6.1. Variants of the Fukaya category for finite type surfaces	29
6.2. Balanced versus exact categories for noncompact surfaces	31
6.3. Restriction functors	32
6.4. Locally finite Fukaya category for non-finite-type Riemann surfaces	32
6.5. Pullback for étale maps	33
6.6. O_π -equivariant branes	36
6.7. Essential surjectivity	37
7. HMS for compact surfaces	42
7.1. HMS for genus two curves and Mumford degenerations of Abelian surfaces	44
References	45

1. INTRODUCTION

The problem of homological mirror symmetry (HMS) where the A-side is taken to be a Riemann surface has been much studied. When the surface is compact of genus $g \geq 2$, this problem is an

instance of mirror symmetry for varieties of general type [KKOY09, GKR17]. For $g = 2$, this form of HMS was proven by Seidel [Sei11], and Efimov [Efi12] extended Seidel’s method to higher genera.

The case of open or punctured Riemann surfaces has also received a lot of attention, including works of Bocklandt [Boc16], Abouzaid-Auroux-Efimov-Katzarkov-Orlov [AAE⁺13], Lee [Lee16], Ruddat [Rud17], Lekili-Polishchuk [LP18], and Cho-Hong-Lau [CHL18]. The structure of Fukaya categories of surfaces was elucidated by Haiden-Katzarkov-Kontsevich [HKK17], and also by the theory of topological Fukaya categories as presented for instance in [DK18]. The viewpoint of this paper and our previous work [PS19] is similar to those of Lee and Cho-Hong-Lau, since they seek to understand the Fukaya category of a surface by decomposing it into pairs of pants (see also [Sei12]).

Returning to the compact case, the mirror geometry is a certain Landau-Ginzburg model defined on a three fold. In each case, the mirror model is presented in a fairly complicated way, and in particular the connection to the geometry of the genus g surface is not readily apparent (of course, by actually reading the proof one sees the connection, but it is based on branched covering arguments and does not seem very intrinsic.)

Part of the reason why the setting of Seidel and Efimov’s results is so intricate is that, although compact Riemann surfaces are among the simplest examples of symplectic manifolds, describing their Fukaya category is not obvious. The main issue is that the Fukaya category is a global invariant which cannot be computed from simpler local data, or at least this is the general expectation.

It turns out that this is not always true: mathematicians have realised that there is a large class of symplectic manifolds for which the Fukaya category has good local-to-global properties. One of the first results in this direction was Nadler and Zaslow’s description of the Fukaya category of cotangent bundles in terms of constructible sheaves on the base [NZ09]. Later in [Kon09] Kontsevich conjectured that the Fukaya category of Weinstein manifolds should localize over a Lagrangian skeleton. This is an area of intense current research; a local-to-global (sheaf-theoretic) description of the Fukaya category of Weinstein symplectic manifolds has now been established in many cases, in [GPS20] and in subsequent articles by the same authors.

This point of view has been applied with great success to HMS. The sheaf-theoretic models of the Fukaya category are, as a rule, much easier to compute since everything relies on relatively simple local calculations; and there are no pseudo-holomorphic discs involved. The HMS statements one is interested in often localize: complicated equivalences can be obtained by putting together more accessible local statements. We followed this strategy in our previous article [PS19] to study Hori–Vafa mirror symmetry for toric Calabi–Yau threefolds. The mirror symplectic manifolds are punctured surfaces, and their Fukaya category can be studied very efficiently via sheaf-theoretic methods.

How much of this picture can be expected to hold in the compact setting? In this article we address this question in the simplest case of compact Riemann surfaces of genus $g \geq 2$. We will give a complete local-to-global description of their Fukaya category which parallels our earlier results for punctured Riemann surfaces. We will use this description to study HMS for Riemann surfaces: we obtain new and simpler proof of Seidel and Efimov’s results and obtain many new HMS equivalences; in fact, in a way our methods allow us to study at once all possible mirror partners of compact Riemann surfaces, and all possible HMS equivalences.

Except for interesting proposals of Tamarkin [Tam15] and Tsygan [Tsy09] from a different point of view, to the best of our knowledge this is the first description of the Fukaya category of a compact manifold in terms of local-to-global data. The compact setting is seemingly very different from the exact setting. One of key players in the exact story are *skeleta*, i.e. half-dimensional CW complexes sitting inside the symplectic manifold as Lagrangian deformation retracts. Kontsevich’s proposal relied precisely on the fact that we could flow Lagrangians in an exact symplectic manifold as close

as we wish to a skeleton, where the symplectic geometry is the same as that of cotangent bundles. However, for obvious topological reasons, compact symplectic manifolds do not admit a skeleta.¹

Nonetheless our results are a proof of concept that a sheaf theoretic approach is feasible also in the compact case, at least under appropriate assumptions. We stress that the restriction on the genus ($g \geq 2$) is not accidental. The hyperbolic/general type setting is similar to the exact setting in that contributions from pseudo-holomorphic disks can be controlled. Formally, this has the consequence that, contrary to the general case, the Fukaya category can be defined over the ground field rather than the Novikov ring. In future work we plan to study generalisations of our techniques to higher dimensions, where some of the tricks we will use in this paper will not be available.

Let us give a more detailed summary of our results and of the main ideas going into their proofs.

1.1. Main results. The starting point for this of project was our earlier work [PS19]. Our main technical result there was a description of the Fukaya category of a Riemann surface with punctures Σ in terms of the Fukaya category of the pairs-of-pants. Let \mathcal{P} be a pants decomposition of Σ , and denote by $P \in \mathcal{P}$ the individual pairs-of-pants. In this model is convenient to view pants as overlapping along cylinders: so the intersection of two neighbouring pants is a symplectic cylinder C . We proved that $\text{Fuk}(\Sigma)$ can be expressed as a (homotopy) equalizer of dg categories

$$(1) \quad \text{Fuk}(\Sigma) \simeq \underset{\leftarrow}{\text{lim}} \left(\prod_{P \in \mathcal{P}} \text{Fuk}(P) \rightrightarrows \prod_{C=P_1 \cap P_2} \text{Fuk}(C) \right)$$

This was the key input in our proof of HMS for Σ which reduces to checking it for the pairs-of-pants P , and then using equivalence (1) to generalize it to all genera and an arbitrary number of punctures.

Equivalence (1) is closely related to, but different from, the local-to-global properties which are built in in the definition of the topological Fukaya category: it is a genuinely different manifestation of the locality of the Fukaya category in the exact setting, since it does not follow from gluing together local descriptions arising from an open cover of the skeleton; instead (1) can be proved by taking *closed* covers of the skeleton of Σ with suitable properties. However the statement finally does not even refer to the skeleton, and only depends on a pants-decomposition: of course, up to equivalence, any pants-decomposition yields the same answer.

We are stressing this point because whereas the more familiar locality statements available in the exact setting depend on working on a skeleton, (1) does not. And in fact, whereas skeleta are not well-defined for compact Riemann surfaces and so approaches based on skeleta cannot extend beyond the exact setting, it turns out that (1) holds also in the compact setting. This is the content of our first main result; we refer the reader to Theorem 6.16 in the main text for a more precise statement:

Theorem 1.1. *Let Σ be a compact Riemann surface of genus $g \geq 2$, equipped with a pants decomposition \mathcal{P} . Then there is an equivalence of categories*

$$(2) \quad \text{Fuk}(\Sigma) \simeq \underset{\leftarrow}{\text{lim}} \left(\prod_{P \in \mathcal{P}} \text{Fuk}(P) \rightrightarrows \prod_{C=P_1 \cap P_2} \text{Fuk}(C) \right)$$

The idea behind the proof of Theorem 1.1 is very simple. It is a variation on a trick which has been used often in symplectic geometry, namely translating between the Fukaya category of a space and that of an unramified covering. Except that, contrary to what usually happens, we work with an *infinite* covering $\pi : \tilde{\Sigma} \rightarrow \Sigma$, which we call the *maximal tropical cover* (see Section 4.1 for the construction). The group of deck transformations of π is a free abelian group O_π of rank equal to

¹Roughly, the choice of skeleton is analogous to choosing a Morse function all of whose critical points have index at most half the real dimension; on a compact manifold any Morse function must have a critical point of index equal to the dimension.

the genus of Σ . The surface $\tilde{\Sigma}$ is in a sense a much more complicated object than Σ : it is not of finite type, and comes with an infinite pants decomposition $\tilde{\mathcal{P}}$ induced by that of Σ . However, as all non-compact Riemann surfaces it is Stein: its Fukaya category is amenable to sheaf-theoretic methods, and we show that equivalence (1) from our previous paper applies.

In fact, since $\tilde{\Sigma}$ is not finite type, one needs to finesse somewhat the kind of Fukaya category one wants to work with, because there are several meaningful options. It turns out that for our argument, the relevant object is what we call the *locally finite* Fukaya category, and define in Section 4.3: this includes both compact and non-compact Lagrangians, but satisfying a local finiteness condition. We denote it $\mathrm{Fuk}^{\mathrm{lf}}(\tilde{\Sigma})$; it carries a O_π -action because O_π acts via deck transformations on $\tilde{\Sigma}$. The key observation is that there is an equivalence (see Theorem 6.16 in the main text)

$$(3) \quad \mathrm{Fuk}^{\mathrm{lf}}(\tilde{\Sigma})^{O_\pi} \simeq \mathrm{Fuk}(\Sigma)$$

between O_π -equivariant objects in $\mathrm{Fuk}^{\mathrm{lf}}(\tilde{\Sigma})$, and the Fukaya category of Σ . The left-hand side is in a way more complicated but we can understand it via sheaf theoretic methods and equivalence (1); while $\mathrm{Fuk}(\Sigma)$ can be defined only in terms of Floer-theory. Equivalence (3) readily implies Theorem 1.1.

Remark 1.2. The strategy to prove Theorem 1.1 presented here is not the only one we can imagine. Heather Lee [Lee16] uses carefully chosen Hamiltonians to prove gluing results for open surfaces, and Auroux-Smith have recently extended Lee's technique to the case where a compact surface is cut along a single simple closed curve [AS20, Corollary 5.8]. It seems highly plausible that these techniques could be extended to the case of cutting a compact surface along several circles simultaneously, which would allow one to deduce Theorem 1.1. Instead of using Lee's techniques, we use more generic properties of Fukaya categories, such as the relationship between covering spaces and equivariance, and Viterbo restriction functors for wrapped categories of open surfaces (in addition to the detailed understanding of the structure of the Fukaya categories of open surfaces coming from Haiden-Katzarkov-Kontsevich [HKK17]).

Additionally our methods allow us to answer an open question from [HKK17]. One of the main results of [HKK17] is a geometrization theorem for the (\mathbb{Z} -graded) Fukaya category of a punctured surface Σ : the authors prove that all objects in $\mathrm{Fuk}(\Sigma)$ are *geometric*, i.e. they are equivalent in the Fukaya category to a union of arcs and immersed curves equipped with a brane structure. Question 2 in Section 7 of [HKK17] asks whether geometrization holds in the compact setting. In this article we give a complete positive answer (see Corollary 6.18 in the main text)

Theorem 1.3. *Let Σ be a closed surface of genus $g \geq 2$. Then all objects in $\mathrm{Fuk}(\Sigma)$ are geometric.*

We remark that the case $g = 0$ of the statement is trivial, the case and $g = 1$ is essentially equivalent via HMS to Atiyah's classification of vector bundles on elliptic curves.

Auroux-Smith [AS20] also considers the question of geometricity of objects in Fukaya categories of compact surfaces. Their result is different in that it has both a stronger hypothesis and a stronger conclusion: it says that a spherical object with nonzero Chern character is represented by a simple closed curve with rank one local system.

Homological mirror symmetry. This description of the Fukaya category of a compact Riemann surface of genus $g \geq 2$ has significant applications to HMS. In particular, we recover Seidel and Efimov's beautiful HMS results for curves of genus $g \geq 2$ [Sei11] [Efi12]. Our method of proof however is different, and can help clarify why these mirror constructions actually work. Both Seidel and Efimov start from carefully designed, and somewhat ad hoc, superpotentials. The proof then goes through familiar but delicate steps in HMS: matching generators, and deformation theory. In particular, as pointed out by Seidel in the Introduction of [Sei11], these proofs do not apply to the other constructions of LG mirrors of curves proposed in the literature.

In order to apply Theorem 1.1 to the problem of HMS, we need to study the singularity category of a normal crossing surface X . We require that X that may be presented as $f^{-1}(0)$ for some morphism $f : Y \rightarrow \mathbb{A}^1$ from a smooth 3-fold Y to the line, and also that the dual intersection complex of X be orientable. We require also that the irreducible components of Z , the singular locus of X , are rational curves. These rational curves correspond to the edges of a trivalent graph $G(X)$ that encodes their intersections. We shall prove that, as long as X satisfies the stated requirements, the singularity category $\mathrm{DSing}(X)$ depends only on the graph $G(X)$ but not on any other details of X . For comparison, there is a more general result of Orlov, that states that $\mathrm{DSing}(X)$ (which in this paper is always taken idempotent complete) only depends on the infinitesimal neighbourhood of Z in X . In our setting, the singularity category is also insensitive to the infinitesimal neighbourhood, and is completely determined by the combinatorics of Z ; we must remark, however, that the requirement that X arises as the fiber of a morphism from a smooth 3-fold does entail a topological restriction on the infinitesimal neighbourhood.

The main ingredient is that $\mathrm{DSing}(-)$ satisfies étale (and in particular Zariski) descent with respect to varieties that are presented as the zero fiber of a morphism. This allows us to write $\mathrm{DSing}(X)$ as a limit that matches the limit calculating the Fukaya category in Theorem 1.1. Our main result is a kind of universal HMS statement for compact Riemann surfaces, which we state as Theorem 1.4 below.

If X is a normal crossing surface with rational singular locus Z as before, we associate to it a trivalent graph $G(X)$ as follows: the vertices of $G(X)$ are the singular points of Z ; two vertices are joined by an edge if they lie on the same irreducible component of Z . Pants decompositions of Riemann surfaces also give rise to trivalent graphs via their dual intersection complex. Up to homeomorphism, there is a unique Riemann surface $\Sigma_{G(X)}$ equipped with a pants decomposition corresponding to $G(X)$: if Z is compact, then $\Sigma_{G(X)}$ is a compact Riemann surface without boundary; the genus of $\Sigma_{G(X)}$ is equal to the rank of $H_1(G(X), \mathbb{Z})$.

The following is our main HMS result, see Theorem 7.1 in the body of the paper.

Theorem 1.4. *There is an equivalence of categories*

$$\mathrm{DSing}(X) \simeq \mathrm{Fuk}(\Sigma_{G(X)})$$

Seidel and Efimov’s results are special cases of Theorem 1.4, for very specific choices of the mirror superpotential W (where $X = W^{-1}(0)$). But contrary to their original proofs, our approach shows that it is only the combinatorics of the singular locus that comes into play. This makes it relatively straightforward to compare different mirror constructions: it is enough to check the shape of the singular locus of the superpotential. This viewpoint has several benefits: our proof of HMS applies to all genera, and works equally well in the compact or punctured setting; and it clarifies that HMS depends on a simple geometric relationship between the surface and its mirror.

As we pointed out earlier, surfaces have actually many different geometrically meaningful mirrors. The techniques developed in this paper allow us to easily check that they indeed give rise to HMS equivalences, which are alternative to Seidel’s and Efimov’s. We focus on two constructions:

- (1) Hori–Vafa mirror symmetry matches a toric CY 3-folds X_Σ , equipped with a toric superpotential W_Σ , to a punctured surface Σ equipped with a pants decomposition. We show that the Hori–Vafa picture holds in complete generality for all toric 3-folds, regardless of the CY assumption, and even in the compact setting where no superpotential is available.
- (2) We show that the Mumford degeneration of abelian surfaces provides a natural mirror LG model for Riemann surfaces of genus 2. The same construction appears, for the other direction of HMS, in beautiful recent work of Cannizzo [Can20].

Notes on the second arXiv version. The principal difference between the first and second arXiv versions is that discussion of the sheaf of categories in Section 3 has been significantly expanded and made more precise. It seemed to us that a full exposition of this sheaf of categories would be of

independent interest, especially since others might wish to adapt this sheaf for other purposes. The new version also clarifies and corrects some issues that were raised regarding the first version. One issue, raised by Ed Segal [Seg21], was an ambiguity in the Knörrer periodicity equivalences used in the definition of the sheaf. In the second arXiv version, we make an auxiliary choice of what we call framing data on the graph G to resolve this ambiguity. This requires the addition of a hypothesis in Theorem 5.6 that the dual intersection complex of X be orientable. Another clarification has to do with how the area dependence of the Fukaya categories of surfaces arises from our sheaf of categories. The notion of weights introduced Section 3 makes this area dependence much more explicit.

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2. PRELIMINARIES

2.1. Notations and conventions. Much of the set-up of the paper will be borrowed from our previous article [PS19]. In this section we briefly recapitulate some of the background from [PS19]. We will specify the categorical setting in which we will place ourselves; and then we will recall some basic notations and constructions in the (topological) Fukaya category of surfaces. The reader is advised to consult [PS19] for additional information.

Let κ be a ring of characteristic 0. We remark that in the course of the article it will be important to take κ to be equal to either

- a fixed ground field k ,
- the field Λ of universal Novikov series over k with parameter q ,

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i q^{\lambda_i} \mid a_i \in k, \lambda_i \in \mathbb{R}, \lim_{i \rightarrow \infty} \lambda_i = \infty \right\},$$

- or the subring $\Lambda^{fin} \subset \Lambda$ of Novikov series with finitely many nonzero terms.

2.1.1. Preliminaries on dg categories. By *dg category* we mean one of the following equivalent notions:

- (1) a $\mathbb{Z}/2\mathbb{Z}$ -graded κ -linear dg category, or
- (2) a \mathbb{Z} -graded $\kappa((u))$ -linear dg category, where u has cohomological degree 2 (a 2-periodic dg category).

The former perspective is common in the literature on Fukaya categories, while the second perspective is used in [Pre11], which is a key reference for us.² There is one subtle difference between these perspectives: in the latter one, the operator u can be rescaled as $u \mapsto \gamma u$ for $\gamma \in \kappa^\times$. This changes the $\kappa((u))$ -linear structure, and this kind of twist will appear in Section 3.

Throughout the paper we will work with dg categories up to *Morita equivalence*. References for the Morita theory of κ -linear $\mathbb{Z}/2\mathbb{Z}$ -graded dg categories are Section 1 of [DK18] and Section 2 of [Dyc17].

- (1) We denote by $\mathrm{DGCat}^{(2), \mathrm{non}\text{-}\mathrm{cocmpl}}$ the ∞ -category of (not necessarily small) dg categories

²Preygel denotes u by β and gives it degree -2 , but his differentials have degree -1 .

- (2) We denote by $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$ the ∞ -category of cocomplete κ -linear dg categories. If C is in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$ we denote by C^ω its full subcategory of compact objects
- (3) We denote by $\mathrm{DGCat}_{\mathrm{small}}^{(2)}$ the ∞ -category of small κ -linear dg categories.

There are natural functors

$$\mathrm{U} : \mathrm{DGCat}_{\mathrm{cont}}^{(2)} \rightarrow \mathrm{DGCat}^{(2), \mathrm{non-cocmpl}} \quad , \quad \mathrm{Ind} : \mathrm{DGCat}_{\mathrm{small}}^{(2)} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}^{(2)}$$

given respectively by the forgetful functor, and by Ind-completion. We recall some useful facts about limits and colimits of dg categories.

Lemma 2.1 ([GR17], Corollary 7.2.7). *The functor $\mathrm{Ind} : \mathrm{DGCat}_{\mathrm{small}}^{(2)} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}^{(2)}$ preserves small colimits.*

Lemma 2.2 ([DG15], Proposition 1.7.5). *Let I be a small ∞ -category and let $\Psi : I \rightarrow \mathrm{DGCat}_{\mathrm{cont}}^{(2)}$ a functor. Assume that for all morphisms $\alpha_{i,j} : i \rightarrow j$ in I the functor $\Psi_{i,j} := \Psi(\alpha_{i,j})$ admits a continuous right adjoint $\Phi_{i,j}$. Then there is a well-defined functor $\Phi : I^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}^{(2)}$ such that for all $\alpha_{i,j} : i \rightarrow j$ in I $\Phi(\alpha_{i,j}) = \Phi_{i,j}$, and a canonical equivalence*

$$\varinjlim_I \Psi \simeq \varprojlim_{I^{\mathrm{op}}} \Phi \in \mathrm{DGCat}_{\mathrm{cont}}^{(2)}$$

Lemma 2.3. *Let I be a small ∞ -category and let $\Phi : I \rightarrow \mathrm{DGCat}_{\mathrm{small}}^{(2)}$ a functor. Consider the composition*

$$\mathrm{Ind}(\Phi) : I \xrightarrow{\Phi} \mathrm{DGCat}_{\mathrm{small}}^{(2)} \xrightarrow{\mathrm{Ind}} \mathrm{DGCat}_{\mathrm{cont}}^{(2)}$$

Assume that for all i the limit functor $\varprojlim_I \mathrm{Ind}(\Phi) \rightarrow \Phi(i)$ preserves compact objects. Then there is a fully faithful functor

$$\left(\varprojlim_I \mathrm{Ind}(\Phi) \right)^\omega \rightarrow \varprojlim_I \Phi \in \mathrm{DGCat}_{\mathrm{small}}^{(2)}$$

Additionally, the functor is an equivalence if I is finite.

Proof. The existence of a functor $\left(\varprojlim_I \mathrm{Ind}(\Phi) \right)^\omega \rightarrow \varprojlim_I \Phi$ follows from the universal property of limits, and it is easy to check that it has to be fully-faithful. The last statement follows because a finite limit of compact objects in a triangulated dg category is also compact: this implies that when I is finite we get a functor in the opposite direction $\varprojlim_I \Phi \rightarrow \left(\varprojlim_I \mathrm{Ind}(\Phi) \right)^\omega$, which is an inverse. \square

We conclude this section by fixing some notations for categories of matrix factorizations. The reader can find in Section 3.1.2 [PS19] a quick primer on matrix factorizations; we will make use of notations and results from [Pre11] to which we refer the reader for more information. Let X be a separated Noetherian scheme with a flat map $f : X \rightarrow \mathbb{A}^1$. We denote by $\mathrm{QCoh}^{(2)}(X)$ and $\mathrm{Perf}^{(2)}(X)$ the \mathbb{Z}_2 -folding of the triangulated dg categories of quasi-coherent sheaves and perfect complexes on X . We associate to the pair (X, f) a category of matrix factorizations $\mathrm{MF}(X, f) \in \mathrm{DGCat}_{\mathrm{small}}^{(2)}$. We denote by $\mathrm{MF}^\infty(X, f)$ its Ind-completion

$$\mathrm{MF}^\infty(X, f) := \mathrm{Ind}(\mathrm{MF}(X, f)) \in \mathrm{DGCat}_{\mathrm{cont}}^{(2)}$$

Let X_0 the central fiber of f . There is an equivalence ([Pre11, Proposition 3.4.1])

$$\mathrm{MF}(X, f) \simeq \mathrm{DSing}(X_0) := \mathrm{D}^b\mathrm{Coh}(X_0) / \mathrm{Perf}(X_0)$$

where $\mathrm{DSing}(X_0)$ is the *singularity category* of X_0 . We denote its Ind-completion by $\mathrm{DSing}^\infty(X_0)$. Since the category of matrix factorizations and the singularity category encode the same data, throughout the paper we will switch freely between the two view-points.

2.1.2. *Preliminaries on the Fukaya category.* We briefly recall some basic facts and conventions on the topological Fukaya category, following [PS19]. Let Σ a non-compact Riemann surface, not necessarily of finite type. It is a classical result of Behnke and Stein [BS47] that all non-compact Riemann surfaces are Stein. Let $\Gamma \subset \Sigma$ be a skeleton: this is a ribbon graph locally of finite type.

- (1) The skeleton Γ carries a canonical cosheaf of \mathbb{Z}_2 -graded small triangulated dg categories, $\mathcal{F}^{\text{top}}(-)$

$$U \subset \Gamma \mapsto \mathcal{F}^{\text{top}}(U) \in \text{DGCat}_{\text{small}}^{(2)}$$

If $U \subset V$ are open subsets, we denote the corestriction

$$C : \mathcal{F}^{\text{top}}(V) \rightarrow \mathcal{F}^{\text{top}}(U)$$

We call the global sections $\mathcal{F}^{\text{top}}(\Gamma)$ the *topological Fukaya category* of Σ , and we denote it $\text{Fuk}^{\text{top}}(\Sigma)$.

- (2) Let $\text{Ind} : \text{DGCat}_{\text{small}}^{(2)} \rightarrow \text{DGCat}_{\text{cont}}^{(2)}$ be the Ind-completion functor. Since $\text{Ind}(-)$ preserves small colimits, applying Ind section-wise to a cosheaf yields a cosheaf. We denote $\mathcal{F}_{\infty}^{\text{top}}$ the cosheaf of presentable ∞ -categories

$$U \subset \Gamma \mapsto \mathcal{F}_{\infty}^{\text{top}}(U) := \text{Ind}(\mathcal{F}^{\text{top}}(U)) \in \text{DGCat}_{\text{cont}}^{(2)}$$

If $U \subset V$ are open subsets, we denote the corestriction functor

$$C_{\infty} = \text{Ind}(C) : \mathcal{F}_{\infty}^{\text{top}}(U) \rightarrow \mathcal{F}_{\infty}^{\text{top}}(V)$$

We denote the global sections of $\mathcal{F}_{\infty}^{\text{top}}$ by $\text{Fuk}_{\text{top}}^{\infty}(\Sigma)$. Note that

$$\text{Fuk}_{\text{top}}^{\infty}(\Sigma) \simeq \text{Ind}(\text{Fuk}^{\text{top}}(\Sigma))$$

- (3) The corestriction functors have *continuous* right adjoints $C_{\infty} \dashv R_{\infty}$

$$U \subset V \subset \Gamma, \quad R_{\infty} : \mathcal{F}_{\infty}^{\text{top}}(V) \rightarrow \mathcal{F}_{\infty}^{\text{top}}(U).$$

It follows from Lemma 2.2 that $\mathcal{F}_{\infty}^{\text{top}}(-)$, equipped with R_{∞} , defines a sheaf on Γ with values in $\text{DGCat}_{\text{cont}}^{(2)}$.

- (4) The Ind-completed topological Fukaya category also carries another type of restriction functors, called *exotic restrictions* in [PS19], and considered also in [Dyc17]. Namely, let $U \subset \tilde{\Gamma}$ be an open subgraph without 1-valent vertices, and let $Z \subset \Gamma$ be its complement.

Since Z is also a locally finite ribbon graph, we can meaningfully evaluate on it the topological Fukaya category $\mathcal{F}^{\text{top}}(-)$. There are restriction functors

$$S : \mathcal{F}^{\text{top}}(\Gamma) \rightarrow \mathcal{F}^{\text{top}}(Z), \quad S_{\infty} = \text{Ind}(S) : \mathcal{F}_{\infty}^{\text{top}}(\Gamma) \rightarrow \mathcal{F}_{\infty}^{\text{top}}(Z)$$

fitting into cofiber sequences

$$\mathcal{F}^{\text{top}}(U) \xrightarrow{C} \mathcal{F}^{\text{top}}(\Gamma) \xrightarrow{S} \mathcal{F}^{\text{top}}(Z), \quad \mathcal{F}_{\infty}^{\text{top}}(U) \xrightarrow{C_{\infty}} \mathcal{F}_{\infty}^{\text{top}}(\Gamma) \xrightarrow{S_{\infty}} \mathcal{F}_{\infty}^{\text{top}}(Z)$$

We denote by $T_{\infty} : \mathcal{F}_{\infty}^{\text{top}}(Z) \rightarrow \mathcal{F}_{\infty}^{\text{top}}(\Gamma)$ the right adjoint of S_{∞} , and refer to it as the exceptional corestriction functor. Since S_{∞} is a Verdier localization, its right adjoint T_{∞} is fully-faithful. The functor $T_{\infty} : \mathcal{F}_{\infty}^{\text{top}}(Z) \rightarrow \mathcal{F}_{\infty}^{\text{top}}(\Gamma)$ preserves colimits, and is therefore an arrow in $\text{DGCat}_{\text{cont}}^{(2)}$.

We also record the following simple observation.

Remark 2.4. Let $U \subset V \subset \Gamma$. The symplectic collars of U and V are open sub-surfaces of Σ , let us denote them Σ_U and Σ_V . Assume that Σ_U and Σ_V are *fully stopped*: that is, the intersection of U and V with each connected component of the boundaries of Σ_U and Σ_V , respectively, is non-empty. Then $R_{\infty} : \mathcal{F}_{\infty}^{\text{top}}(V) \rightarrow \mathcal{F}_{\infty}^{\text{top}}(U)$ restricts to compact objects, and in fact $R_{\infty} = \text{Ind}(R)$ where $R : \mathcal{F}^{\text{top}}(V) \rightarrow \mathcal{F}^{\text{top}}(U)$.

Let us briefly explain why this is the case. In the fully stopped setting, the topological Fukaya category is smooth and proper (see [Dyc17]). Since the corestriction $C : \mathcal{F}^{\text{top}}(V) \rightarrow \mathcal{F}^{\text{top}}(U)$ is a functor between smooth and proper categories, it admits a right adjoint

$$R : \mathcal{F}^{\text{top}}(U) \rightarrow \mathcal{F}^{\text{top}}(V).$$

Ind-completion preserves adjoints: that is, an adjunction $C \dashv R$ can be Ind-completed to an adjunction $\text{Ind}(C) \dashv \text{Ind}(R)$. The uniqueness of adjoints then implies that $\text{Ind}(R) \simeq R_\infty$.

By [HKK17] there is an equivalence between the topological Fukaya category $\text{Fuk}^{\text{top}}(\Sigma)$ and the wrapped Fukaya category of Σ

$$(4) \quad \text{Fuk}^{\text{top}}(\Sigma) \simeq \text{Fuk}^w(\Sigma).$$

Ganatra–Pardon–Shende, starting from [GPS20], establish this for more general Weinstein manifolds. Equivalence (4) is key to our approach in this paper. It extends more generally to the partially wrapped setting. Let $\Gamma \subset \Sigma$ be a skeleton with non-compact edges, that is such $S = \Gamma \cap \partial\Sigma$ is non-empty. Let $\text{Fuk}^w(\Sigma, S)$ the partially wrapped Fukaya category relative to the set of stops given by S . Then their work gives an equivalence

$$(5) \quad \mathcal{F}^{\text{top}}(\Gamma) \simeq \text{Fuk}^w(\Sigma, S)$$

3. GRAPHS AND CATEGORIES

Let G be a graph. We assume throughout this section that G has no loops and that all vertices of G have valency 1 or 3. We denote by

- V_G , the set of vertices of G
- H_G , the set of half-edges of G
- E_G , the set of edges of G

If v is a trivalent vertex in V_G we denote by $x_{v,1}, x_{v,2}$ and $x_{v,3}$ the half-edges incident to v ; similarly, if v has valency one we denote x_v the half-edge incident to v . For every $v \in V_G$ we denote H_v the set of half-edges incident to v . There is a map $\alpha : H_G \rightarrow E_G$ that associates to an half-edge the corresponding edge. We allow G to have non-compact edges, i.e. edges that are incident to a vertex only at one of their ends but not the other: $t \in E_G$ is non-compact if and only if the preimage $\alpha^{-1}(t)$ is a singleton. We say that a half-edge x is *incident* to an edge t if $x \in \alpha^{-1}(t)$.

Our graphs will also be decorated with the following data:

- *Weights*, consisting of functions $\alpha : V_G \rightarrow \kappa^\times$ and $\beta : E_G \rightarrow \kappa^\times$ that assign invertible scalars to each vertex and edge of G .
- *Framings*, consisting of a cyclic ordering of the edges at each 3-valent vertex, and an orientation of each edge.

Remark 3.1. The weights are present for two reasons that are intertwined.

- On the A-side, the weights take into account the fact that the Fukaya category depends on a parameter q in the Novikov field Λ .³ When comparing the sheaf of categories we construct to the Fukaya category of a surface with a pants decomposition, we will take $\kappa = \Lambda$ and set $\alpha(v) = q^{-A(v)}$ and $\beta(t) = q^{B(t)}$ for some functions $A : V_G \rightarrow \mathbb{R}$ and $B : E_G \rightarrow \mathbb{R}$ that encode the areas of various parts of the pants decomposition.

³When considering noncompact surfaces, the dependence on q may be trivialized, and when considering compact surfaces of genus $g \geq 2$, there are constructions that allow one to set $q = 1$; in these situations the weights will disappear, see Section 6. However, the covering space argument in Section 6 requires one to work with categories where q is present as a parameter since the q -dependence cannot be equivariantly trivialized.

- On the B-side, the weights take into account the fact that categories such as $\mathrm{MF}(X, f)$ and $\mathrm{DSing}(X_0)$ are 2-periodic dg categories, meaning that there is a prescribed isomorphism from the double shift to the identity. Equivalently, one may regard these categories as being linear over $\kappa((u))$ where u has cohomological degree 2.⁴ Rescaling the potential f leads to an equivalent category, but with a rescaled 2-periodicity structure.

We shall eventually show that, up to equivalence, the category of global sections of our sheaf depends only on the product $\prod_{v \in V_G} \alpha(v) \prod_{t \in E_G} \beta(t)$, and in many cases it does not even depend on that.

Remark 3.2. The framings are necessary in order to address a certain $\mathbb{Z}/2\mathbb{Z}$ -grading ambiguity pointed out in [Seg21]. The fact that this ambiguity arises in the present construction is due to our choice to present the sheaf in terms of B-side categories. On the A-side there is a canonical way of resolving the ambiguity: in brief, this is because the Viterbo or Lee restriction functor preserves the orientations of Lagrangians (rather than reversing them). The framings are the data that allow us to transfer this preferred choice to the B-side. Up to equivalence, the category of global sections does not depend on these data; they are purely auxiliary.

From the HMS perspective, the framings are necessary for the following reason. Let P be the pair of pants. Then there exist HMS equivalences of categories

$$\mathrm{Fuk}(P) \cong \mathrm{MF}(\mathbb{A}^3, x_1 x_2 x_3).$$

There are several such equivalences. The symmetric group S_3 acts on P by symplectomorphisms that permute the three holes of P , and it acts on \mathbb{A}^3 by permuting the variables x_1, x_2, x_3 . Thus S_3 acts on both $\mathrm{Fuk}(P)$ and $\mathrm{MF}(\mathbb{A}^3, x_1 x_2 x_3)$, and this induces an action of S_3 on the set of (homotopy classes of) equivalences between these categories. The key point is that this action is not trivial: even permutations act trivially, but odd permutations act by composing a given equivalence with the shift functor.

Let C be a cylinder. Then we have $\mathrm{Fuk}(C) \cong \mathrm{Perf}^{(2)}(\mathbb{G}_m)$. Now S_2 acts on C by permuting the holes, and it acts on \mathbb{G}_m by the inversion morphism. Again, the action of S_2 on the set of such equivalences is not trivial: it acts by composition with a shift.

3.1. Local model for the restriction functors. Before setting up the sheaf of categories on graphs, we will present a detailed construction of the restriction functors in a local model consisting of a single trivalent vertex. This allows us to suppress some indices and make the essential points clearer.

So let G consist of a single trivalent vertex v , and three edges. Equip G with a cyclic ordering of the edges, and an orientation of each edge. Let us name the edges t_1, t_2, t_3 so that the chosen cyclic order of the edges is compatible with the standard cyclic order of the indices 1, 2, 3. Then we may choose weights consisting of invertible scalars $\alpha = \alpha(v) \in \kappa^\times$ at the vertex v and $\beta_i = \beta(t_i) \in \kappa^\times$ at the edges t_i .

The stalk at the vertex v is

$$\mathcal{B}_v = \mathrm{MF}^\infty(\mathbb{A}^3, \alpha x_1 x_2 x_3).$$

Note that the weight appears in the potential; the indexing of the variables is meant to reflect a bijection between these variables and the set of edges. For each edge t_i , construct the ring $\kappa[x_i, \tilde{x}_i]/(x_i \tilde{x}_i - \beta_i)$. Note that the weight appears in the relation. We write $\mathbb{G}_{m, x_i, \tilde{x}_i, \beta_i}$ for the spectrum of this ring. The stalk at the edge t_i is

$$\mathcal{B}_{t_i} = \mathrm{QCoh}^{(2)}(\mathbb{G}_{m, x_i, \tilde{x}_i, \beta_i}).$$

⁴The Fukaya category may also be regarded this way, but we usually treat the Fukaya category as a $\mathbb{Z}/2\mathbb{Z}$ -graded category, meaning that the double shift is strictly equal to identity.

What we need to do is specify precisely a collection of restriction functors $\mathcal{B}_v \rightarrow \mathcal{B}_{t_i}$. For each $i = 1, 2, 3$, we can consider a composition

$$R_i : \mathcal{B}_v = \mathrm{MF}^\infty(\mathbb{A}^3, \alpha x_1 x_2 x_3) \rightarrow \mathrm{MF}^\infty(\mathbb{A}^3 \setminus \{x_i = 0\}, \alpha x_1 x_2 x_3) \rightarrow \mathrm{QCoh}^{(2)}(\mathbb{G}_{m, x_i, \tilde{x}_i, \beta_i}) = \mathcal{B}_{t_i},$$

where the first functor is restriction to a Zariski open subset, and the second functor is a Knörrer periodicity equivalence. Two points deserve emphasis:

- (1) This functor is to be chosen so that the variable x_i appearing in the notation for the source category matches the variable x_i appearing in the notation for the target category. In particular, these two variables cannot be rescaled independently.
- (2) There are two choices for the Knörrer periodicity equivalence that differ from each other by a shift (note that our categories are $\mathbb{Z}/2\mathbb{Z}$ -graded so the double shift is the identity).

The latter point means that we need to specify a preferred choice of Knörrer periodicity equivalence for each half-edge. We do this using the framing data. Recall that these data determine a cyclic ordering of the edges at v , and an orientation of each edge. For each *unordered* pair of distinct indices $i, j \in \{1, 2, 3\}$, let $k \in \{1, 2, 3\}$ be the third index (so that $\{i, j\} \cup \{k\} = \{1, 2, 3\}$), and construct the matrix factorization

$$(6) \quad F_{ij} = \left(\mathcal{O}_{\mathbb{A}^3} \begin{array}{c} \xrightarrow{x_i x_j} \\ \xleftarrow{\alpha x_k} \end{array} \mathcal{O}_{\mathbb{A}^3} \right)$$

where the left-hand $\mathcal{O}_{\mathbb{A}^3}$ is placed in even degree. When restricting to the i -th or j -th edge, the object F_{ij} is mapped by Knörrer periodicity to an object isomorphic to either $\mathcal{O}_{\mathbb{G}_m}$ or $\mathcal{O}_{\mathbb{G}_m}[1]$. We stipulate the following rule:

- Suppose j follows i with respect to the given cyclic order at v , then:
- If t_i is oriented into v , we require $R_i(F_{ij}) \cong \mathcal{O}_{\mathbb{G}_m}$.
- If t_i is oriented out of v , we require $R_i(F_{ij}) \cong \mathcal{O}_{\mathbb{G}_m}[1]$.
- If t_j is oriented into v , we require $R_j(F_{ij}) \cong \mathcal{O}_{\mathbb{G}_m}[1]$.
- If t_j is oriented out of v , we require $R_j(F_{ij}) \cong \mathcal{O}_{\mathbb{G}_m}$.

For instance, the condition $R_i(F_{ij}) \cong \mathcal{O}_{\mathbb{G}_m}$ obtains when the inverse functor

$$\mathrm{QCoh}^{(2)}(\mathbb{G}_{m, x_i, \tilde{x}_i, \beta_i}) \rightarrow \mathrm{MF}^\infty(\mathbb{A}^3 \setminus \{x_i = 0\}, \alpha x_1 x_2 x_3)$$

is given by tensoring with the restriction of F_{ij} . The reason why this convention is consistent is that if $\{i, j, k\}$ is an unordered triple of pairwise distinct indices, then $R_j(F_{ij}) = R_j(F_{jk})[1]$ with respect to *any* choice of Knörrer periodicity equivalence at the edge j . It is evident that reversing the orientation of an edge modifies the corresponding restriction by a shift, and reversing the cyclic ordering at the vertex modifies all of the restrictions at that vertex by a shift.

3.2. Dependence of the local model on the weights. We now turn to the question: In what sense do the local models depend on the weights? One may argue that the categories $\mathrm{MF}(\mathbb{A}^3, \alpha x_1 x_2 x_3)$ are all equivalent as α varies, and the rings $\kappa[x, \tilde{x}]/(x\tilde{x} - \beta)$ are all isomorphic as β varies, so it would seem that these weights could all be set to 1 once and for all. However, this is not appropriate in our setting because, as mentioned above, we regard the coordinates x_1, x_2, x_3 as fixed; these coordinates serve to “mark” the categories and are used in the construction of the restriction functors.

The case of $\kappa[x, \tilde{x}]/(x\tilde{x} - \beta)$ is the more transparent one: it is clear that any automorphism of this ring that fixes the coordinates x, \tilde{x} cannot change the value of β : indeed, the value of β may be extracted as the product of the coordinates.

The case of $\mathrm{MF}(\mathbb{A}^3, \alpha x_1 x_2 x_3)$ is more subtle. A famous theorem of Orlov [Orl03] implies that there is an equivalence of categories⁵

$$\mathrm{MF}(\mathbb{A}^3, \alpha x_1 x_2 x_3) \cong \mathrm{DSing}(\{x_1 x_2 x_3 = 0\}).$$

⁵All our categories are idempotent complete.

The right-hand side appears not to depend on α , but in fact there is a finer structure that does depend on α , namely the 2-periodic or $\kappa((u))$ -linear structure.

The way this works can be seen using Preygel's approach to $\text{MF}(X, f)$ [Pre11]. Preygel begins by considering the category of tuples (\mathcal{E}, d, B) where (\mathcal{E}, d) is a perfect complex on the total space X and $B \in \text{hom}^{-1}(\mathcal{E}, \mathcal{E})$ is a nullhomotopy for the multiplication by f such that $B^2 = 0$; this is a dg model for $\text{Coh}(X_0)$. Next, tensor everything with $\kappa[[u]]$, where u has cohomological degree 2, and replace (\mathcal{E}, d, B) with $(\mathcal{E}, d + uB)$. Note that $(d + uB)^2 = uf$, so the term "matrix factorization" is justified. Lastly, localize by inverting u to obtain the $\kappa((u))$ -linear category $\text{MF}(X, f)$. This category is 2-periodic, with the periodicity given by multiplication by u . The formulae make clear the effect of rescaling f : if we rescale f to αf for some $\alpha \in \kappa^\times$, then to get an equivalent category we must rescale u to $\alpha^{-1}u$.

There is also a way to view this in terms Orlov-style $\mathbb{Z}/2\mathbb{Z}$ -graded matrix factorizations. In this construction, we start with pairs

$$(7) \quad F = \left(\mathcal{E}_0 \begin{array}{c} \xrightarrow{t_0} \\ \xleftarrow{t_1} \end{array} \mathcal{E}_1 \right)$$

such that $t_1 t_0 = f \cdot 1_{\mathcal{E}_0}$ and $t_0 t_1 = f \cdot 1_{\mathcal{E}_1}$. For two such objects F, F' , the differential d on the complex $\text{hom}(F, F')$ is the graded commutator with the internal differentials t, t' . Now suppose that f is rescaled by the factor c^2 for some $c \in \kappa^\times$. There are several ways to convert a matrix factorization for f into one for $c^2 f$. The simplest is to rescale all internal differentials such as t_0, t_1 by c . This has the effect of rescaling the differential d on $\text{hom}(F, F')$ by c , but it does not change the chain-level composition. Since the cocycles and coboundaries for d and cd are the same, there is a natural identification of the cohomology categories.

To see the effect of rescaling the differential in terms of A_∞ -structures, we use the homological perturbation method. This involves, for each hom complex, choosing a projection Π onto its cohomology as well as a homotopy operator P such that

$$Pd + dP = I - \Pi.$$

Because this equation is inhomogeneous, when we rescale d by c we should rescale P by c^{-1} . The homological perturbation method then produces an A_∞ -structure on the cohomology category with vanishing differential. For each $d \geq 2$, the formula for the d -th operation m_d uses the homotopy operator $(d - 2)$ times, and hence the operator m_d is rescaled by c^{2-d} . There is no canonical way to trivialize this dependence on c in the $\mathbb{Z}/2\mathbb{Z}$ -graded setting.

This shows that $\text{MF}(X, f)$ and $\text{MF}(X, \alpha f)$ are not canonically equivalent with their 2-periodic structures, but of course there can be other ways to construct an equivalence between these categories. For instance, if X carries a \mathbb{G}_m -action such that f is homogeneous with nonzero weight, we can rescale the coordinates on X to absorb the scaling factor c^2 . This already applies to the case $(X, f) = (\mathbb{A}^3, x_1 x_2 x_3)$ (in several ways).

Using the classification of A_∞ -structures presented in [AAE⁺13], it is possible to show that any equivalence $\text{MF}(\mathbb{A}^3, \alpha x_1 x_2 x_3) \cong \text{MF}(\mathbb{A}^3, \alpha' x_1 x_2 x_3)$ for $\alpha \neq \alpha'$ that is compatible with the 2-periodic structure *must* involve rescaling the coordinates x_1, x_2, x_3 in some way.

To make a precise statement, consider the matrix factorizations

$$(8) \quad F_{ij}(\sqrt{\alpha}) = \left(\mathcal{O}_{\mathbb{A}^3} \begin{array}{c} \xrightarrow{\sqrt{\alpha} x_i x_j} \\ \xleftarrow{\sqrt{\alpha} x_k} \end{array} \mathcal{O}_{\mathbb{A}^3} \right)$$

These objects generate $\text{MF}(\mathbb{A}^3, \alpha x_1 x_2 x_3)$, and they correspond to each other for various values of α according to our identifications. Furthermore, the coordinates x_i actually appear in the cohomology category as generators of $H^0(\text{hom}(F_{ij}(\sqrt{\alpha}), F_{ij}(\sqrt{\alpha})) \cong \kappa[x_i, x_j]/(x_i x_j))$. Recall that a *strict homotopy equivalence* between A_∞ -categories whose cohomologies have been identified is an A_∞ -equivalence that induces identity on the cohomology category. We use the term *semi-strict*

homotopy equivalence to refer to an A_∞ -equivalence that preserves the objects $F_{ij}(\sqrt{\alpha})$ and the morphisms denoted x_i , but which may behave arbitrarily on other morphisms.

Proposition 3.3. *Suppose that $\alpha, \alpha' \in \kappa^\times$ are such that $\text{MF}(\mathbb{A}^3, \alpha x_1 x_2 x_3)$ and $\text{MF}(\mathbb{A}^3, \alpha' x_1 x_2 x_3)$ are semi-strictly homotopy equivalent as $\kappa((u))$ -linear categories. Then $\alpha = \alpha'$.*

Suppose that $\alpha'/\alpha = c^2$ for some $c \in \kappa^\times$. Let \mathcal{C} be the A_∞ -category obtained from the minimal model of $\text{MF}(\mathbb{A}^3, \alpha x_1 x_2 x_3)$ by rescaling m_d by c^{2-d} . Then \mathcal{C} is semi-strictly homotopy equivalent to $\text{MF}(\mathbb{A}^3, \alpha' x_1 x_2 x_3)$.

Proof. Our argument is based on the results of [AAE⁺13]. Consider the generating objects $F_{ij}(\sqrt{\alpha})$ defined in (8), which are preserved by a semi-strict homotopy equivalence. The cohomology category of these objects is a certain quiver algebra with relations which the authors of [AAE⁺13] denote by $A_{(p,q)}$, where p and q are certain grading data that are irrelevant in the $\mathbb{Z}/2\mathbb{Z}$ -graded setting. The variables x_1, x_2, x_3 then serve as names for certain loops of length two in the quiver $A_{(p,q)}$. The A_∞ -structures on $A_{(p,q)}$ are classified up to strict homotopy equivalence by two scalars a, b , which are extracted from certain m_3 operations. These structures are denoted $m^{a,b}$. Rescaling m_d by c^{2-d} thus changes $m^{a,b}$ to $m^{c^{-1}a, c^{-1}b}$. It is demonstrated that the categories $\text{MF}(\mathbb{A}^3, \alpha x_1 x_2 x_3)$ correspond to nonzero values of a and b . By rescaling the arrows in $A_{p,q}$, all of the structures $m^{a,b}$ with $a \neq 0$ and $b \neq 0$ are equivalent to $m^{1,1}$. However, such rescalings may modify the morphisms named x_1, x_2, x_3 . If we only allow rescalings that do not modify these morphisms, then $m^{a,b}$ is only equivalent to $m^{\lambda a, \lambda^{-1}b}$, and so the product ab of the two scalars is invariant with respect to this class of rescalings. Since $m^{a,b}$ and $m^{c^{-1}a, c^{-1}b}$ give different values for this product, they are not equivalent with respect to the class of rescalings that fix x_1, x_2, x_3 . \square

3.3. Relationship of the local model to the Fukaya category. We shall now describe how the local model for the restriction defined in Section 3.1 is compatible with the theory of Fukaya categories.

Let the notation be as in Section 3.1: let v be a trivalent vertex with incident edges t_1, t_2, t_3 , with weights $\alpha, \beta_1, \beta_2, \beta_3$, and with orientations on the edges and a cyclic ordering at the vertex that is compatible with our indexing of the edges. We take $\kappa = \Lambda$, the Novikov field, and we assume that the weights are of the form $\alpha = q^{-A}$ and $\beta_i = q^{B_i}$ ($i = 1, 2, 3$) for some real numbers A, B_1, B_2, B_3 to be determined.

To the vertex v we associate a pair of pants (sphere with three holes) P , whose boundary components $(\partial P)_i$ are in bijection with the edges t_i , and to each edge t_i we associate a cylindrical neighborhood or collar C_i around the corresponding boundary component of P . The weights A, B_1, B_2, B_3 will be related to the size of the symplectic form on the pieces P, C_1, C_2, C_3 respectively, and the framing data are used to single out certain collections of generating objects for the associated Fukaya categories.

For each pair of boundary components of P , there is a Lagrangian arc that connects them. The cyclic ordering of the edges (part of the framing) induces a cyclic ordering of the boundary components, and we orient these arcs so as to be compatible with this cyclic ordering. We notate these objects as L_{12}, L_{23}, L_{31} , where L_{ij} is an arc connecting boundary component i to boundary component j and oriented from i to j . (Reversing the cyclic ordering of the edges at v reverses the orientations on these objects, which categorically speaking is the same as applying the shift functor.)

For each C_i , we take an arc K_i joining the two boundary components of C_i . We choose the orientation on K_i to be induced from the orientation on the edge t_i : if the orientation on t_i points into the vertex v , then the orientation on K_i points toward the interior of the pair of pants, and vice versa. Note that this orientation may differ from the orientation of $L_{ij} \cap C_i$.

Next, form the (wrapped) Fukaya categories $\text{Fuk}(P)$ and $\text{Fuk}(C_i)$ over Λ , and apply HMS [AAE⁺13]. There is an equivalence of categories

$$\Phi : \text{Fuk}(P) \cong \text{MF}(\mathbb{A}^3, q^{-A}x_1x_2x_3)$$

That sends generating objects L_{ij} to $F_{ij}(q^{-A/2})$ as in (8), where A is the area of the pair of pants. Note that we have used the cyclic ordering. In terms of [AAE⁺13], this is the model for the Fukaya category that is generated by odd morphisms u_{ij} and v_{ji} for cyclically ordered i, j such that $m_3(u_{12}, u_{23}, u_{31})$ and $m_3(v_{13}, v_{32}, v_{21})$ have constant term $q^{A/2}$.

Now consider $\text{Fuk}(C_i)$. There is an equivalence of categories

$$\Psi_i : \text{Fuk}(C_i) \cong \text{Perf}^{(2)}(\mathbb{G}_{m, x_i, \tilde{x}_i, B_i})$$

where B_i is the area of the annulus. $\text{Fuk}(C_i)$ is generated by the arc K_i with a specific orientation determined by the edge orientation, and we require that it corresponds to the structure sheaf under Ψ_i . The variables x_i and \tilde{x}_i are to be interpreted as the basic Reeb chords that generate the endomorphism of this object.

We now explain the functor

$$R_i : \text{Fuk}(P) \rightarrow \text{Fuk}(C_i)$$

which we will define as a variant of the Viterbo restriction functor. To define it, we need to make use of a Liouville one-form θ on P , which is a one form such that $d\theta = \omega$ is the symplectic form, and such that the corresponding Liouville vector field points outward along the boundary. The choice of θ is not unique, since two choices θ, θ' differ by a closed one form that represents a class $[\theta' - \theta] \in H^1(P, \mathbb{R})$. For each i , we choose a form θ_i such that $C_i \subset P$ forms a Liouville subdomain. The core or skeleton associated to θ_i will then consist of a circle contained in C_i with some other 1-cells attached. It is not possible to choose a single form θ such that all three annuli C_1, C_2, C_3 form Liouville subdomains simultaneously, because P does not retract onto its boundary plus a collection of 1-cells; a 2-cell is required. However, it is possible to choose a form such that two of the three annuli are Liouville subdomains (the skeleton is then a dumbbell graph).

Algebraically, the choice of θ_i trivializes the categories $\text{Fuk}(P)$ and $\text{Fuk}(C_i)$ with respect to the Novikov parameter q . This is done by rescaling all morphisms by certain q -powers. Denote by $\text{Fuk}^{ex}(P, \theta_i)$ and $\text{Fuk}^{ex}(C_i, \theta_i)$ the resulting categories over the ground field k . Geometrically, the existence of a Liouville embedding $C_i \rightarrow P$ gives a Viterbo restriction functor

$$R_i^{ex} : \text{Fuk}^{ex}(P, \theta_i) \rightarrow \text{Fuk}^{ex}(C_i, \theta_i).$$

that induces a functor $R_i : \text{Fuk}(P) \rightarrow \text{Fuk}(C_i)$ once we restore the trivialized q parameter.

Remark 3.4. Even though R_i is obtained from a functor over k by base change to Λ , the diagram consisting of all three restrictions R_1, R_2, R_3 is *not* the base change to Λ of a diagram over k , precisely because we cannot choose a single Liouville form that works for all three annuli.

It remains to verify that the corresponding functor between B-side categories does have the properties required in Section 3.1. The first requirement is that the coordinate x_i should correspond under R_i to the coordinate of the same name. This holds because, when we trivialize the q parameter using θ_i , the morphisms in $\text{Fuk}(P)$ and $\text{Fuk}(C_i)$ named by x_i are rescaled in the same way.

The second requirement has to do with the shift ambiguity in the Knörrer periodicity equivalence. The key observation is that Viterbo restriction does not reverse the orientation of the Lagrangians. We spell out the details: Choose a pair of indices ij so that j follows i in the cyclic order at v . Then the matrix factorization F_{ij} is mapped by Φ^{-1} to L_{ij} , where L_{ij} is the arc on P that connects boundary i to boundary j , and that is oriented into P at i and out of P at j .

- If t_i is oriented into v , the orientations on K_i and L_{ij} match, and $R_i(F_{ij}) \cong \mathcal{O}_{\mathbb{G}_m}$.
- If t_i is oriented out of v , the orientations on K_i and L_{ij} differ, and $R_i(F_{ij}) \cong \mathcal{O}_{\mathbb{G}_m}[1]$.
- If t_j is oriented into v , the orientations on K_j and L_{ij} differ, and $R_j(F_{ij}) \cong \mathcal{O}_{\mathbb{G}_m}[1]$.

- If t_j is oriented out of v , the orientations on K_j and L_{ij} match, and $R_j(F_{ij}) \cong \mathcal{O}_{\mathbb{G}_m}$.

3.4. Sheaves of categories on graphs. In Section 3.1.3 of [PS19] we explained how to associate to a graph G a locally constant sheaf of categories $\mathcal{B}(-)$, with values in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$. In this section we enlarge slightly the set-up of [PS19]: in [PS19] we required G to be trivalent and planar, but in fact this is not necessary for the construction, and it will be useful to consider more general graphs.

Note that specifying $\mathcal{B}(-)$ means more concretely giving the following data:

- for each vertex v of G , a category \mathcal{B}_v , which is the stalk of $\mathcal{B}(-)$ at v
- for each edge t of G , a category \mathcal{B}_t , which is the stalk of $\mathcal{B}(-)$ at an interior point of t
- if t is incident to v , a restriction functor $\mathcal{B}_v \rightarrow \mathcal{B}_t$

The information can be encoded equivalently in terms of the *exit-path quiver* of G : the exit-path quiver has as vertices the strata of G (that is, vertices and edges), while the arrows keep track of inclusions of strata. We find it convenient to work in fact with a small variation of the exit-path quiver of G , which is bicolored. Namely, we associate to G a bipartite quiver Q_G with black and white vertices:

- (1) The set of black vertices of Q_G is given by V_G ; the set of white vertices is given by E_G
- (2) All the arrows in Q_G go from black to white vertices: if v is in V_G and t is in E_G , the set of arrows between v and t is the set of half-edges x incident to v such that $\alpha(x) = t$

Note that there are either one or no arrows between a black vertex \bullet_v and a white vertex \circ_t ; an arrow $\bullet_v \rightarrow \circ_t$ is naturally labeled by an half-edge incident to v . We will write $q \in Q_G$ to refer to any vertex q in Q_G , without specifying the color. As we said, if we forget the coloring of the vertices of Q_G , we get back the usual exit path quiver of G .

The data defining the constructible sheaf of categories \mathcal{B} can be phrased in terms of a functor between Q_G and $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$, which with small abuse of notation we keep denoting $\mathcal{B}(-)$. For clarity let us make a concrete example: if t is an edge of G , then this corresponds to a white vertex \circ_t of Q_G ; the value of the functor $\mathcal{B}(-)$ on the vertex \circ_t coincides with the stalk \mathcal{B}_t at an interior point of t . In the following we shall switch between these two perspectives according to what is more convenient in a given context.

Before defining \mathcal{B} we need to fix some notations. Let T be a finite set with n elements: we will regard the elements of T as variables by working with the free commutative algebra over T , denoted $\kappa[t, t \in T]$, and subalgebras spanned by subsets $T' \subset T$. We set

$$\mathbb{A}_T^n := \mathrm{Spec}(\kappa[t, t \in T]), \quad \text{and} \quad W_T := \times_{t \in T} t : \mathbb{A}_T^n \longrightarrow \mathbb{A}^1.$$

If t is in T we denote $D(t) = \mathbb{A}_T^n - \{t = 0\}$ the corresponding principal open subset. We set $\mathbb{G}_{m,t} := \mathrm{Spec}(\kappa[t, t^{-1}])$, and $\mathbb{A}_t^1 = \mathrm{Spec}(\kappa[t])$. We denote $j : \mathbb{G}_{m,t} \rightarrow \mathbb{A}_t^1$ the inclusion. If $t, t' \in T$ and $\beta \in \kappa^\times$, we set $\mathbb{G}_{m,t,t',\beta} := \mathrm{Spec}(\kappa[t, t'] / t t' - \beta)$. We have inclusions $\mathbb{A}_t^1 \xleftarrow{j} \mathbb{G}_{m,t,t',\beta} \xrightarrow{j} \mathbb{A}_{t'}^1$. The canonical isomorphisms $\mathbb{G}_{m,t,t',\beta} \cong \mathbb{G}_{m,t'}$ and $\mathbb{G}_{m,t,t',\beta} \cong \mathbb{G}_{m,t}$ determine equivalences

$$\mathrm{QCoh}^{(2)}(\mathbb{G}_{m,t'}) \simeq \mathrm{QCoh}^{(2)}(\mathbb{G}_{m,t,t',\beta}) \quad \mathrm{QCoh}^{(2)}(\mathbb{G}_{m,t}) \simeq \mathrm{QCoh}^{(2)}(\mathbb{G}_{m,t,t',\beta})$$

Now we define the sheaf of categories \mathcal{B} :

- (1) We define $\mathcal{B}(-)$ on the vertices of Q_G as follows:

- If v is a trivalent vertex of G ,

$$\mathcal{B}_v = \mathcal{B}(\bullet_v) := \mathrm{MF}^\infty(\mathbb{A}_{H_v}^3, \alpha(v)W_{H_v})$$

- If v has valency one,

$$\mathcal{B}_v = \mathcal{B}(\bullet_v) := \mathrm{QCoh}^{(2)}(\mathbb{A}_{x_v}^1)$$

- If t is a non-compact edge of G , and x is the unique half-edge incident to t ,

$$\mathcal{B}_t = \mathcal{B}(\circ_t) := \mathrm{QCoh}^{(2)}(\mathbb{G}_{m,x})$$

- if t is a compact edge of G , and x and y are the two half-edges incident to t , we set $\mathcal{B}(\circ_t) := \mathrm{QCoh}^{(2)}(\mathbb{G}_{m,x,y,\beta(t)})$
- (2) We define $\mathcal{B}(-)$ on the arrows of Q_G as follows:
- If v is trivalent and $\bullet_v \xrightarrow{x} \circ_t$ is an arrow,

$$\mathcal{B}(x) : \mathcal{B}(\bullet_v) = \mathrm{MF}^\infty(\mathbb{A}_{H_v}^3, \alpha(v)W_{H_v}) \xrightarrow{\iota^*} \mathrm{QCoh}^{(2)}(\mathbb{G}_{m,x}) \simeq \mathcal{B}(\circ_t)$$

is the functor defined in Section 3.1, which uses the framings to determine the Knörrer periodicity equivalence.

- If v has valency one and $\bullet_v \xrightarrow{x} \circ_t$ is an arrow,

$$\mathcal{B}(x) : \mathcal{B}(\bullet_v) = \mathrm{QCoh}^{(2)}(\mathbb{A}_x^1) \xrightarrow{j^*} \mathrm{QCoh}^{(2)}(\mathbb{G}_{m,x}) \simeq \mathcal{B}(\circ_t)$$

Remark 3.5. The quiver Q_G is not actually a category. Formally speaking the source of $\mathcal{B}(-)$ is the ∞ -category which the nerve of the 1-category associated with Q_G . However since as are no composable arrows in Q_G , to define such a functor it is enough to specify what it does on vertices and arrows.

The global sections of \mathcal{B} over G can be computed in the usual way by taking a limit over the quiver Q_G

$$(9) \quad \mathcal{B}(G) \simeq \varprojlim_{q \in Q_G} \mathcal{B}(q) \in \mathrm{DGCat}_{\mathrm{cont}}^{(2)}$$

Remark 3.6. Let us explain more concretely how to interpret (9). Assume for simplicity that all vertices of G have valency 3. If v_1, v_2 are vertices of G joined by an edge t we have a diagram

$$\mathcal{B}_{v_1} \times \mathcal{B}_{v_2} \rightrightarrows \mathcal{B}_t$$

where the arrows are given by projection followed by restriction. Running over the vertices and edges of G yields a Čech diagram in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$ and $\mathcal{B}(G)$ is the following equalizer

$$(10) \quad \mathcal{B}(G) \rightarrow \left[\bigoplus_{v \in V_G} \mathcal{B}_v \rightrightarrows \bigoplus_{t \in E_G} \mathcal{B}_t \right]$$

Formula (9) encodes the same information: it states that $\mathcal{B}(G)$ is equivalent to the equalizer

$$(11) \quad \mathcal{B}(G) \rightarrow \left[\bigoplus_{\bullet_v \in Q_G} \mathcal{B}(\bullet_v) \rightrightarrows \bigoplus_{\circ_t \in Q_G} \mathcal{B}(\circ_t) \right]$$

which is the same as (10). Throughout the paper we will switch freely between the formalisms (10) and (11) to compute the global sections of \mathcal{B} , depending on what is more convenient in a given situation.

The assignment above allow us to uniquely define the sheaf \mathcal{B} over any graph with 1 or 3-valent vertices and no loops. In more technical terms, \mathcal{B} defines a sheaf over the site made up by these graphs and open inclusions between them. If $U \subset V$ is an open inclusion of graphs we denote the restrictions functor $S_{\mathcal{B}} : \mathcal{B}(V) \rightarrow \mathcal{B}(U)$.

We record the following elementary fact.

Lemma 3.7. *Let $U \subset V$ be an open inclusion of graphs with vertices of valency one or three, and let $S_{\mathcal{B}} : \mathcal{B}(V) \rightarrow \mathcal{B}(U)$ be the restriction functor. Then the left and the right adjoint of $S_{\mathcal{B}}$ are naturally equivalent: that is, we have ambidexterous adjunctions*

$$T_{\mathcal{B}} \dashv S_{\mathcal{B}} \dashv T_{\mathcal{B}}, \quad S_{\mathcal{B}} \dashv T_{\mathcal{B}} \dashv S_{\mathcal{B}}$$

Proof. Clearly, it is sufficient to prove that we have

$$S_{\mathcal{B}} \dashv T_{\mathcal{B}} \dashv S_{\mathcal{B}}.$$

By construction, $\mathcal{B}(U)$ and $\mathcal{B}(V)$ can be realized as limits of the categories associated to the vertices \mathcal{B}_v and the edges \mathcal{B}_t . The corestriction functor $T_{\mathcal{B}}$ can be naturally defined in terms of this presentation. Indeed we can write $T_{\mathcal{B}}$ as a limit of basic functors only involving the categories \mathcal{B}_v and \mathcal{B}_t : namely $\mathcal{B}_v \xrightarrow{\cong} \mathcal{B}_v$ if v belongs to both U and V ; the zero functor $0 \rightarrow \mathcal{B}_v$, or the right adjoint $\mathcal{B}_t \rightarrow \mathcal{B}_v$ to the restriction $\mathcal{B}_v \rightarrow \mathcal{B}_t$, when v is a vertex which does not belong to U .

It is therefore sufficient to show that these three basic functors admit ambidexterous adjoints. This clearly holds for the identity and the zero functor. So we only need to check the corestriction $\mathcal{B}_t \rightarrow \mathcal{B}_v$. We will explain the case when v is 3-valent, the 1-valent case being similar. Let $X = \text{Spec}(\kappa[x, y]/xy)$. Note that we can rewrite this as

$$\begin{array}{ccc} \mathcal{B}_t & \longrightarrow & \mathcal{B}_v \\ \cong \downarrow & & \downarrow \cong \\ \text{IndCoh}^{(2)}(\mathbb{G}_m) & \xrightarrow{i_*} & \text{IndCoh}^{(2)}(X) \end{array}$$

where $i : \mathbb{G}_m \rightarrow X$ is the open inclusion of an axis minus the origin. Then the fact that we have an ambidexterous adjunction $i^* \dashv i_* \dashv i^! \simeq i^*$ is well-known; a reference is for instance Appendix A.2 of [Pre11] (see the proof of Proposition A.2.3). \square

Remark 3.8. It will also be useful for us to work with small categories. We will consider the constructible pre-sheaf of small categories \mathcal{B}^ω defined as follows

- the stalk of \mathcal{B}^ω on a vertex v of G is given by \mathcal{B}_v^ω
- the stalk of \mathcal{B}^ω on an edge t of G is given by \mathcal{B}_t^ω
- the restrictions are given by $S_{\mathcal{B}}^\omega : \mathcal{B}_v^\omega \rightarrow \mathcal{B}_t^\omega$

The pre-sheaf \mathcal{B}^ω is well defined, since the restrictions functors preserve compact objects: e.g. if v is trivalent we can rewrite $S_{\mathcal{B}}^\omega$ in terms of matrix factorization categories as

$$\begin{array}{ccc} \mathcal{B}_v^\omega & \xrightarrow{S_{\mathcal{B}}^\omega} & \mathcal{B}_e^\omega \\ \cong \downarrow & & \downarrow \cong \\ \text{MF}(\mathbb{A}_{x,y,z}^3, W = xyz) & \xrightarrow{i^*} & \text{Perf}^{(2)}(\mathbb{G}_m) \end{array}$$

If G is a graph, we will denote by $\mathcal{B}^\omega(G)$ the sections of \mathcal{B}^ω on G . Note that, as we did with \mathcal{B} , we can encode \mathcal{B}^ω as a functor $Q_G \rightarrow \text{DGCat}_{\text{small}}^{(2)}$.

Let us make two observations on the differences between \mathcal{B} and \mathcal{B}^ω . First, \mathcal{B}^ω is actually a sheaf as long as G is finite, but it is only a presheaf if G is not finite. Second, the corestrictions $T_{\mathcal{B}}$ do not preserve compact objects and therefore are not well defined in the setting of \mathcal{B}^ω .

3.5. Dependence of $\mathcal{B}(G)$ on the weights and framings. Let G be a graph as above with weights $\alpha : V_G \rightarrow \kappa^\times$ and $\beta : E_G \rightarrow \kappa^\times$ and framings f . Denote by $\mathcal{B}(G, \alpha, \beta, f)$ the category of global sections constructed above.

Proposition 3.9. *Suppose G is connected. Up to $\kappa((u))$ -linear equivalence, the category $\mathcal{B}(G, \alpha, \beta, f)$ depends only on the underlying graph G and the product of the weights $\gamma = \prod_{v \in V_G} \alpha(v) \prod_{t \in E_G} \beta(t)$. If the graph G has a noncompact edge or a 1-valent vertex, this category does not even depend on the weights.*

Proof. First we show that $\mathcal{B}(G, \alpha, \beta, f)$ does not depend on the framings f . Any two framings f, f' may be obtained from one another by a sequence of moves where the orientation of an edge is reversed, or the cyclic ordering at a trivalent vertex is reversed.

- If f and f' differ only in the orientation of the edge t , then the diagrams computing $\mathcal{B}(G, \alpha, \beta, f)$ and $\mathcal{B}(G, \alpha, \beta, f')$ are equivalent via the functor that acts on $\mathcal{B}(\circ_t)$ by the shift [1] and the identity elsewhere.
- If f and f' differ only in the cyclic ordering at the vertex v , then the diagrams computing $\mathcal{B}(G, \alpha, \beta, f)$ and $\mathcal{B}(G, \alpha, \beta, f')$ are equivalent via the functor that acts on $\mathcal{B}(\bullet_v)$ by the shift [1] and the identity elsewhere.

Next we address the dependence on the weights. This has to do with rescaling the coordinates in the presentation. Recall the local model for the restriction

$$R_i : \mathrm{MF}^\infty(\mathbb{A}^3, \alpha(v)x_1x_2x_3) \rightarrow \mathrm{QCoh}^{(2)}(\mathrm{Spec} \kappa[x_i, \tilde{x}_i]/(x_i\tilde{x}_i - \beta(t_i)))$$

If the variable x_i is rescaled to $x'_i = \lambda x_i$, then $\alpha(v)$ changes to $\alpha'(v) = \lambda^{-1}\alpha(v)$, while $\beta(t_i)$ becomes $\beta'(t_i) = \lambda\beta(t_i)$. Thus the product of all weights is unchanged. In essence, the weights define a 0-chain on the quiver Q_G with values in κ^\times , $(\alpha, \beta) \in C_0(Q_G, \kappa^\times)$, while the set of possible rescalings λ is the set of 1-chains on Q_G , $\lambda \in C_1(Q_G, \kappa^\times)$. The effect of performing a rescaling is to add the boundary of λ : $(\alpha', \beta') = (\alpha, \beta) + \partial\lambda$. Thus any two sets of weights that define the same class in $H_0(Q_G, \kappa^\times) = \kappa^\times$ lead to equivalent categories.

Moreover, this argument shows that by rescaling, all weights can be set to 1 except for a single one, which may be at a vertex or an edge. In the case where G has a noncompact edge or a 1-valent vertex, the weights may be concentrated at the corresponding edge, and it amounts to a specific choice of coordinate along that edge, which may be disregarded after the limit has been taken. \square

Definition 3.10. For a graph G and a weight $\gamma \in \kappa^\times$, we shall use the abbreviated notation $\mathcal{B}(G, \gamma)$ to denote $\mathcal{B}(G, \alpha, \beta, f)$ where α and β are a set of weights whose product is γ , and f is any choice of framings. The previous proposition implies that this category is well-defined up to $\kappa((u))$ -linear equivalence.

When comparing $\mathcal{B}(G, \gamma)$ with the Fukaya category of surface, we take $\alpha(v) = q^{-A(v)}$ and $\beta(t) = q^{B(t)}$ where $A(v)$ is the area of the pair of pants at v and $B(t)$ is the area of the annulus at t . We then have

$$-\mathrm{val}_q(\gamma) = -\mathrm{val}_q \left(\prod_{v \in V_G} \alpha(v) \prod_{t \in E_G} \beta(t) \right) = \sum_{v \in V_G} A(v) - \sum_{t \in E_G} B(t),$$

which is the total area of the surface: the sum of the areas of the pairs of pants double-counts the area of each annulus.

When comparing $\mathcal{B}(G, \gamma)$ with categories of matrix factorizations, it is useful to think of the parameter γ as twisting the 2-periodicity structure. That is to say, the categories $\mathcal{B}(G, \gamma)$ are not necessarily equivalent as $\kappa((u))$ -linear categories ($\deg u = 2$), but they are equivalent as κ -linear categories. More precisely, $\mathcal{B}(G, \gamma)$ is equivalent to $\mathcal{B}(G, 1)$ with the substitution $u \mapsto \gamma u$.

Remark 3.11. As shown in [Seg21], it is possible to twist the construction of $\mathcal{B}(G)$ using a $\mathbb{Z}/2\mathbb{Z}$ -valued cocycle $\gamma \in C^1(G, \mathbb{Z}/2\mathbb{Z})$. The effect is to modify the gluing along the edge t to incorporate a shift $[\gamma(t)]$. The result depends up to equivalence on the cohomology class $[\gamma] \in H^1(G, \mathbb{Z}/2\mathbb{Z})$. We refer the reader to [Seg21] for further discussion of these twists, as well as the \mathbb{Z} -graded case where these twists have a more fundamental importance.

4. TOPOLOGICAL FUKAYA CATEGORY AND PANTS DECOMPOSITION

The main technical result of [PS19] was a description of the topological Fukaya category of a punctured Riemann surface of finite type in terms of pants decomposition. In this section we revisit that result, and extend it to some non-finite type Riemann surfaces arising as covers of compact Riemann surfaces. In the non-finite type setting the kind of Fukaya category which will be most relevant for us is the *locally finite* Fukaya category, which will be introduced in Section 4.3. We

remark that not all the results in this section will actually be needed for the later sections. However we believe that giving an account of the theory of [PS19] in the more general setting of non-finite type Riemann surfaces might be of some independent interest.

In this section all of the weights are set to 1. When dealing with topological Fukaya categories of open Riemann surfaces, this does not entail a loss of generality.

4.1. Riemann surfaces and maximal tropical covers. Let Σ be a compact Riemann surface of genus $g \geq 2$, equipped with a pants decomposition \mathcal{P} . We can encode the pants decomposition \mathcal{P} into a trivalent graph G . The graph G is the dual intersection complex of the pants decomposition: that is, vertices of G are in bijection with the pants making up the decomposition, and the half-edges incident to a vertex v correspond to the boundary components of the corresponding pair of pants \mathcal{P}_v ; two vertices v_1, v_2 are connected by an edge if the pants \mathcal{P}_{v_1} and \mathcal{P}_{v_2} are glued along the corresponding boundary components. For simplicity, we will assume that no pairs-of-pant in \mathcal{P} self-intersects, i.e. G contains no loops. The graph G is a kind of combinatorial tropicalization of Σ . In particular we can define a map of topological spaces $\Sigma \rightarrow G$, such that the preimages of the vertices of G are homeomorphic to pairs-of-pants, and the preimages of points in the interior of the edges of G are S^1 .

We will work with an infinite-sheeted unramified cover $\tilde{\Sigma}$ of Σ , which we define next. Let $\Sigma \rightarrow G$ be the tropicalization map. Let \tilde{G} be the maximal abelian cover of G , and let $q : \tilde{G} \rightarrow G$ be the covering map. Then $\tilde{\Sigma}$ can be defined as the fiber product in spaces

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\pi} & \Sigma \\ \downarrow & & \downarrow \\ \tilde{G} & \xrightarrow{q} & G \end{array}$$

Equivalently, we can consider the natural homomorphism $\pi_1(\Sigma) \rightarrow \pi_1(G)$: then the regular covering space $\tilde{\Sigma}$ corresponds to quotient of $\pi_1(\Sigma)$ given by

$$\pi_1(\Sigma) \rightarrow \pi_1(G) \rightarrow \pi_1(G)/[\pi_1(G), \pi_1(G)] \cong \mathbb{Z}^g$$

Definition 4.1. We call $\pi : \tilde{\Sigma} \rightarrow \Sigma$ a *maximal tropical cover* of Σ .

By construction $\tilde{\Sigma}$ is an infinite type Riemann surface that comes equipped with a regular covering map $\pi : \tilde{\Sigma} \rightarrow \Sigma$ with group of deck transformation given by $O_\pi := \mathbb{Z}^g$. A pants decomposition of Σ induces a pants decomposition $\tilde{\mathcal{P}}$ of $\tilde{\Sigma}$. The trivalent graph \tilde{G} encoding the pants decomposition $\tilde{\mathcal{P}}$ is \tilde{G} . The action by the group of deck transformations \mathbb{Z}^g preserves the pants decomposition.

Since $\tilde{\Sigma}$ is an infinite-genus Riemann surface, it is in particular non-compact. By a classical result of Behnke and Stein [BS47] all non-compact Riemann surfaces are Stein. As a consequence, we can study the Fukaya category of $\tilde{\Sigma}$ via the topological models which become available in the Stein setting. In particular $\tilde{\Sigma}$ can be equipped with a skeleton $\tilde{\Gamma}$, which is an infinite but locally finite ribbon graph: $\tilde{\Gamma}$ is homotopy equivalent to an enumerable wedge of circles.

4.1.1. Covers by finite type subsurfaces. Let $\mathcal{U} = \{\tilde{\Sigma}_i\}_{i \in \mathbb{N}}$ be an cover of $\tilde{\Sigma}$ by open subsurfaces of finite type having the following properties:

- (1) for all i , $\tilde{\Sigma}_i \subset \tilde{\Sigma}_{i+1}$
- (2) $\tilde{\Sigma} = \bigcup_{i \in \mathbb{N}} \tilde{\Sigma}_i$
- (3) for all i , $\tilde{\Sigma}_i$ is a connected open Riemann surface of finite genus, which is the interior of a union of pants P in $\tilde{\mathcal{P}}$.

We say that \mathcal{U} is *compatible* with $\tilde{\mathcal{P}}$.

Definition 4.2. Let $\tilde{\Gamma}$ be a skeleton of $\tilde{\Sigma}$. We say that $\tilde{\Gamma}$ is *adapted to \mathcal{U}* if for all $i \in \mathbb{N}$, $\tilde{\Gamma}$ intersects transversely the boundary of $\tilde{\Sigma}_i$ and

$$\Gamma_i := \tilde{\Sigma}_i \cap \tilde{\Gamma}$$

is a skeleton for $\tilde{\Sigma}_i$, possibly with *stops*: i.e. edges incident to the boundary $\partial\tilde{\Sigma}_i$.

Lemma 4.3. *Given an open cover \mathcal{U} of $\tilde{\Sigma}$ satisfying properties (1), (2) and (3) above there exist skeleta adapted to \mathcal{U} .*

Proof. For concreteness we will briefly explain the case $g = 2$, where we work with the minimal pants decomposition. The general case can be dealt with similarly. We make first a preliminary observation. Let $\tilde{\Sigma}$ be a non-compact Riemann surface such that can be written as $\tilde{\Sigma} = \tilde{\Sigma}_1 \amalg_Z \tilde{\Sigma}_2$ where

- (1) $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are Riemann surfaces with boundary, and $Z \subset \partial\tilde{\Sigma}_1 \cap \partial\tilde{\Sigma}_2$
- (2) $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are equipped with skeleta Γ_1 and Γ_2 such that $Z \subset \Gamma_1 \cap \Gamma_2$.

Then $\Gamma := \Gamma_1 \amalg_Z \Gamma_2$ is a skeleton for $\tilde{\Sigma}$.

Now let Σ be a compact Riemann surface of genus 2. Let $\tilde{\Sigma} \rightarrow \Sigma$ be its maximal tropical cover. There is unique pants decomposition of Σ made up of two pairs of pants which do not self-intersect: we denote these pairs-of-pants Σ_1 and Σ_2 and write $\Sigma = \Sigma_1 \cup_D \Sigma_2$, where $D \cong S^1 \amalg S^1 \amalg S^1$. The maximal abelian cover of Σ_i can be described explicitly as follows.

Consider the equilateral simplicial tessellation of \mathbb{R}^2 . We can view its boundary as a locally finite ribbon graph Δ embedded in \mathbb{R}^2 : let T be a spanning tree for this graph. Let B be the union of the open balls $B(x, \epsilon)$ with radius $\epsilon \ll 1$, where x ranges over the vertices of Δ . Let ∂B be the boundary of B : this is an enumerable union of circles with centers in the vertices of Δ . Then $\tilde{\Sigma}_i$ can be realized as $\mathbb{R}^2 - B$. A skeleton for $\tilde{\Sigma}_i$ is given by

$$\Gamma_i = (T - D) \cup \partial B.$$

In words, Γ_i is a connected ribbon graph obtained by drawing edges between the components of ∂B , but in such a way that no new loops are created.

The covering $\tilde{\Sigma} \rightarrow \Sigma$ can be realized via the identification

$$\tilde{\Sigma} \cong \tilde{\Sigma}_1 \amalg_{\partial B} \tilde{\Sigma}_1 \longrightarrow \Sigma_1 \amalg_D \Sigma_2 \cong \Sigma.$$

Note that $\tilde{\Sigma}$ carries a natural pants decomposition $\tilde{\mathcal{P}}$. Also $\tilde{\Sigma}$ inherits a skeleton $\tilde{\Gamma} = \Gamma_1 \amalg_{\partial B} \Gamma_2$. Then the skeleton $\tilde{\Gamma}$ is adapted to \mathcal{U} , for any open cover \mathcal{U} compatible with $\tilde{\mathcal{P}}$.

Indeed, let U be an open subset of $\tilde{\Sigma}$ which is the interior of a union of pants. Then $U \cap \Gamma_i$ is a skeleton (possibly with non-compact edges) of $U \cap \tilde{\Sigma}_i$: indeed $U \cap \tilde{\Sigma}_i$ retracts onto a bouquet of N circles for some N , and $\Gamma_i \cap \tilde{\Sigma}_i$ has the same homotopy type, which can be easily checked by inspection. Then it follows that

$$U \cong U \cap \tilde{\Sigma}_1 \amalg_{U \cap \partial B} U \cap \tilde{\Sigma}_2 \text{ retracts onto } U \cap \Gamma = U \cap \Gamma_1 \amalg_{U \cap \partial B} U \cap \Gamma_2$$

which is what we needed to prove. \square

Remark 4.4. If $\mathcal{U} = \{\tilde{\Sigma}_i\}_{i \in \mathbb{N}}$ is an open cover as above, and $\tilde{\Gamma}$ is a skeleton adapted to \mathcal{U} , then $\tilde{\Gamma}$ will have at least one stop on each boundary component of $\tilde{\Sigma}_i$, for all i . That is, for each connected component B of $\partial\tilde{\Sigma}_i$, the intersection $\Gamma_i \cap B$ is non-empty.

4.2. The topological Fukaya category of $\tilde{\Sigma}$. In this Section we explain how to compute the Fukaya category of $\tilde{\Sigma}$ via the sheaf of categories on graphs introduced in Section 3.4. This follows closely [PS19] except for the fact that, since $\tilde{\Sigma}$ is not of finite type, some extra argument is needed.

We fix once and for all a covering $\mathcal{U} = \{\tilde{\Sigma}_i\}_{i \in \mathbb{N}}$ compatible with \mathcal{P} , and a skeleton Γ adapted to \mathcal{U} . It follows immediately from the definition that

$$\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma}) = \varinjlim_i \mathcal{F}^{\mathrm{top}}(\Gamma_i), \quad \mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}) = \varinjlim_i \mathcal{F}_\infty^{\mathrm{top}}(\Gamma_i).$$

By definition Γ_i is a skeleton for $\tilde{\Sigma}_i$, possibly with stops. The closure of $\tilde{\Sigma}_i$ is a Riemann surface with boundary equipped with a pants decomposition. Let $G_i \subset \tilde{G}$ be the open sub-graph encoding the pants decomposition of $\tilde{\Sigma}_i$. We can evaluate the sheaf $\mathcal{B}(-)$ on G_i . By Theorem 8.3 of [PS19] we have that $\mathcal{B}(G_i)$ is equivalent to the Ind-completion of the topological Fukaya category of $\tilde{\Sigma}_i$

$$\mathcal{B}(G_i) \simeq \mathrm{Ind}(\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma}_i))$$

Now let Γ'_i be the compact skeleton of $\tilde{\Sigma}_i$ obtained by removing the non-compact edges of Γ_i . Since the topological Fukaya category of a surface can be computed by evaluating $\mathcal{F}^{\mathrm{top}}$ on a compact skeleton we have that $\mathcal{B}(G_i) \simeq \mathcal{F}_\infty^{\mathrm{top}}(\Gamma'_i)$.

We have inclusions

$$\Gamma'_i \subset \Gamma_i, \quad \Gamma'_i \subset \Gamma'_{i+1}, \quad \Gamma_i \subset \Gamma'_{i+1}$$

where Γ'_i is a closed subgraph of Γ_i and of Γ'_{i+1} , while Γ_i is an open subgraph of Γ'_{i+1} . This yields functors

$$T_\infty : \mathcal{B}(G_i) \simeq \mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}_i) \longrightarrow \mathcal{F}_\infty^{\mathrm{top}}(\Gamma_i), \quad T_\infty : \mathcal{B}(G_i) \simeq \mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}_i) \longrightarrow \mathcal{B}(G_{i+1}) \simeq \mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}_{i+1}),$$

$$C_\infty : \mathcal{F}_\infty^{\mathrm{top}}(\Gamma_i) \longrightarrow \mathcal{B}(G_{i+1}) \simeq \mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}_{i+1})$$

Note that $S_\infty : \mathcal{B}(G_{i+1}) \rightarrow \mathcal{B}(G_i)$ coincides with the restriction functor $S_{\mathcal{B}}$ of the sheaf \mathcal{B} discussed in Section 3.4. Thus $T_\infty : \mathcal{B}(G_i) \rightarrow \mathcal{B}(G_{i+1})$ is the same as the corestriction functor $T_{\mathcal{B}}$ introduced in Lemma 3.7. For all i , we have factorizations

$$\mathcal{F}_\infty^{\mathrm{top}}(\Gamma_i) \xrightarrow{C_\infty} \mathcal{B}(G_{i+1}) \xrightarrow{T_\infty} \mathcal{F}_\infty^{\mathrm{top}}(\Gamma_{i+1}) \quad \mathcal{B}(G_i) \xrightarrow{T_\infty} \mathcal{F}_\infty^{\mathrm{top}}(\Gamma_i) \xrightarrow{C_\infty} \mathcal{B}(G_{i+1})$$

Proposition 4.5. *There is an equivalence*

$$\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}) \simeq \varinjlim_i \mathcal{F}_\infty^{\mathrm{top}}(\Gamma_i) \simeq \varinjlim_i \mathcal{B}(G_i)$$

Proof. Consider the directed system in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$

$$C_1 := \mathcal{B}(G_1) \xrightarrow{T_\infty} C_2 := \mathcal{F}_\infty^{\mathrm{top}}(\Gamma_1) \xrightarrow{C_\infty} C_3 := \mathcal{B}(G_2) \xrightarrow{T_\infty} C_4 := \mathcal{F}_\infty^{\mathrm{top}}(\Gamma_2) \xrightarrow{C_\infty} \dots$$

Then by cofinality

$$\varinjlim_i \mathcal{B}(G_i) \simeq \varinjlim_i C_i \simeq \varinjlim_i \mathcal{F}_\infty^{\mathrm{top}}(\Gamma_i) \simeq \mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}).$$

□

Now consider the inductive system in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$

$$\mathcal{B}(G_1) \xrightarrow{S_\infty} \mathcal{B}(G_2) \xrightarrow{S_\infty} \mathcal{B}(G_3) \xrightarrow{S_\infty} \dots$$

Proposition 4.6. *There is an equivalence*

$$\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}) \simeq \varinjlim_i \mathcal{B}(G_i) \simeq \mathcal{B}(\tilde{G})$$

Proof. By Lemma 3.7, S_∞ is also the right adjoint of T_∞ . By Proposition 4.5, we have that $\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}) \simeq \varinjlim_i \mathcal{B}(G_i)$, where the colimit is computed in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$. By Lemma 2.2 taking right adjoint yields

$$\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}) \simeq \varprojlim_i \mathcal{B}(G_i) \text{ in } \mathrm{DGCat}_{\mathrm{cont}}^{(2)}.$$

Now, since $\{G_i\}_{i \in \mathbb{N}}$ is an open cover of \tilde{G} , the limit $\varprojlim_i \mathcal{B}(G_i)$ computes the global sections of \mathcal{B} , and is therefore equivalent to $\mathcal{B}(\tilde{G})$. \square

As a consequence of Proposition 4.6 we can rewrite $\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma})$ as a limit of a diagram in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$ having

- (1) as vertices, the categories \mathcal{B}_v and \mathcal{B}_e , where v and e vary over the vertices and edges of \tilde{G}
- (2) as structure morphisms, the restrictions $\mathcal{B}_v \rightarrow \mathcal{B}_e$, whenever e is incident to v .

More precisely, let us denote by \tilde{L} be the poset of strata of \tilde{G} , where \tilde{G} is stratified by its vertices and edges. Since \mathcal{B} is a constructible sheaf we can compute its global sections $\mathcal{B}(\tilde{G})$ by taking a limit over its stalks on strata: this limit is naturally indexed by the poset \tilde{L} . We can express this by writing

$$\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}) \simeq \mathcal{B}(\tilde{G}) \simeq \varprojlim_{l \in \tilde{L}} \mathcal{B}_l$$

where $l \in \tilde{L}$ correspond to the strata of \tilde{G} (i.e. its vertices and edges), and \mathcal{B}_l denotes the stalk of $\mathcal{B}(-)$ on the stratum l .

The group O_π of deck transformations of $\pi : \tilde{\Sigma} \rightarrow \Sigma$ acts on $\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma})$. We will use the presentation of $\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma})$ we have just given to compute the O_π -equivariant category $\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma})^{O_\pi}$. Recall that O_π is also the group of deck transformation of the maximal abelian cover $q : \tilde{G} \rightarrow G$.

Proposition 4.7. *There is an equivalence*

$$\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma})^{O_\pi} \simeq \mathcal{B}(\tilde{G})^{O_\pi} \simeq \mathcal{B}(G)$$

Proof. As we explained, $\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma})$ can be realized as the limit of a diagram of categories in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$ indexed by the poset of strata of \tilde{G} , which we denoted \tilde{L} . Additionally, the O_π -action on $\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma})$ can be easily understood via this presentation. Namely the group O_π acts on \tilde{L} , and therefore on the right-hand side of

$$\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma}) \simeq \varprojlim_{l \in \tilde{L}} \mathcal{B}_l$$

and this induces the action of O_π on $\mathrm{Fuk}_\infty^{\mathrm{top}}(\tilde{\Sigma})$. Note that clearly $\tilde{L}/O_\pi \cong L$, where L is the poset of strata of G . Note also that

$$\mathcal{B}(G) \simeq \varprojlim_{l \in L} \mathcal{B}_l$$

Since passing to the invariant category is a limit, and limits commute with limits, we can write

$$\mathcal{B}(\tilde{G})^{O_\pi} \simeq \left(\varprojlim_{l \in \tilde{L}} \mathcal{B}_l \right)^{O_\pi} \simeq \varprojlim_{l \in (\tilde{L}/O_\pi)} \mathcal{B}_l \simeq \mathcal{B}(G)$$

and this concludes the proof. \square

4.3. The locally finite topological Fukaya category. The account of the topological Fukaya category of non-finite type Riemann surfaces which we have given so far focused on the Ind-completed Fukaya category. However for the applications we have in mind it will be useful to work with small categories. It turns out that for our purposes the topological Fukaya category of $\tilde{\Sigma}$

$$\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma}) = \mathrm{Fuk}_{\infty}^{\mathrm{top}}(\tilde{\Sigma})^{\omega} \simeq \mathcal{B}(\tilde{G})^{\omega}$$

is not quite the right category, it is too small. In particular, although the O_{π} -action restricts to $\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma})$, the O_{π} -equivariant category $\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma})^{O_{\pi}}$ is the zero-category.

The key to resolving the issue is the following observation: whereas $\mathcal{F}_{\infty}^{\mathrm{top}}(-)$ with restrictions R_{∞} defines a sheaf with values in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$ (see Section 2.1.2 (3)), this is no longer true after restricting to compact objects. To fix this we need to sheafify the assignment

$$(12) \quad U \subset \tilde{\Gamma} \mapsto (\mathcal{F}_{\infty}(U))^{\omega} \in \mathrm{DGCat}_{\mathrm{small}}^{(2)}.$$

This yields a sheaf $\mathcal{F}^{\mathrm{lf}}(-)$ with values in $\mathrm{DGCat}_{\mathrm{small}}^{(2)}$: the superscript *lf* stands for *locally finite*, meaning that locally on the skeleton the sections of $\mathcal{F}^{\mathrm{lf}}(-)$ are the compact objects inside $\mathcal{F}_{\infty}^{\mathrm{top}}(-)$.

Definition 4.8. We call the global sections of the sheaf $\mathcal{F}^{\mathrm{lf}}(-)$ the *locally finite topological Fukaya category* of $\tilde{\Sigma}$, and we denote it $\mathrm{Fuk}^{\mathrm{top,lf}}(\tilde{\Sigma})$.

Remark 4.9. The category $\mathrm{Fuk}^{\mathrm{top,lf}}(\tilde{\Sigma})$ is, by construction, the Morita dual of $\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma})$: that is

$$\mathrm{Fuk}^{\mathrm{top,lf}}(\tilde{\Sigma}) \simeq \mathrm{Fun}_{\mathrm{DGCat}_{\mathrm{small}}^{(2)}}(\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma}), \mathrm{Perf}^{(2)}(k)).$$

As such it was considered in [STZ14] and [DK18] as a model for the compact, or *infinitesimally wrapped* Fukaya category of a punctured Riemann surface.

We are going to prove for $\mathrm{Fuk}^{\mathrm{top,lf}}(\tilde{\Sigma})$ some of the structure results we have established for $\mathrm{Fuk}_{\infty}^{\mathrm{top}}(\tilde{\Sigma})$ in the previous section. Since the arguments are very similar for the two cases, we will limit ourselves to a somewhat abbreviated treatment.

Recall from Remark 3.8 that \mathcal{B}^{ω} is the presheaf of small categories obtained from \mathcal{B} by taking compact objects section-wise.

Proposition 4.10. *There is an equivalence*

$$\mathrm{Fuk}^{\mathrm{top,lf}}(\tilde{\Sigma}) \simeq \varprojlim_i \mathcal{B}^{\omega}(G_i) \simeq \varprojlim_i \mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma}_i)$$

Proof. By Remark 4.4 and Remark 2.4, for all i the category $\mathcal{F}^{\mathrm{top}}(\Gamma_i)$ is a smooth and proper, and the restriction functor $R_{\infty} : \mathcal{F}_{\infty}^{\mathrm{top}}(\Gamma_{i+1}) \rightarrow \mathcal{F}_{\infty}^{\mathrm{top}}(\Gamma_i)$ restricts to compact objects, that is we can write $R_{\infty} = \mathrm{Ind}(R)$. By definition we can write

$$\mathrm{Fuk}^{\mathrm{top,lf}}(\tilde{\Sigma}) \simeq \varprojlim_i \mathcal{F}^{\mathrm{top}}(\Gamma_i).$$

As we discussed, there is an equivalence $\mathcal{B}^{\omega}(G_i) \simeq \mathcal{F}^{\mathrm{top}}(\Gamma'_i)$. This yields factorizations

$$\begin{array}{ccc} & \xleftarrow{R} & \\ \mathcal{F}^{\mathrm{top}}(\Gamma_i) & \xleftarrow{R} \mathcal{B}(G_{i+1}) \xleftarrow{S} \mathcal{F}^{\mathrm{top}}(\Gamma_{i+1}) & \xleftarrow{S_{\mathcal{B}^{\omega}}} \mathcal{B}(G_i)^{\omega} \xleftarrow{S} \mathcal{F}^{\mathrm{top}}(\Gamma_i) \xleftarrow{R} \mathcal{B}(G_{i+1})^{\omega} \end{array}$$

Then we can conclude by a cofinality argument exactly as in the proof of Proposition 4.5, by looking at the inverse system

$$\mathcal{B}(G_1)^{\omega} \xleftarrow{S} \mathcal{F}^{\mathrm{top}}(\Gamma_1) \xleftarrow{R} \mathcal{B}(G_2)^{\omega} \xleftarrow{S} \mathcal{F}^{\mathrm{top}}(\Gamma_2) \xleftarrow{R} \dots$$

□

Remark 4.11. Note that there are fully-faithful embeddings

$$\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma}) \subset \mathrm{Fuk}^{\mathrm{top},\mathrm{lf}}(\tilde{\Sigma}) \subset \mathrm{Fuk}_{\infty}^{\mathrm{top}}(\tilde{\Sigma}).$$

The second inclusion is an immediate consequence of Proposition 4.10. As for the first inclusion, recall that by Proposition 4.6 $\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma}) \simeq (\varprojlim \mathcal{B}(G_i))^{\omega}$. Then by Lemma 2.3,

$$\mathrm{Fuk}^{\mathrm{top}}(\tilde{\Sigma}) \subset \varprojlim \mathcal{B}(G_i)^{\omega} \simeq \mathrm{Fuk}^{\mathrm{top},\mathrm{lf}}(\tilde{\Sigma}).$$

Proposition 4.12. *The O_{π} -action on $\mathrm{Fuk}_{\infty}^{\mathrm{top}}(\tilde{\Sigma})$ restricts to $\mathrm{Fuk}^{\mathrm{top},\mathrm{lf}}(\tilde{\Sigma})$ and there is an equivalence*

$$\mathrm{Fuk}^{\mathrm{top},\mathrm{lf}}(\tilde{\Sigma})^{O_{\pi}} \simeq \mathcal{B}^{\omega}(G)$$

Proof. The proof is the same as the proof of Proposition 4.7, from which we import notations. First note that we can rewrite the equivalence from Proposition 4.10 as

$$\mathrm{Fuk}^{\mathrm{top},\mathrm{lf}}(\tilde{\Sigma}) \simeq \varprojlim_{l \in \tilde{L}} (\mathcal{B}_l)^{\omega}$$

As in the proof of Proposition 4.7, this allows us to understand very concretely the O_{π} -action: indeed the group O_{π} acts on \tilde{L} , and therefore on the limit $\varprojlim_{l \in \tilde{L}} \mathcal{B}_l$. Now $\tilde{L}/G \cong L$, where L is the poset of strata of G , and we conclude that

$$\mathrm{Fuk}^{\mathrm{top},\mathrm{lf}}(\tilde{\Sigma})^{O_{\pi}} \simeq \varprojlim_{l \in (\tilde{L}/O_{\pi})} (\mathcal{B}_l)^{\omega} \stackrel{(*)}{\simeq} \mathcal{B}^{\omega}(G).$$

Let us comment briefly on equivalence (*): since \mathcal{B}^{ω} is in general just a pre-sheaf, it is not a priori clear that its sections on G should satisfy descent, and therefore could be expressed as $\varprojlim_{l \in L} (\mathcal{B}_l)^{\omega}$. However \mathcal{B}^{ω} is indeed a sheaf when G is finite, as in our case: see Remark 3.8. \square

5. SINGULARITY CATEGORIES AND GRAPHS

In this section we explain how to express the singularity category of a normal crossing surface in terms of the combinatorial sheaves of categories on graphs introduced in Section 3.4. This is a key ingredient in the proof of HMS for compact surfaces which will be carried out in Section 7.

5.1. Singularity categories of normal crossing surfaces. Let T be a smooth variety of dimension 3 and let $X \subset T$ be a simple normal crossing divisor of the form $X = f^{-1}(0)$, where $f : T \rightarrow \mathbb{A}^1$ is a morphism. Denote by Z the singular locus of X , and by S the singular locus of Z . We make the following assumptions

- (1) The irreducible component of Z are rational curves isomorphic to either \mathbb{G}_m , \mathbb{A}^1 or \mathbb{P}^1
- (2) Let C be a irreducible component of Z . Then the intersection between C and S is empty if $C \cong \mathbb{G}_m$, has cardinality at most one if $C \cong \mathbb{A}^1$, and has cardinality at most two if $C \cong \mathbb{P}^1$

Definition 5.1. If these assumptions are satisfied we say that X has a *graph-like* singular locus.

Remark 5.2. Note that if $X \subset Y$ is a toric divisor inside a smooth toric 3-fold, then X has graph-like singular locus.

We fix once and for all parametrizations of the irreducible components of Z compatible, in an appropriate sense, with the stratification of Z given by S :

- Let C be a component of Z isomorphic to \mathbb{A}^1 , then we fix a parametrization $\phi : \mathbb{A}^1 \xrightarrow{\cong} C$ such that $\phi(\mathbb{A}^1 - \{0\}) \cap S = \emptyset$
- Let C be a component of Z isomorphic to \mathbb{P}^1 then we fix a parametrization $\psi : \mathbb{P}^1 \xrightarrow{\cong} C$ such that $\psi(\mathbb{P}^1 - \{0, \infty\}) \cap S = \emptyset$

Now let S' be the subset of Z obtained as the union of the images $\phi(\{0\})$, and $\psi(\{0, \infty\})$, as ϕ and ψ run over the fixed parametrizations of the components of Z . Note that $S \subset S'$.

Definition 5.3. We will refer to the points in S' as *marked points*.

Example 5.4. (1) Let $T = \mathbb{P}^1 \times \mathbb{A}^2$, and let $X = X_1 \cup X_2$ where $X_1 = \mathbb{P}^1 \times \mathbb{A}^1 \times \{0\}$ and $X_2 = \mathbb{P}^1 \times \{0\} \times \mathbb{A}^1$. Then $Z \cong \mathbb{P}^1$, $S = \emptyset$, and $S' = \{(0, 0, 0), (\infty, 0, 0)\} \in Z$.
 (2) Let $T = T^*\mathbb{P}^1 \times \mathbb{A}^1$, and let $X = X_1 \cup X_2 \cup X_3$ where $X_1 = T_0^*\mathbb{P}^1 \times \mathbb{A}^1$, $X_2 = \mathbb{P}^1 \times \mathbb{A}^1$ and $X_3 = T_\infty^*\mathbb{P}^1 \times \mathbb{A}^1$. Then Z has five irreducible components, and $S = S'$ has cardinality two and is given by $(0, 0), (\infty, 0) \in \mathbb{P}^1 \times \mathbb{A}^1 \subset T^*\mathbb{P}^1 \times \mathbb{A}^1$.

Let $G(X)$ be the graph having as vertices the points of S' , and as edges the irreducible components of Z : if C is an irreducible component of Z and $s \in S'$, then the edge t_C is incident to the vertex v_s if and only if s lies on C . Note that $G(X)$ has no loops and the vertices of $G(X)$ have valency equal to either three or one.

Note that the graph $G(X)$ does not depend on the choices of parametrization of the irreducible components of Z : any such choice gives rise to equivalent graphs.

Example 5.5. Let us follow up on Example 5.4. In case (1), $G(X)$ has two 1-valent vertices and one edge joining them. In case (2), $G(X)$ has two 3-valent vertices, one compact edge joining them, and four non-compact ones.

Observe that the dual intersection complex of the simple normal crossing divisor X is naturally a triangulated real two-dimensional topological manifold.

The following theorem is the main result of this section.

Theorem 5.6. *Suppose that the dual intersection complex of X is orientable. Then there is an κ -linear equivalence of categories*

$$\mathrm{DSing}^\infty(X) \simeq \mathcal{B}(G(X))$$

We prove Theorem 5.6 in Section 5.1.1 below.

Remark 5.7. Before proceeding, we should clarify that in the statement of Theorem 5.6, the 2-periodicity of $\mathrm{DSing}^\infty(X)$ is regarded as a *property* rather than a *structure*. If we choose a way of presenting X as the zero fiber of a map $f : T \rightarrow \mathbb{A}^1$, then, via the equivalence $\mathrm{MF}^\infty(T, f) \simeq \mathrm{DSing}^\infty(X)$, the singularity category is endowed with a 2-periodic *structure*. The version of Theorem 5.6 that takes this structure into account is a $\kappa((u))$ -linear equivalence

$$\mathrm{MF}^\infty(T, f) \simeq \mathcal{B}(G(X), \gamma)$$

where $\gamma \in \kappa^\times$ accounts for a possible rescaling of the 2-periodic structure (See Section 3.5). In the proof of Theorem 5.6 below, the weight γ arises because we are rescaling the coordinates in the local models.

Remark 5.8. In the case where the dual intersection complex of X is not orientable, the first Stiefel-Whitney class of this 2-manifold restricts to a class $w_1 \in H^1(G, \mathbb{Z}/2\mathbb{Z})$. We expect that the analog of Theorem 5.6 holds when the category $\mathcal{B}(G(X))$ is twisted by w_1 as in [Seg21] (See Remark 3.11 above).

5.1.1. Matrix Factorizations and completions. Let $H \subset \mathbb{A}^n$ be the union of the coordinate hyperplanes. We fix an open subset $Y \subset \mathbb{A}^n$. The intersection $H \cap Y$ is a normal crossing divisor in Y , which with small abuse of notation we keep denoting H .

Lemma 5.9. *Let T be a scheme of dimension n and let $X \subset T$ be a normal crossing divisor. Let $Z_X \subset X$ be the singular locus of X . Assume that Z_X is isomorphic to the singular locus Z_{H_X} of a normal crossing divisor $H_X \subset Y$ given by a subset of the components of H . Then the completion of Z_X in X is isomorphic to the completion of Z_{H_X} in H_X .*

Proof. The key input will be Theorem 1.5 of [CM03], which gives a criterion comparing the infinitesimal neighborhoods of a scheme in different ambient spaces.⁶

Consider the two embeddings

$$j_1 : Z_X \subset X \subset T, \quad j_2 : Z_X \cong Z_{H_X} \subset H_X \subset Y$$

Let $N_{Z_X/T}$ and $N_{Z_X/Y}$ be the normal sheaves relative to the embedding of Z_X in T and in Y . Let Z_1, \dots, Z_l be the irreducible components of Z_X , and denote $\iota_i : Z_i \rightarrow Z_X$ the embedding of the i -th irreducible component in Z_X . There is a commutative diagram of exact sequences

$$\begin{array}{ccccccc} \bigoplus_i \iota_{i,*} T_{Z_i} & \longrightarrow & j_1^* T_T \cong \mathcal{O}_{Z_X}^n & \longrightarrow & N_{Z_X/T} & \longrightarrow & 0 \\ \cong \downarrow & & \downarrow \cong & & & & \\ \bigoplus_i \iota_{i,*} T_{Z_i} & \longrightarrow & j_2^* T_Y \cong \mathcal{O}_{Z_X}^n & \longrightarrow & N_{Z_X/Y} & \longrightarrow & 0 \end{array}$$

where T_T and T_Y are the tangent sheaves over T and over Y respectively. This yields an isomorphism of normal sheaves $N_{Z_X/T} \simeq N_{Z_X/Y}$. Since Z_X is affine, higher cohomology of quasi-coherent sheaves vanish: thus [CM03, Theorem 1.5] implies that the completions of T and Y along Z_X are isomorphic

$$(13) \quad \widehat{Z}_X^T \cong \widehat{Z}_X^Y.$$

Now consider the restriction to \widehat{Z}_X^T of local functions cutting out the irreducible components of X . Under isomorphism (13), they correspond to functions on \widehat{Z}_X^Y . Further, up to a change of coordinates on \widehat{Z}_X^Y , these coincide with the functions on Y cutting out the components of H_X . Thus we deduce that (13) restricts to an equivalence

$$\widehat{Z}_X^X \simeq \widehat{Z}_X^T \times_T X \cong \widehat{Z}_X^Y \times_Y H_X \simeq \widehat{Z}_X^{H_X}$$

and this concludes the proof. \square

We are now ready to prove Theorem 5.6.

The proof of Theorem 5.6. We will make use of two key properties of $\text{DSing}^\infty(-)$. For reference, we state them below

- Let U be a quasi-projective scheme. By Theorem 2.10 [Orl11], $\text{DSing}^\infty(U)$ only depends on the formal neighborhood of the singular locus of U .⁷
- By Proposition A.3.1 of [Pre11] $\text{DSing}^\infty(-)$ satisfies étale (and thus Zariski) descent with respect to varieties that are presented as the zero fiber of a morphism.

We will use the notations of Section 5.1: X is a 2-dimensional simple normal crossing divisor, Z is its singular locus, S is the singular locus of Z , and S' is the set of marked points on Z . We fix a parametrization of the components of Z . Note that if S' is empty then the statement is trivial and, if S is empty, the statement just follows from Zariski descent for $\text{QCoh}^{(2)}(-)$. We will assume that $S = S' \neq \emptyset$, the case when $S \subsetneq S'$ follows in a similar way.

Let $s \in S$. There are exactly three components of Z that are incident to s , we denote them $Z_{s,1}, Z_{s,2}, Z_{s,3}$ and set $Z_s = \bigcup_{i=1}^3 Z_{s,i}$. We denote by Z_s^c the union of the components of Z that are not incident to s , $Z_s^c = \overline{Z - Z_s}$. The fixed parametrization of the components of Z yields an isomorphism between $Z - Z_s^c$ and the coordinate lines in \mathbb{A}^3 which sends s to $(0, 0, 0) \in \mathbb{A}^3$. If s

⁶The reference [CM03] is about complex analytic varieties and therefore also Theorem 1.5 is formulated in that generality: but the proof relies entirely on general sheaf cohomology techniques, which apply without variations to schemes over any ground field κ .

⁷In fact, Orlov proves that this is true only up to idempotent completion: however throughout the paper we work with categories up to Morita equivalence, and so in particular up to idempotent completion.

and s' are adjacent vertices in $G(X)$, then there is exactly one irreducible component $Z_{s,s'}$ of Z such that $s, s' \in Z_{s,s'}$.

For every $s \in S$, let $U_s = X - Z_s^c$. Then $\mathcal{U} = \{U_i\}_{s \in S}$ is an open cover of X . Note that \mathcal{U} does not have any non-trivial triple overlaps. Since $\mathrm{DSing}^\infty(-)$ satisfies Zariski descent we can realize $\mathrm{DSing}^\infty(X)$ as a limit in $\mathrm{DGCat}_{\mathrm{cont}}^{(2)}$

$$\mathrm{DSing}^\infty(X) \rightarrow \left[\bigoplus_{s \in S} \mathrm{DSing}^\infty(U_s) \rightrightarrows \bigoplus_{s,s' \in S} \mathrm{DSing}^\infty(U_s \cap U_{s'}) \right]$$

We make the following observations:

- (a) U_s is a normal crossing surface, with singular locus Z_s . By Lemma 5.9 the formal neighborhood of Z_s inside U_s is isomorphic to the formal neighborhood of the coordinate axes inside the coordinate hyperplanes $H \subset \mathbb{A}^3$. Orlov's results from [Orl11] imply

$$\mathrm{DSing}^\infty(U_s) \simeq \mathrm{DSing}^\infty(H) \simeq \mathrm{MF}^\infty(\mathbb{A}^3, x_1 x_2 x_3)$$

- (b) If $s \neq s'$ and s and s' are adjacent in $G(X)$, then $U_s \cap U_{s'} = X - (Z_s^c \cup Z_{s'}^c)$. The singular locus of $U_s \cap U_{s'}$ is isomorphic to \mathbb{G}_m . Using again Lemma 5.9, and Orlov's Theorem we deduce that

$$\mathrm{DSing}^\infty(U_s \cap U_{s'}) \simeq \mathrm{QCoh}^{(2)}(\mathbb{G}_m)$$

- (c) If $s \neq s'$, and s and s' are not adjacent vertices in $G(X)$, then $U_s \cap U_{s'} = X - Z$. Since $X - Z$ is smooth

$$\mathrm{DSing}^\infty(U_s \cap U_{s'}) \simeq 0$$

Using these observations we can rewrite, in terms of the quiver $Q_{G(X)}$, the equalizer that computes $\mathrm{DSing}^\infty(X)$ as

$$(14) \quad \mathrm{DSing}^\infty(X) \rightarrow \left[\bigoplus_{\bullet_v \in Q_{G(X)}} \mathcal{B}(\bullet_v) \rightrightarrows \bigoplus_{\circ_t \in Q_{G(X)}} \mathcal{B}(\circ_t) \right]$$

Now, diagram (14) has the same shape as diagram (11) from Remark 3.6 which computes $\mathcal{B}(G(X))$ (see Remark 3.6). As the last step of the proof, we need to show that we can choose the equivalences from our observations (a) and (b) above, in such a way that the functors appearing in (14) match the ones appearing in (11).

The first issue to address is whether these functors are compatible with the conventions for resolving the shift ambiguity of the Knörrer periodicity that we set down in Section 3.1. This is where we use the hypothesis that the dual intersection complex $C(X)$ of X is orientable. Choose an orientation of $C(X)$. Since the graph G embeds into $C(X)$, the orientation of $C(X)$ induces a cyclic ordering of the edges at each vertex of G . Taking these cyclic orderings together with arbitrarily chosen orientations of each edge, we obtain framing data for G as in Section 3.1.

Consider now an irreducible component D of X ; the structure sheaf \mathcal{O}_D gives rise to an object of $\mathrm{DSing}^\infty(X)$. This object therefore gives rise to a compatible system of objects in the diagram (14). For an adjacent pair of vertices v, v' that lie on the boundary of D , and using the notation of Section 3.1, we see that \mathcal{O}_D maps to objects of form F_{ij} with i, j cyclically ordered (with respect to the chosen framing data) at both v and v' . These must therefore map to the same object over the edge t joining v to v' , and this is consistent with the convention set out in Section 3.1.⁸

It remains to show that we can rescale the local coordinates on each patch so that the functors in the diagram (14) match the diagram defining $\mathcal{B}(G(X))$. Let $s \in S$, and let $Z_{s,1}, Z_{s,2}$ and $Z_{s,3}$ be the three components of Z incident to s . Let us assume, for ease of exposition, that $Z_{s,1}, Z_{s,2}$ and

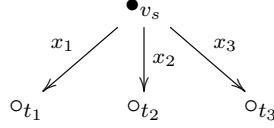
⁸This argument shows moreover that the convention of Section 3.1 is essentially determined by the requirement that \mathcal{O}_D should give rise to an object of $\mathcal{B}(G(X))$ in the present setting.

$Z_{s,3}$ are all isomorphic to \mathbb{P}^1 : this implies that, in addition to s , each component carries a second marked point. Let us name these points s_1, s_2 and s_3 .⁹ For all $i = 1, 2, 3$ we denote

$$j^* : \mathrm{DSing}^\infty(U_s) \rightarrow \mathrm{DSing}^\infty(U_s \cap U_{s_i})$$

the pull-back functor.

Let us recall the dictionary translating between X and the quiver $Q_{G(X)}$. The marked point s corresponds to a black vertex \bullet_v and the components $Z_{s,1}, Z_{s,2}$ and $Z_{s,3}$ correspond to white vertices \circ_{t_1}, \circ_{t_2} and \circ_{t_3} . Additionally we have three arrows



It is easy to see that we can make compatible choices of equivalences *locally* around each vertex s of $G(X)$. Namely for every $s \in S$ we can choose equivalences

$$\rho_s : \mathrm{DSing}^\infty(U_s) \simeq \mathcal{B}(\bullet_{v_s}) \quad \sigma_s : \mathrm{DSing}^\infty(U_s \cap U_{s_i}) \simeq \mathcal{B}(\circ_{t_i})$$

in such a way that for all i we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{DSing}^\infty(U_s) & \xrightarrow{\rho_s} & \mathcal{B}(\bullet_{v_s}) \\ j^* \downarrow & & \downarrow \mathcal{B}(x_i) \\ \mathrm{DSing}^\infty(U_s \cap U_{s_i}) & \xrightarrow{\sigma_s} & \mathcal{B}(\circ_{t_i}) \end{array}$$

Indeed, by definition, the functors j^* and $\mathcal{B}(x_i)$ are both given by restriction to the i -th component of the singular locus.

The key point is that we can actually make a *global* choice of compatible equivalences. Let us take two adjacent vertices in $G(X)$, s and s' . They both lie on a component C of Z , $C \cap S = \{s, s'\}$. The points s, s' and the curve C give rise to a subquiver of $Q_{G(X)}$ of the form

$$\bullet_{v_s} \xrightarrow{x} \circ_{t_C} \xleftarrow{y} \bullet_{v_{s'}}$$

This corresponds to a commutative diagram of categories

$$\begin{array}{ccccc} \mathrm{DSing}^\infty(U_s) & \xrightarrow{\rho_s} & \mathcal{B}(\bullet_{v_s}) & & \mathcal{B}(\bullet_{v_{s'}}) \xleftarrow{\rho_{s'}} \mathrm{DSing}^\infty(U_{s'}) \\ j^* \downarrow & & \searrow \mathcal{B}(x) & & \swarrow \mathcal{B}(y) \\ \mathrm{DSing}^\infty(U_s \cap U_{s'}) & \xrightarrow{\sigma_s} & \mathcal{B}(\circ_{t_C}) & \xleftarrow{\sigma_{s'}} & \mathrm{DSing}^\infty(U_s \cap U_{s'}) \\ & & & & \downarrow j^* \end{array}$$

For clarity, let us assume that we have numbered the components of the singular locus of $U_{s'}$ and that

$$j^* : \mathrm{DSing}^\infty(U_{s'}) \longrightarrow \mathrm{DSing}^\infty(U_s \cap U_{s'})$$

corresponds to restriction to the *first component*. The compatibility of the equivalences ρ_s, σ_s with $\rho_{s'}, \sigma_{s'}$ amounts to the condition that

$$\sigma_s \circ \sigma_{s'}^{-1} = \mathrm{Id}_{\mathrm{DSing}^\infty(U_s \cap U_{s'})}$$

Note that the composition $\sigma_s \circ \sigma_{s'}^{-1}$ is an auto-equivalence of $\mathrm{DSing}^\infty(U_s \cap U_{s'}) \simeq \mathrm{QCoh}^{(2)}(\mathbb{G}_m)$. We can be more precise than that: since $\sigma_s \circ \sigma_{s'}^{-1}$ is a composition of pull-backs along isomorphisms of schemes we have that

$$\sigma_s \circ \sigma_{s'}^{-1} = \lambda^*$$

⁹The general case where some or all of the Z_{s_i} are isomorphic to \mathbb{A}^1 is proved similarly.

where we denote by $\lambda : \mathbb{G}_m \rightarrow \mathbb{G}_m$ the multiplication by $\lambda \in \mathbb{G}_m$. We will see below how we can set λ equal to 1 by modifying $\rho_{s'}$ and $\sigma_{s'}$ by an appropriate auto-equivalence of $\mathrm{DSing}^\infty(U_{s'})$.

The natural action of $(\mathbb{G}_m)^3$ on \mathbb{A}^3 restricts to H , and via pull-back induces an action on $\mathrm{DSing}^\infty(H)$. We denote

$$(\lambda_1, \lambda_2, \lambda_3)^* : \mathrm{DSing}^\infty(H) \rightarrow \mathrm{DSing}^\infty(H)$$

the pull-back along the multiplication by $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{G}_m)^3$. For all $i = 1, 2, 3$ there is a commutative diagram

$$\begin{array}{ccc} \mathrm{DSing}^\infty(H) & \xrightarrow{\iota_i^*} & \mathrm{QCoh}^{(2)}(\mathbb{G}_m) \\ (\lambda_1, \lambda_2, \lambda_3)^* \downarrow & & \downarrow \lambda_i^* \\ \mathrm{DSing}^\infty(H) & \xrightarrow{\iota_i^*} & \mathrm{QCoh}^{(2)}(\mathbb{G}_m) \end{array}$$

where ι_i^* is the restriction to the i -th component of the singular locus.

The equivalence $\mathrm{DSing}^\infty(U_{s'}) \simeq \mathrm{DSing}^\infty(H)$ induces a $(\mathbb{G}_m)^3$ -action on $\mathrm{DSing}^\infty(U_{s'})$. Recall that we stipulated that $j^* : \mathrm{DSing}^\infty(U_{s'}) \rightarrow \mathrm{DSing}^\infty(U_s \cap U_{s'})$ is the restriction to the first component of the singular locus of $U_{s'}$. Then it is enough to set

$$\rho_{s'} := \rho_{s'} \circ (1, \lambda^{-1}, 1)^* \quad , \quad \sigma_{s'} := \sigma_{s'} \circ (\lambda^{-1})^*$$

to obtain that $\sigma_s \circ \sigma_{s'}^{-1} = \mathrm{Id}_{\mathrm{DSing}^\infty(U_s \cap U_{s'})}$, as desired.

Acting with \mathbb{G}_m^3 on $\mathrm{DSing}^\infty(U_{s'})$ allows us to modify, by scalar multiplication, the restriction functors to the three components of the singular locus of $U_{s'}$. In this way, the equivalences $\rho_{s'}$ and $\sigma_{s'}$ at the vertex s' can be made compatible with the equivalences ρ_s and σ_s coming from any neighboring vertex s . An induction on the cardinality of S shows that there are *global* choices of compatible equivalences, and this concludes the proof. \square

Remark 5.10. Theorem 5.6 has the remarkable consequence that, as long as X satisfies the stated requirements (it is the fiber of a morphism from a smooth variety to \mathbb{A}^1 , it has graph-like singular locus, and the dual intersection complex is orientable), the category $\mathrm{DSing}^\infty(X)$ does not actually depend on the infinitesimal neighborhood of the singular locus Z of X , but only on the combinatorics of $G(X)$. We must remark, however, that the requirement that X arises as the fiber of a morphism $f : T \rightarrow \mathbb{A}^1$ with T smooth does entail a topological restriction on the infinitesimal neighborhood.

We stress that the singularity categories consider in Theorem 5.6 are exceptional in this respect. In general the singular locus Z is not by itself sufficient to determine $\mathrm{DSing}^\infty(-)$: one does need information on the infinitesimal neighborhood of Z . As an example consider a quadratic bundle E over a scheme X . Note that E carries a natural superpotential W given by the quadratic form q . The singular locus of W is always X , independently on the quadratic form; however by Theorem 9.3.4 of [Pre11] the category of matrix factorizations $\mathrm{MF}(E, W)$ is equivalent to modules over a sheaf of Clifford algebras which does depend on q .

6. COVERING SPACES AND EQUIVARIANCE

In this section we set up the theory of the Fukaya category of a compact Riemann surface and develop the connection between covering spaces, anchored Lagrangians, and some notions of equivariance for categories.

6.1. Variants of the Fukaya category for finite type surfaces. We shall now recall several different variants of the Fukaya category for compact and noncompact surfaces. All of the versions are closely related, but having different ways of presenting the Fukaya category can make some arguments easier.

We fix a ground field k of characteristic 0. Let Σ be surface of finite type, meaning that Σ is either compact or it embeds into a compact surface, with symplectic form ω . (We shall consider non-finite type surfaces below.) Recall the universal (over k) Novikov field Λ ,

$$(15) \quad \Lambda = \left\{ \sum_{i=0}^{\infty} a_i q^{\lambda_i} \mid a_i \in k, \lambda_i \in \mathbb{R}, \lim_{i \rightarrow \infty} \lambda_i = \infty \right\},$$

the field of formal series with real exponents such that there are only finitely many nonzero terms with exponent less than any given threshold. The most canonical version of the Fukaya category of Σ is a $\mathbb{Z}/2$ -graded triangulated split-closed A_∞ -category defined over Λ . We will denote it simply by $\text{Fuk}(\Sigma)$. The generating objects are unobstructed immersed Lagrangians equipped with orientations and spin structures; an immersed Lagrangian is unobstructed when it is not null-homotopic and does not bound any teardrops.¹⁰ In the case where Σ is noncompact, we also include noncompact properly embedded arcs that are treated in the manner of wrapped Floer cohomology. Most importantly, the A_∞ structure maps count pseudoholomorphic polygons weighted by q^{Area} ; it is the possibility of having infinitely many terms that necessitates using the Novikov field Λ .

In certain situations, it is possible to define versions of the Fukaya category over smaller fields such as k .

First we introduce the balanced Fukaya category as follows. Let $\pi : S(T\Sigma) \rightarrow \Sigma$ be the circle bundle associated to the tangent bundle of Σ , defined as the set of oriented real lines in the tangent spaces. The form $\pi^*(\omega)$ defines a cohomology class in $H^2(S(T\Sigma), \mathbb{R})$. This class vanishes provided that either

- Σ is compact of genus $g \neq 1$, or
- Σ is not compact.

Suppose we are in such a case, so that we may choose a one-form θ on $S(T\Sigma)$ such that $d\theta = \pi^*\omega$. Given a oriented connected Lagrangian submanifold $L \subset \Sigma$, which is to say an oriented simple closed curve, we may form a tangent lift $\sigma : L \rightarrow S(TM)|_L$. We say L is *balanced* if $\int_L \sigma^*(\theta) = 0$. Contractible curves are never balanced, and every isotopy class of noncontractible simple closed curves contains a balanced representative, which is unique up to Hamiltonian isotopy. The generating objects we consider consist of a balanced curve equipped with an orientation and spin structure, and in the case where Σ is non compact, we also include noncompact properly embedded arcs as before. The morphism complexes are defined over k , and the A_∞ structure maps count pseudoholomorphic polygons *without* weighting by the area. We enlarge this category to a $\mathbb{Z}/2$ -graded, triangulated, split-closed A_∞ -category over k , which we call the *balanced Fukaya category* $\text{Fuk}^{bal}(\Sigma, \theta)$.

The construction of $\text{Fuk}^{bal}(\Sigma, \theta)$ depends on θ modulo exact one-forms on $S(T\Sigma)$. Given two choices θ and θ' , there is a class $[\theta' - \theta] \in H^1(S(T\Sigma), \mathbb{R})$ measuring the difference. If Σ is compact with $g \neq 1$, then $H^1(S(T\Sigma), \mathbb{R}) \cong H^1(\Sigma, \mathbb{R})$, and after moving θ' by a symplectomorphism of Σ with appropriate flux, we can ensure that this class vanishes. This symplectomorphism then induces an equivalence of categories $\text{Fuk}^{bal}(\Sigma, \theta) \cong \text{Fuk}^{bal}(\Sigma, \theta')$. On the other hand, if Σ is not compact, then $H^1(S(T\Sigma), \mathbb{R}) \cong H^1(\Sigma, \mathbb{R}) \oplus \mathbb{R}$, and it is not necessarily possible to make the difference class vanish using a symplectomorphism of Σ or its completion.

The exact Fukaya category is another variant defined over k that is only defined when Σ is noncompact. Here we choose a one-form λ on Σ such that $d\lambda = \omega$. We say a Lagrangian L is *exact* if $\lambda|_L$ is an exact one-form. This is the same thing as saying that L is balanced with respect to the form $\theta = \pi^*\lambda$, and the construction of the *exact Fukaya category* $\text{Fuk}^{ex}(\Sigma, \lambda)$ follows exactly the same lines as the balanced Fukaya category in that case. Note that once again this is a “wrapped”

¹⁰This condition is equivalent to the condition that the pullback of the Lagrangian under the universal covering of Σ is embedded. In the case where Σ is a sphere, $\text{Fuk}(\Sigma)$ as defined here is the zero category: it is well-known that in order to have a good Fukaya category for the sphere one must include *weakly unobstructed* Lagrangians.

variant of the Fukaya category. This construction depends on λ modulo exact one-forms on Σ , and it is again possible to relate any two choices by a symplectomorphism of Σ or its completion.

For noncompact Σ , all of the categories introduced so far are “wrapped” versions; if we wish to restrict to objects supported on compact Lagrangians we use the subscript “c”, so we have $\text{Fuk}^c(\Sigma)$ (over Λ), $\text{Fuk}^{bal,c}(\Sigma, \theta)$, and $\text{Fuk}^{ex,c}(\Sigma, \lambda)$ (over k).

All variants of the Fukaya category are closely related to one another. The relationship between the balanced Fukaya category $\text{Fuk}^{bal}(\Sigma, \theta)$ and $\text{Fuk}(\Sigma)$ works as follows. Let $\mathcal{C}(\Sigma, \theta)$ be the full subcategory of $\text{Fuk}(\Sigma)$ whose objects are branes supported on balanced curves, so that \mathcal{C} is a Λ -linear category where the operations weight curves by area. It is known that $\mathcal{C}(\Sigma, \theta)$ split generates $\text{Fuk}(\Sigma)$ over Λ (for example, by combining Proposition 2.15 of [AS20] with the observation that the curves used in that proposition can be taken to be balanced, since every isotopy class has a balanced representative). The condition that the curves are balanced means that all series appearing in the operations are finite, so that it is possible to set $q = 1$ in all operations, obtaining a k -linear category $\mathcal{C}_{q=1}(\Sigma, \theta)$, and this category embeds into $\text{Fuk}^{bal}(\Sigma, \theta)$ and split generates the latter category over k . Because of these relationships, $\text{Fuk}^{bal}(\Sigma, \theta)$ can be thought of as giving a model of $\text{Fuk}(\Sigma)$ over the smaller field k .

Since the exact Fukaya category $\text{Fuk}^{ex}(\Sigma, \lambda)$ is a special case of the balanced Fukaya category (with $\theta = \pi^*\lambda$), everything in the previous paragraph is still valid. However in this case there is also a k -linear embedding of $\text{Fuk}^{ex}(\Sigma, \lambda)$ into $\text{Fuk}(\Sigma)$. This functor is identity on objects and rescales every morphism by t to its Floer action. Since this rescaling becomes trivial upon setting $q = 1$, this embedding followed by specialization recovers the identity on $\text{Fuk}^{ex}(\Sigma, \lambda)$. Since $\text{Fuk}^{ex}(\Sigma, \lambda)$ generates $\text{Fuk}(\Sigma)$ over Λ , we conclude there $\text{Fuk}(\Sigma)$ is equivalent to $\text{Fuk}^{ex}(\Sigma, \lambda) \otimes_k \Lambda$.

6.2. Balanced versus exact categories for noncompact surfaces. We now turn to the relationship between the various balanced and exact categories for noncompact surfaces. We have already seen that for compact surfaces, only the balanced category is defined, and it is independent of the choice of balancing form θ up to equivalence, since any two choices θ and θ' may be related by a symplectic isotopy of Σ . For noncompact surfaces the situation is a bit more subtle.

Recall that the difference between θ and θ' defines a class $[\theta' - \theta] \in H^1(S(T\Sigma), \mathbb{R})$. Since Σ is assumed noncompact, we may choose a trivialization $\eta : S(T\Sigma) \rightarrow \Sigma \times S^1$. Then η induces a splitting $H^1(S(T\Sigma), \mathbb{R}) \cong H^1(\Sigma, \mathbb{R}) \oplus H^1(S^1, \mathbb{R})$. By applying a symplectic isotopy of Σ to θ' , say, it is possible to eliminate the component in $H^1(\Sigma, \mathbb{R})$, so that the difference class lies in the subspace $H^1(S^1, \mathbb{R})$, but we cannot necessarily eliminate this latter component by the same method.

Let θ be given, and chose some $\lambda \in \Omega^1(\Sigma)$ that $d\lambda = \omega$, and consider the class $[\theta - \pi^*\lambda] \in H^1(\Sigma, \mathbb{R}) \oplus H^1(S^1, \mathbb{R})$. By adding a closed 1-form to λ , we may assume that this class lies in $H^1(S^1, \mathbb{R})$. Suppose this class is $[\alpha\phi]$, where ϕ is the standard angular form on S^1 such that $\int_{S^1} \phi = 1$. Using the trivialization η , the form $\alpha\phi$ can be regarded as a form on $S(T\Sigma)$. Then we have that $[\theta - (\pi^*\lambda + \alpha\phi)] = 0$. As a result of this, we conclude that $\text{Fuk}^{bal}(\Sigma, \theta)$ is equivalent to $\text{Fuk}^{bal}(\Sigma, \pi^*\lambda + \alpha\phi)$ for some Liouville form λ and some $\alpha \in \mathbb{R}$. Of course $\text{Fuk}^{ex}(\Sigma, \lambda) = \text{Fuk}^{bal}(\Sigma, \pi^*\lambda)$, so we now pose the problem of relating the categories $\text{Fuk}^{bal}(\Sigma, \pi^*\lambda)$ and $\text{Fuk}^{bal}(\Sigma, \pi^*\lambda + \alpha\phi)$.

A compact Lagrangian L gives rise to an object of $\text{Fuk}^{bal}(\Sigma, \pi^*\lambda)$ when $\int_L \lambda = 0$, while it gives rise to an object of $\text{Fuk}^{bal}(\Sigma, \pi^*\lambda + \alpha\phi)$ when $\int_L \lambda + \alpha \int_L \phi = 0$. Observe that $\int_L \phi$ is nothing but the rotation number of the curve L with respect to the trivialization η . The trivialization η (which is essentially a vector field on Σ) determines a line field Σ , and this rotation number is one-half of the Maslov class of L computed with respect to that line field: we write $\int_L \phi = \mu_L/2$. In summary the condition for a compact Lagrangian to be balanced with respect to $\pi^*\lambda + \alpha\phi$ is the relation

$$(16) \quad \int_L \lambda + (\alpha/2)\mu_L = 0.$$

In particular, an η -gradable ($\mu_L = 0$) Lagrangian is balanced if and only if it is exact. Conversely, any non-gradable exact Lagrangian must be deformed by an amount proportional to μ_L in order to become balanced.

Let L_0 be an exact Lagrangian, and let L_1 be the balanced Lagrangian obtained from L_0 by an isotopy with flux equal to $(-\alpha/2)\mu_L$. Then as modules over the HKK generators, L_1 (with the trivial local system) is equivalent to L_0 equipped with a local system whose holonomy is $q^{(-\alpha/2)\mu_L}$. In other words, every balanced object can be interpreted as an exact object with a local system that depends only on μ_L . Upon setting $q = 1$, these local systems become trivial, and so $\text{Fuk}^{bal}(\Sigma, \pi^*\lambda + \alpha\phi) \cong \text{Fuk}^{ex}(\Sigma, \lambda)$.

6.3. Restriction functors. A key element in our analysis of HMS for noncompact Riemann surfaces is the Viterbo restriction functor on exact categories. For a Liouville embedding $(\Sigma, \lambda) \rightarrow (\Sigma', \lambda')$, there is functor $r : \text{Fuk}^{ex}(\Sigma', \lambda') \rightarrow \text{Fuk}^{ex}(\Sigma, \lambda)$, which roughly speaking takes an exact Lagrangian in Σ' to its intersection with the subsurface Σ . By extending scalars to Λ , we also obtain a restriction functor $r : \text{Fuk}(\Sigma') \rightarrow \text{Fuk}(\Sigma)$.

We remark that there is a bit of subtlety in computing the functor r on a nonexact object of $\text{Fuk}(\Sigma')$: it is not necessarily true that a Lagrangian L in Σ' disjoint from Σ maps to the zero object, since after replacing L with an isomorphic complex of exact Lagrangians, those Lagrangians may very well intersect Σ .

We may assume that L is in minimal position with respect to the boundary components of Σ (L and the boundary components do not bound a disk), since this may always be achieved by a Hamiltonian isotopy. To be more precise about how restriction works, we distinguish two cases: either the contracting Liouville flow eventually pushes $L \subset \Sigma'$ so as to lie entirely within Σ , or not. Objects for which the contracting Liouville flow push L entirely into Σ include objects contained in Σ , and also curves in Σ' that are parallel to a boundary component of Σ . Objects for which the contracting Liouville flow never pushes L entirely into Σ are either taken to zero or to some collection of arcs lying in Σ .

Lemma 6.1. *Assume that $L \subset \Sigma'$ is in minimal position with respect to the boundary components of Σ .*

- (1) *If the contracting Liouville flow eventually pushes L entirely into Σ , then restriction to Σ maps L to the same object thought of as an object in the Liouville completion of Σ .*
- (2) *otherwise, restriction takes L to an object isomorphic to one supported on $L \cap \Sigma$.*

Proof. We prove the second part. We express L in terms of exact Lagrangians by applying contracting Liouville flow ϕ_t to collapse L onto the core of Σ' , and then intersect with Σ . In the contraction process, new components of the intersection $\phi_t(L) \cap \Sigma$ may be created, due to portions of L being pushed in to Σ . (If L were not assumed to be in minimal position with respect to $\partial\Sigma$, it would also be possible for components of $\phi_t(L) \cap \Sigma$ to merge together.) However, any components of $\phi_t(L) \cap \Sigma$ that get created will bound a disk with $\partial\Sigma$, and hence will represent the zero object in $\text{Fuk}(\Sigma)$. \square

6.4. Locally finite Fukaya category for non-finite-type Riemann surfaces. Let $\tilde{\Sigma}$ be a connected Riemann surface not of finite type. Choose an exhaustion of $\tilde{\Sigma}$ by open subsurfaces of finite type,

$$(17) \quad \tilde{\Sigma} = \bigcup_{N=1}^{\infty} \tilde{\Sigma}_N, \quad \tilde{\Sigma}_1 \subset \tilde{\Sigma}_2 \subset \cdots \subset \tilde{\Sigma}.$$

Since each subsurface $\tilde{\Sigma}_N$ is a punctured Riemann surface, and in particular an exact symplectic manifold, we may associate to it a Fukaya category, namely the wrapped Fukaya category $\text{Fuk}^w(\tilde{\Sigma}_N)$. Recall that this category contains both exact compact Lagrangian branes as well as

properly embedded arcs, and that the morphism complexes are computed by wrapping around the punctures of $\tilde{\Sigma}_N$. Next, for a pair of indices $N \leq M$, we have an embedding $\tilde{\Sigma}_N \rightarrow \tilde{\Sigma}_M$ (which may be taken to be an exact embedding), so there is a restriction functor,

$$(18) \quad r_{M,N} : \text{Fuk}^w(\tilde{\Sigma}_M) \rightarrow \text{Fuk}^w(\tilde{\Sigma}_N),$$

known as the Viterbo restriction (constructed by Abouzaid-Seidel). As N tends to infinity, the categories $\text{Fuk}^w(\tilde{\Sigma}_N)$ with the functors $r_{M,N}$ form an inverse system of A_∞ -categories, and we may pass to the homotopy limit, which we denote as

$$(19) \quad \text{Fuk}^{\text{lf}}(\tilde{\Sigma}) = \varprojlim_{N \rightarrow \infty} \text{Fuk}^w(\tilde{\Sigma}_N).$$

We call $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ the *locally finite Fukaya category* of $\tilde{\Sigma}$. It is potentially different from other Fukaya categories one might reasonably associate to $\tilde{\Sigma}$: for instance, an object of $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ may be supported on a Lagrangian submanifold with infinitely many connected components.

We remark that it is not difficult to show that $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ does not depend up to equivalence on the choice of exhaustion.

For our purposes $\tilde{\Sigma}$ will always be an infinite-sheeted covering of a compact Riemann surface Σ : namely, we take $\tilde{\Sigma}$ to be the *maximal tropical cover* of Σ as defined in Section 4.1. Also we take the exhaustion by open subsurface of $\tilde{\Sigma}$ to be compatible with the pants decomposition \tilde{P} of $\tilde{\Sigma}$ in the sense specified in Section 4.1.1. In this setting we have defined also the topological locally finite Fukaya category of $\tilde{\Sigma}$. Recall that [HKK17] gives a dictionary between the wrapped and the topological Fukaya categories of the finite type subsurfaces. Then, using Proposition 4.10, we obtain also an equivalence between the locally finite Fukaya category and its topological counterpart. We record this simple observation in the following Proposition.

Proposition 6.2. *There is an equivalence*

$$\text{Fuk}^{\text{lf}}(\tilde{\Sigma}) \simeq \text{Fuk}^{\text{top,lf}}(\tilde{\Sigma})$$

where the Liouville structure of $\tilde{\Sigma}$ is used to trivialize the dependence of the Fukaya category on the Novikov parameter.

Proof. The statement follows from the chain of equivalences

$$\text{Fuk}^{\text{lf}}(\tilde{\Sigma}) = \varprojlim_{N \rightarrow \infty} \text{Fuk}^w(\tilde{\Sigma}_N) \simeq \varprojlim_{N \rightarrow \infty} \text{Fuk}^{\text{top}}(\tilde{\Sigma}_N) \stackrel{(*)}{\simeq} \text{Fuk}^{\text{top,lf}}(\tilde{\Sigma})$$

where equivalence $(*)$ is given by Proposition 4.10. □

6.5. Pullback for étale maps. We wish to understand how Fukaya categories pullback along $\pi : \tilde{\Sigma} \rightarrow \Sigma$. Since $\tilde{\Sigma}$ is not of finite-type, we consider an exhaustion $\tilde{\Sigma} = \bigcup_{N=1}^{\infty} \tilde{\Sigma}_N$ by open subsurfaces $\tilde{\Sigma}_N$ that are of finite-type. Then the covering map $\pi : \tilde{\Sigma} \rightarrow \Sigma$ restricts to maps $\pi_N : \tilde{\Sigma}_N \rightarrow \Sigma$; π_N is a local diffeomorphism (an étale map) but not a covering map. Given a Lagrangian brane L , we construct an object $\pi_N^{-1}(L)$ in $\text{Fuk}^w(\tilde{\Sigma}_N)$ by pulling back the Lagrangian and brane structure under π_N . By construction, the objects $\pi_N^{-1}(L)$ are mapped to one another under the functors $r_{M,N}$, and so this system of objects gives rise to an object of the limit $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$. It remains to show that this object-by-object construction can be extended to a pullback functor, and to understand compatibility with the restriction functors.

In order to show that this assignment is functorial, we need to recall how morphism complexes and A_∞ operations are computed in a wrapped category like $\text{Fuk}^w(\tilde{\Sigma}_N)$. Given two objects L_0 and L_1 , the wrapped Floer complex $CW(L_0, L_1)$ is generated by chords of a Hamiltonian flow generated by a function H . In order to qualify as a “wrapping Hamiltonian”, the function H must be chosen so that it has convex growth on the cylindrical ends of $\tilde{\Sigma}_N$; we also assume for convenience that H

is C^2 -small away from the cylindrical ends. With this setup, the chords that generate $CW(L_0, L_1)$ naturally fall into two classes: the *boundary chords* that live in one of the cylindrical ends of $\tilde{\Sigma}_N$, and the rest, which are called *interior chords*. The distinction between boundary chords and interior chords is not intrinsic to $\text{Fuk}^w(\tilde{\Sigma}_N)$ as a category; it is something having to do with the specific geometrically-defined cochain complexes we use to construct the wrapped Fukaya category using Floer theory. This distinction allows us to make arguments at chain level thanks to the following lemma about Floer-theoretic operations:

Lemma 6.3. *If all of the inputs to an A_∞ operation are interior chords, then the output is also an interior chord.*

Proof. Given a pseudo-holomorphic curve contributing to such an operation, where all inputs are interior chords, the maximum principle applies to the radial coordinates on the cylindrical ends. This demonstrates that such curves cannot enter the cylindrical ends. \square

Now we consider the problem of extending the association $L \mapsto \pi_N^{-1}(L)$ to an A_∞ -functor. Let L_0 and L_1 be objects of $\text{Fuk}(\Sigma)$. The question is to relate $CF(L_0, L_1)$ with $CW(\lambda_N(L_0), \lambda_N(L_1))$. Now $CF(L_0, L_1)$ is generated by the chords of some Hamiltonian function H_0 defined on Σ . When we pull back H_0 to $\tilde{\Sigma}_N$, we do not get a wrapping Hamiltonian, since H_0 remains bounded on the ends; to remedy this we add a wrapping Hamiltonian H_1 that is zero away from the ends. This has the neat property that the interior chords of $(H_0 \circ \pi_N) + H_1$ on $\tilde{\Sigma}_N$ are nothing but the lifts with respect to π_N of chords of H_0 on Σ .

Proposition 6.4. *The association $L \mapsto \pi_N^{-1}(L)$ may be extended to an A_∞ -functor $\lambda_N^{\text{naive}} : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^w(\tilde{\Sigma}_N)$.*

Proof. The functor on objects is $L \mapsto \pi_N^{-1}(L)$. On morphism complexes, λ_N^{naive} takes a chord $x \in CF(L_0, L_1)$ to the sum of all its lifts under π_N . By the discussion above, this is a well-defined element of $CW(\lambda_N(L_0), \lambda_N(L_1))$. Observe that $\lambda_N^{\text{naive}}(x)$ is always a sum of interior chords.

It remains to show that the mapping $x \mapsto \lambda_N(x)$ directly matches all A_∞ operations, so that λ_N is an A_∞ -functor with vanishing higher-order components. To show this claim, observe that any A_∞ -operation contributing with inputs of the form $\lambda_N^{\text{naive}}(x)$ counts pseudoholomorphic curves whose images stay in the interior region (away from the ends of $\tilde{\Sigma}_N$). In that region, all of the perturbations are pulled back under π_N , so these curves are nothing but the lifts of the pseudoholomorphic curves in Σ that contribute to the corresponding operation in $\text{Fuk}(\Sigma)$. (Note that every such curve does indeed lift, since its domain is simply connected, being a disk.) This establishes the existence of the functor $\lambda_N^{\text{naive}} : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^w(\tilde{\Sigma}_N)$. \square

We now introduce some notation having to do with the covering $\pi : \tilde{\Sigma} \rightarrow \Sigma$. Recall that this is the normal covering space of Σ associated to the kernel of the homomorphism

$$(20) \quad \pi_1(\Sigma) \rightarrow \pi_1(G) \mapsto H_1(G) = O_\pi \cong \mathbb{Z}^g,$$

where G is the graph associated to a pants decomposition. Now G is a trivalent graph with first Betti number g ; if we choose a spanning tree of G and collapse it to a point, we obtain a graph G' that is a wedge of g circles. The map $G \rightarrow G'$ is a homotopy equivalence, and we have a map $\Sigma \rightarrow G \rightarrow G'$, so we may as well regard π as the covering associated to the kernel of

$$(21) \quad \pi_1(\Sigma) \rightarrow H_1(G') \cong O_\pi.$$

Furthermore, the edges of G' give us a natural basis for the lattice O_π . We may also consider the maximal abelian covering of G' , call it \tilde{G}' ; the graph \tilde{G}' is the Cayley graph of O_π with respect to the chosen basis, so it is a g -dimensional rectangular lattice. Because $\pi : \tilde{\Sigma} \rightarrow \Sigma$ is isomorphic to the pullback of $\tilde{G}' \rightarrow G'$ along $\Sigma \rightarrow G'$, we have a map

$$(22) \quad \psi : \tilde{\Sigma} \rightarrow \tilde{G}'$$

that will be useful.

In terms of the topology of Σ , the choice of a spanning tree in G , and hence the choice of G' , corresponds to the choice of a *cut system* on Σ . A cut system is a collection of g simple closed curves such that cutting along these circles reduces Σ to a $2g$ -punctured sphere. From a spanning tree T in G , we obtain a cut system by taking, for each edge in $G \setminus T$, the corresponding circle in Σ from the pants decomposition. Conversely, any cut system contained in the given pants decomposition will determine a spanning tree in G .

The picture of $\tilde{\Sigma}$ that follows from these considerations is as follows: We think of $\tilde{\Sigma}$ as consisting of a collection of $2g$ -punctured spheres arranged at the sites of a g -dimensional rectangular lattice \tilde{G}' , and glued together according to that lattice. We shall assume that our exhaustion $\tilde{\Sigma}_N$ of $\tilde{\Sigma}$ consists of those punctured spheres at the sites whose coordinates lie in the cube $[-N, N]^g$.

Now we return to the construction of the pullback functor. Let L_1 and L_2 be two Lagrangian branes in Fuk . In order to understand the compatibility of the functors λ_N^{naive} with restriction, as well as to compute morphism complexes in the limit category $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$, we want to understand geometrically the restriction map, for $M > N$,

$$(23) \quad r_{M,N} : CW(\pi_M^{-1}(L_1), \pi_M^{-1}(L_2)) \rightarrow CW(\pi_N^{-1}(L_1), \pi_N^{-1}(L_2)).$$

Each closed curve in Σ has a corresponding class in O_π . We distinguish two types of Lagrangians L in Σ : Either

- (1) The class of L in O_π is zero, in which case every connected component of $\pi^{-1}(L)$ is a compact curve, or,
- (2) The class of L in O_π is nonzero, in which case every connected component of $\pi^{-1}(L)$ is a noncompact arc.

Given a pair of Lagrangians L_1 and L_2 , we distinguish the following cases:

- (1) The class of at least one of L_1 or L_2 in O_π is zero,
- (2) The classes of both L_1 and L_2 in O_π are nonzero, and distinct,
- (3) The classes of L_1 and L_2 in O_π are nonzero and equal, but L_1 is not isotopic to L_2 ,
- (4) L_1 is isotopic to L_2 , and they represent a nonzero class in O_π .

In each case, we wish to understand whether two components C_i of $\pi^{-1}(L_i)$ may intersect or support boundary chords in $\tilde{\Sigma}_N$. Whenever there is such an intersection point or chord, the images $\psi(C_1)$ and $\psi(C_2)$ must intersect: intersection points correspond to the situation where the images $\psi(C_i)$ touch the same vertex, while chords correspond to the situation where they touch the same edge.

Lemma 6.5. *Suppose that L_1 and L_2 are such that there is a component C_1 of $\pi^{-1}(L_1)$ and a component C_2 of $\pi^{-1}(L_2)$ such that $\psi(C_1) \cap \psi(C_2) \subset \tilde{G}'$ is not compact. Then the classes of L_1 and L_2 in O_π are nonzero and equal.*

Proof. If the class of either L_1 or L_2 in O_π is zero, then one of the components C_1 or C_2 is compact, so $\psi(C_1) \cap \psi(C_2)$ must be compact as well.

If the classes of L_1 and L_2 in O_π are nonzero, then C_1 and C_2 are noncompact arcs. If the classes are distinct, then $\psi(C_1)$ and $\psi(C_2)$ eventually diverge from one another in the lattice \tilde{G}' , so $\psi(C_1) \cap \psi(C_2)$ must again be compact. \square

Lemma 6.6. *Consider components C_i of $\pi^{-1}(L_i)$ ($i = 1, 2$) and their intersections $C_{i,N}$ with $\tilde{\Sigma}_N$. Suppose that L_1 is not isotopic to L_2 . Then for every N and for sufficiently large $M > N$, the restriction map*

$$(24) \quad r_{M,N} : CW(C_{1,M}, C_{2,M}) \rightarrow CW(C_{1,N}, C_{2,N})$$

vanishes on boundary chords.

Proof. First note that Lemma 6.5 implies that the hypothesis can only be satisfied if L_1 and L_2 represent the same nonzero class in O_π , since otherwise for M sufficiently large the set of boundary chords is empty.

In order for $r_{M,N}$ to be nonzero on a boundary chord in $\tilde{\Sigma}_M$, there must be a strip with boundary on C_1 and C_2 joining that chord to some generator in $\tilde{\Sigma}_N$. Applying the map $\psi : \tilde{\Sigma} \rightarrow \tilde{G}'$, we obtain a map from the strip to \tilde{G}' with boundary on $\psi(C_1)$ and $\psi(C_2)$. If we fix N and allow M to increase, then this strip necessarily becomes longer in the sense that its boundaries must traverse longer portions of C_1 and C_2 . Since C_1 and C_2 are lifts of compact circles in Σ , for large enough M relative to N the boundary of the strip will traverse a whole fundamental domain for the coverings $C_i \rightarrow L_i$. Then we may use this strip to construct an isotopy of L_1 to L_2 . \square

It remains to consider the case where L_1 is isotopic to L_2 , and the represent a nonzero class in O_π ; in this case the same conclusion as above holds.

Lemma 6.7. *Consider components C_i of $\pi^{-1}(L_i)$ ($i = 1, 2$) and their intersections $C_{i,N}$ with $\tilde{\Sigma}_N$. Suppose that L_1 is isotopic to L_2 . Then for every N and for sufficiently large $M > N$, the restriction map*

$$(25) \quad r_{M,N} : CW(C_{1,M}, C_{2,M}) \rightarrow CW(C_{1,N}, C_{2,N})$$

vanishes on boundary chords.

Proof. Supposing the conclusion is not true, then as in the proof of the previous lemma, we see that C_1 and C_2 must themselves be isotopic in $\tilde{\Sigma}$. In that case, we may focus our attention on a neighborhood of these arcs in $\tilde{\Sigma}$ that is an infinite cylinder with a periodic sequence of punctures. Then the conclusion follows from a direct analysis of this case. \square

Now we turn to question of compatibility with restriction, that is, whether $\lambda_N^{\text{naive}} \cong r_{M,N} \circ \lambda_M$ for $M \geq N$. On the one hand, it is obvious from the construction that compatibility is true at the level of objects, but for morphisms is less clear. In fact, the functors λ_N^{naive} need to be ‘‘corrected’’ to include contributions from boundary chords in order to achieve compatibility, since it is possible that an interior chord maps to a boundary chord under restriction.

To define the corrected functor λ_N on morphisms, we apply λ_M^{naive} for M sufficiently large compared to N , and then apply Viterbo restriction $r_{M,N}$:

$$(26) \quad \lambda_N = r_{M,N} \circ \lambda_M^{\text{naive}} \quad (M \gg N).$$

Thus, for each N , λ_N may differ from λ_N^{naive} by contributions from boundary generators in $\tilde{\Sigma}_N$. Now Lemmas 6.5, 6.6, and 6.7 imply that, after a further restriction, these boundary generators are sent to zero, so the difference disappears:

$$(27) \quad r_{M,N} \circ \lambda_M = r_{M,N} \circ \lambda_M^{\text{naive}} = \lambda_N.$$

This proves the following proposition.

Proposition 6.8. *The functors $\lambda_N : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^w(\tilde{\Sigma}_N)$ are compatible with Viterbo restriction, and hence their limit defines a functor $\lambda : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma})$.*

6.6. O_π -equivariant branes. Now we seek to establish the equivalence $\text{Fuk}(\Sigma) \cong \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$. We shall take a two-step approach: In this section we show there is a fully faithful embedding $\text{Fuk}(\Sigma) \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$, and in the next section we show that this functor is essentially surjective.

Begin by considering a Lagrangian submanifold $L \subset \Sigma$. Let \tilde{L} be the pullback of L with respect to the covering $\pi : \tilde{\Sigma} \rightarrow \Sigma$, so that \tilde{L} is a (possibly disconnected) Lagrangian submanifold of $\tilde{\Sigma}$. It is clear that \tilde{L} is invariant under the action of O_π ; we will proceed by showing that \tilde{L} can be made into a O_π -equivariant object of $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$, and that the association $L \mapsto \tilde{L}$ realizes, at the level of objects, a functor $\lambda : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$.

We can be a bit more explicit about how the functor $\lambda : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ works when applied to objects and morphisms. Recall that λ is the limit of $\lambda_N : \text{Fuk} \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma}_N)$, which on objects takes L to $\pi^{-1}(L) \cap \tilde{\Sigma}_N$, and it takes a morphism $x \in CW(L_0, L_1)$ to the sum of the lifts of x that lie in $\tilde{\Sigma}_N$, possibly plus some boundary generators in $\tilde{\Sigma}_N$. As $N \rightarrow \infty$, the chain complexes $CW(\lambda_N(L_0), \lambda_N(L_1))$ form an projective system of chain complexes, and we can pass to the limit complex $\lim_{N \rightarrow \infty} CW(\lambda_N(L_0), \lambda_N(L_1))$.

Lemma 6.9. *The projective system $CW(\lambda_N(L_0), \lambda_N(L_1))$ satisfies the Mittag-Leffler condition. Thus \lim^1 vanishes, and $\lim_{N \rightarrow \infty} CW(\lambda_N(L_0), \lambda_N(L_1))$ computes the homotopy limit of the system.*

Proof. For a projective system of groups $(A_N, r_{M,N})$, the Mittag-Leffler condition is the statement that, for each N , the image of $r_{M,N}$ stabilizes for large M . Lemmas 6.5, 6.6, and 6.7 imply that, for large M , the image of $r_{M,N}$ equal to the image of the interior generators under $r_{M,N}$, and this set only increases as M increases. \square

We introduce the notation CF^{lf} for the limit complex, which is the complex computing morphisms in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$:

$$(28) \quad CF^{\text{lf}}(\lambda(L_0), \lambda(L_1)) = \lim_{N \rightarrow \infty} CW(\lambda_N(L_0), \lambda_N(L_1))$$

On the other hand, the limit on the right-hand can be described geometrically in terms of intersection points of the full preimages $\pi^{-1}(L_0), \pi^{-1}(L_1)$, since it is the direct product (not direct sum) of one-dimensional vector spaces associated to each intersection point:

$$(29) \quad CF^{\text{lf}}(\lambda(L_0), \lambda(L_1)) = \prod_{x \in \pi^{-1}(L_0) \cap \pi^{-1}(L_1)} \Lambda_x,$$

where Λ_x denotes the one-dimensional vector space incorporating orientation lines and possibly local systems on L_0 and L_1 .

There is a natural functor $\iota : \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi} \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ given by forgetting the O_π -equivariant structure.

Proposition 6.10. *The functor λ factors through ι , so there is a functor $\lambda^{O_\pi} : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$ such that $\lambda = \iota \circ \lambda^{O_\pi}$. The functor λ^{O_π} is a quasi-equivalence onto its image.*

Proof. For any Lagrangian brane L , $\lambda(L)$ is a O_π -invariant object, since it is the limit of restrictions of supported on the O_π -invariant Lagrangian $\pi^{-1}(L)$, and $\lambda(L)$ carries a natural O_π -equivariant structure owing to this fact. If we take two Lagrangian branes L_0 and L_1 , the the O_π -equivariant morphisms between the corresponding objects of $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$ is nothing but the O_π -invariant subspace of $CF^{\text{lf}}(\lambda(L_0), \lambda(L_1)) = \prod_{x \in \pi^{-1}(L_0) \cap \pi^{-1}(L_1)} \Lambda \cdot x$, which is naturally identified with $CF(L_0, L_1)$, the morphism complex in $\text{Fuk}(\Sigma)$. Thus the functor λ maps $CF(L_0, L_1)$ isomorphically onto the subspace $CF^{\text{lf}}(\lambda(L_0), \lambda(L_1))^{O_\pi}$. \square

6.7. Essential surjectivity. In this section we prove that the functor $\lambda^{O_\pi} : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$ is essentially surjective. One of the key ingredients is the geometrization result of Haiden–Katzarkov–Kontsevich [HKK17], and its extension to the \mathbb{Z}_2 -graded setting given in [AS20]. Let us recall the statement that we will need.

Let S be a finite type Riemann surface either compact or with punctures. Following [AS20] we say that an object L in $\text{Fuk}(S)$ is *geometric* if it is quasi-isomorphic to a union of immersed arcs or curves in S .¹¹

Theorem 6.11 ([HKK17], [AS20]). *If S has more than three punctures then every object in $\text{Fuk}^w(S)$ is geometric.*

¹¹In fact as in [AS20] if S is non-compact one should also consider curves in the Liouville completion of S , but this will not affect our argument.

Proof. Theorem 4.2 of [AS20] states that if S is a finite type Riemann surface with punctures every object of the category of twisted complexes of Lagrangian branes is geometric. In fact this was proved in [HKK17] in the \mathbb{Z} -graded case, and Auroux–Smith show that the proof applies also to the \mathbb{Z}_2 -graded setting. Since our $\text{Fuk}^w(S)$ is the split closure of the category of twisted complexes, this is not quite yet the statement we need. However Corollary 4.12 of [AS20] shows that if S has more than three punctures, then the category of twisted complexes of Lagrangian branes is split closed, and this concludes the proof. \square

Corollary 6.12. *Every object in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ is geometric.*

Proof. Note that the notion of geometricity applies equally well to $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ even though $\tilde{\Sigma}$ is not of finite-type. The only difference with the finite-type case is that objects in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ might consist of infinite (but locally finite) unions of arcs and immersed curves. Recall from [HKK17] and [AS20] that the geometric models for an objects in $\text{Fuk}(S)$ given by Theorem 6.11 are in fact unique: more precisely, they are unique up to a unique isotopy.

Let us place ourselves in the setting of Section 6.4. Recall that we consider the limit

$$\text{Fuk}^{\text{lf}}(\tilde{\Sigma}) = \varprojlim_{N \rightarrow \infty} \text{Fuk}^w(\tilde{\Sigma}_N)$$

The surfaces $\tilde{\Sigma}_N$ are finite type Riemann surfaces with more than 3 punctures, so they fall within the scope of Theorem 6.11. That is, all objects in $\text{Fuk}^w(\tilde{\Sigma}_N)$ are geometric. The Viterbo restriction functors

$$r_{M,N} : \text{Fuk}^w(\tilde{\Sigma}_M) \rightarrow \text{Fuk}^w(\tilde{\Sigma}_N),$$

admit an especially simple description on geometric objects due to Lemma 6.1: they send a geometric object $L \in \text{Fuk}^w(\tilde{\Sigma}_M)$ to its intersection with $\tilde{\Sigma}_N$, or, in the case of an object parallel to a boundary component of $\tilde{\Sigma}_N$, to the corresponding object sitting inside the Liouville completion of $\tilde{\Sigma}_N$.

Now, by definition an object in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ is given by a tuple

$$L := (\{L_N\}, \{\alpha_N\})_{N \in \mathbb{N}}$$

where L_N is an object in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma}_N)$ and $\alpha_N : L_{N-1} \xrightarrow{\cong} r_{N,N-1}(L_N)$ is a quasi-equivalence. Geometrization for $\tilde{\Sigma}_N$ and $\tilde{\Sigma}_{N-1}$ tells us that if we fix a geometric model C_{N-1} for L_{N-1} we can find a compatible geometric model C_N for L_N , in the sense that C_{N-1} is obtained by intersecting C_N with Σ_{N-1} . The idea is the following: we pick a geometric representative C_N of L_N ; then the intersection of C_N with Σ_{N-1} provides another geometric model for L_{N-1} , possibly different from C_{N-1} . By uniqueness however this new geometric model must differ from C_{N-1} only up to an isotopy, which we can lift up to Σ_N so as to get a new geometric representative of L_N with the desired compatibility property with C_{N-1} . This allows us to build up recursively a geometric model for an object L in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$: namely, at each stage $\tilde{\Sigma}_N$ we pick a geometric model which is compatible with the geometric model selected at stage $N-1$. This concludes the proof. \square

Proposition 6.13. *Let L be an object of $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$, and let \bar{L} be the object of $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ obtained by forgetting the equivariant structure. Then any indecomposable component C of \bar{L} is isomorphic to either a compact immersed curve with local system, or else an arc that is fixed by some subgroup H of O_π isomorphic to \mathbb{Z} . In either case, C is isomorphic to a lift of a compact immersed curve in Σ .*

Proof. Let L be an object of $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$. By forgetting the O_π -equivariant structure, we obtain an O_π -invariant object \bar{L} in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$. Applying geometrization to \bar{L} , we see that \bar{L} is a direct sum of immersed curves with local system and arcs in $\tilde{\Sigma}$, such that the whole collection is O_π -invariant.

Let C be a component of \bar{L} . Let H be the subgroup of O_π that fixes C (H may be trivial). Then \bar{L} must contain the direct sum $\bigoplus_{g \in O_\pi/H} g \cdot C$. Once again we may regard $\tilde{\Sigma}$ as a collection of $2g$ -punctured spheres arranged on the lattice O_π and glued together. Let $\tilde{\Sigma}_0$ be one of the punctured spheres in this decomposition: it is a fundamental domain for the O_π -action. Because the object $\bigoplus_{g \in O_\pi/H} g \cdot C$ is assumed to be locally finite, the intersection of this object with $\tilde{\Sigma}_0$ consists of finitely many immersed curves and arcs on $\tilde{\Sigma}_0$. Matching the corresponding boundaries of $\tilde{\Sigma}_0$ gives the original surface Σ , and the immersed curves and arcs must match up to give a compact immersed submanifold of Σ . It is then clear that $\bigoplus_{g \in O_\pi/H} g \cdot C$ is nothing but the pullback of this submanifold under the covering map $\pi : \tilde{\Sigma} \rightarrow \Sigma$. This shows that our original component C is a lift with respect to π of an immersed curve in Σ .

Recall that there is a homomorphism $\tau : \pi_1(\Sigma) \rightarrow O_\pi$. If $\iota : S^1 \rightarrow \Sigma$ is an immersion, we may consider $\tau \circ \iota_* : \pi_1(S^1) \rightarrow O_\pi$. Either $\tau \circ \iota_*$ vanishes or it does not. If it vanishes, then ι can be lifted to an immersion $\tilde{\iota} : S^1 \rightarrow \tilde{\Sigma}$; if it does not vanish, then it may be lifted to an immersion $\tilde{\iota} : \mathbb{R} \rightarrow \tilde{\Sigma}$. In the former case, the lift is not fixed by any subgroup of O_π , while in the latter case, the lift is fixed by a subgroup $H \cong \mathbb{Z}$: indeed, H is nothing but the image of $\tau \circ \iota_*$. This establishes that any noncompact component of \bar{L} must be fixed by a subgroup of O_π that is isomorphic to \mathbb{Z} . \square

To understand the next step in the argument, consider the following. Let L be an object of $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$, and let \bar{L} be the underlying object in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$. By Proposition 6.13, we know that \bar{L} decomposes into a direct sum of objects of the form $\bar{C} = \bigoplus_{g \in O_\pi/H} g \cdot C$, where C is a lift of an immersed curve \bar{C} in Σ (equipped with a local system), and H is either trivial or isomorphic to \mathbb{Z} .

Now we remember the O_π -equivariant structure on our original object L . It is possible that the equivariant structure, which is a certain collection of isomorphisms $\Phi_g : \bar{L} \rightarrow \bar{L}$, could mix together different components in the direct sum decomposition. This does indeed occur in the case where L is the pullback of a simple closed curve, with nontrivial class in O_π , equipped with higher-rank local system. Nevertheless, we wish to show that any equivariant structure on \bar{L} is isomorphic to one where there is no mixing between components corresponding to nonisotopic immersed curves in Σ . For this we use again the geometrization result applied to intermediate coverings $\tilde{\Sigma}/H$ for various subgroups $H \subset O_\pi$. Recall that a saturate sublattice $H \subset O_\pi$ is a subgroup such that if $n \in \mathbb{Z}$ and $ng \in H$ then $g \in H$. For a saturated sublattice we have $O_\pi \cong H \times O_\pi/H$.

Lemma 6.14. *Let \bar{L} be as above, suppose that $H \subset O_\pi$ is a proper saturated sublattice. The components C of \bar{L} into two kinds:*

- (1) *those such that the class of $\pi(C)$ in O_π lies in H , and*
- (2) *those such that the class of $\pi(C)$ does not lie in H .*

Then any equivariant structure on \bar{L} is isomorphic to one for which there is no mixing between these two kinds, and furthermore there is no mixing between summands of the first kind corresponding to nonisotopic immersed curves.

Proof. Consider the surface $\tilde{\Sigma}/H$. Because H is a proper saturated sublattice, it has rank less than g , and so $\tilde{\Sigma}/H$ is a non-finite-type Stein surface, equipped with a pants decomposition. Hence the geometry of this surface may be expressed in terms of topological Fukaya categories, and so several of the results we have shown for $\tilde{\Sigma}$ hold just as well for $\tilde{\Sigma}/H$. In particular, every object of $\text{Fuk}^{\text{lf}}(\tilde{\Sigma}/H)$ is geometric, and we have a descent equivalence

$$(30) \quad \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^H \cong \text{Fuk}^{\text{lf}}(\tilde{\Sigma}/H).$$

From this it follows that

$$(31) \quad \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi} \cong \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{H \times O_\pi/H} \cong \text{Fuk}^{\text{lf}}(\tilde{\Sigma}/H)^{O_\pi/H},$$

So an O_π -equivariant structure on \bar{L} is obtained from an O_π/H -equivariant object L' on $\tilde{\Sigma}/H$ by pullback.

Now the underlying object \bar{L}' in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma}/H)$ is geometric; components C of the first kind correspond to compact components of \bar{L}' , while those of the second kind correspond to noncompact components, which are fixed by some subgroup of O_π/H isomorphic to \mathbb{Z} . We refer to the components of \bar{L}' as being of the first or second kind respectively. Now in $\tilde{\Sigma}/H$, there is no mixing between components of the first kind, simply by geometricity. Furthermore, because the group O_π/H freely permutes the summands of the first kind (none are fixed by any nontrivial subgroup), the O_π/H -equivariant structure on \bar{L}' is isomorphic to one for which there is no mixing between components of the first and second kinds. Pulling back such an equivariant structure gives the desired result. \square

Proposition 6.15. *Any object L in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$ splits O_π -equivariantly into a direct sum of objects of the form $\tilde{C}^{\oplus r}$, where $\tilde{C} = \bigoplus_{g \in O_\pi/H} g \cdot C$, where C is a lift of an immersed curve \bar{C} in Σ equipped with a local system, and H is either trivial or isomorphic to \mathbb{Z} .*

Proof. Since O_π is a lattice of rank $g \geq 2$, every element of O_π is contained in some nontrivial proper saturated sublattice H . Applying Lemma 6.14 to all such sublattices, we see that L splits equivariantly into a direct sum according to different sublattices in O_π , and furthermore that within each sublattice there is a splitting according to isotopy classes of the projection $\pi(C)$. \square

Theorem 6.16. *The functor $\lambda^{O_\pi} : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$ is an equivalence.*

Proof. Since we have already proved that λ^{O_π} is fully-faithful, it remains to show that it is essentially surjective. By 6.7, it suffices to show that an equivariant object of the form $\tilde{C}^{\oplus r}$ where $\tilde{C} = \bigoplus_{g \in O_\pi/H} g \cdot C$ and C is a lift of an immersed curve \bar{C} is in the image. In the case where H is trivial, L is equivariantly isomorphic to the pullback of \bar{C} with its natural equivariant structure. If $H \cong \mathbb{Z}$, then the O_π -equivariant structure on $\tilde{C}^{\oplus r}$ induces an H -equivariant structure on $C^{\oplus r}$. Since $\bar{C} = C/H$, this equivariant structure defines a rank r local system on C , such that $\tilde{C}^{\oplus r}$ is equivariantly isomorphic to the pullback of C with this local system. \square

Corollary 6.17 below is our main local-to-global result for the Fukaya category of a Riemann surface Σ of genus $g \geq 2$. We will actually prove two parallel results: one for the Fukaya category (which is defined over the Novikov field Λ) and one for the balanced Fukaya category (which is defined over a fixed ground field k). The statement is phrased in terms of the sheaf of categories on graphs \mathcal{B}^ω which we introduced in Section 3, and that it is defined over a ground ring κ .

Let G be the graph associated to the pants decomposition of Σ . We equip G with weights $\alpha(v) = q^{-A(v)}$ and $\beta(t) = q^{B(t)}$ where $A(v)$ is the area of the pair of pants corresponding to the vertex v and $B(t)$ is the area of the annulus corresponding to the edge t . Set γ equal to the product of these weights, so that $\gamma = q^{-\text{Area}(\Sigma)}$. We also choose a set of framings f . The covering graph \tilde{G} is given the pull-back weights and framings.

From Definition 3.10, we have $\mathcal{B}(G, \gamma) = \mathcal{B}(G, \alpha, \beta, f)$. We denote the compact objects in this category by $\mathcal{B}_\Lambda^\omega(G, \gamma)$ to emphasize the coefficient field. We can also set all weights to 1 and work over k to get a category $\mathcal{B}_k^\omega(G, 1)$.

Corollary 6.17. *There are $\kappa((u))$ -linear equivalences of categories*

$$\text{Fuk}(\Sigma) \simeq \mathcal{B}_\Lambda^\omega(G, \gamma), \quad \text{Fuk}^{\text{bal}}(\Sigma, \theta) \simeq \mathcal{B}_k^\omega(G, 1)$$

where in the former case $\kappa = \Lambda$ and in the latter $\kappa = k$.

Proof. We will show first that $\text{Fuk}(\Sigma) \simeq \mathcal{B}_\Lambda^\omega(G, \gamma)$. We begin with the equivalence

$$\text{Fuk}^{\text{lf}}(\tilde{\Sigma}) \simeq \text{Fuk}_\Lambda^{\text{top,lf}}(\tilde{\Sigma})$$

from Proposition 6.2, where the Liouville structure $\tilde{\Sigma}$ has been used to trivialize the dependence on the Novikov parameter. This equivalence is not compatible with the O_π action on $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ because the action of O_π does not preserve the Liouville structure. To remedy this, recall the equivalence from Proposition 4.10

$$\varprojlim \mathcal{B}^\omega(G_i) \simeq \text{Fuk}^{\text{top,lf}}(\tilde{\Sigma}).$$

where G_i is an exhaustion of \tilde{G} by finite-type subgraphs, and all weights are set to 1 in this equivalence. Combining these, we obtain an equivalence

$$\text{Fuk}^{\text{lf}}(\tilde{\Sigma}) \simeq \varprojlim \mathcal{B}^\omega(G_i)$$

Now we may restore the weights on the right-hand side and the q -dependence on the left-hand side, so that this equivalence is O_π -equivariant. Then we have a chain of equivalences

$$\text{Fuk}(\Sigma) \stackrel{(*)}{\simeq} \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi} \stackrel{(**)}{\simeq} (\varprojlim \mathcal{B}(G_i)^\omega)^{O_\pi} \stackrel{(***)}{\simeq} \mathcal{B}_\Lambda^\omega(G),$$

where equivalence $(*)$ was proved in Proposition 6.16, equivalence $(**)$ was just proven, and equivalence $(***)$ follows from the argument in Proposition 4.12. Composing them we obtain $\text{Fuk}(\Sigma) \simeq \mathcal{B}_\Lambda^\omega(G)$, which is what we wanted to show.

Now let us pass to the balanced case. Recall that to construct the balanced category one considers first a version of the balanced Fukaya category which is linear over $\Lambda^{\text{fin}} \subset \Lambda$, i.e. the subring of Novikov series with finitely many non-vanishing terms. This category was denoted $\mathcal{C}(\Sigma, \theta)$ in Section 6.1; extension of scalars gives a Λ^{fin} -linear functor

$$\mathcal{C}(\Sigma, \theta) \rightarrow \text{Fuk}(\Sigma) \simeq \mathcal{C}(\Sigma, \theta) \otimes_{\Lambda^{\text{fin}}} \Lambda$$

We get a diagram

$$\text{Fuk}(\Sigma) \leftarrow \mathcal{C}(\Sigma, \theta) \xrightarrow{(a)} \text{Fuk}^{\text{bal}}(\Sigma, \theta).$$

where (a) is specialization at $q = 1$.

The statement we care about is for $\text{Fuk}^{\text{bal}}(\Sigma, \theta)$; but it is actually more convenient to deduce it from the analogous statement for $\mathcal{C}(\Sigma, \theta)$. Namely, we will prove that

$$(32) \quad \mathcal{C}(\Sigma, \theta) \simeq \mathcal{B}_{\Lambda^{\text{fin}}}^\omega(G).$$

To get the result we want for $\text{Fuk}^{\text{bal}}(\Sigma, \theta)$ is then enough to specialize at $q = 1$.

To prove equivalence (32) we need to work around the following issue: being balanced is a global property, so gluing together local balanced branes we do not necessarily get a balanced brane. In other words, it is not immediately obvious that $\mathcal{C}(\Sigma, \theta)$ is the limit of the categories $\mathcal{C}(P, \theta)$, where P are the pants making up the decomposition of Σ . However we have the following diagram

$$\begin{array}{ccc} \text{Fuk}(\Sigma) & \xrightarrow{\simeq} & \mathcal{B}_\Lambda^\omega(G) \\ \uparrow & & \uparrow \\ \mathcal{C}(\Sigma, \theta) & \xrightarrow{\iota} & \mathcal{B}_{\Lambda^{\text{fin}}}^\omega(G) \end{array}$$

where

- the vertical arrows are extension of scalars
- the top equivalence was showed in the first part of the proof
- and ι is fully-faithful.

The fully-faithfulness of ι can be deduced just from the local statement that, if P is the pair-of-pants, then $\mathcal{C}(P, \theta) \simeq \mathcal{B}_{\Lambda^{\text{fin}}}^\omega(G_P)$, which can be easily checked directly.¹² Using that we can write

$$\varprojlim \mathcal{C}(P, \theta) \simeq \varprojlim \mathcal{B}_{\Lambda^{\text{fin}}}^\omega(G_P) \simeq \mathcal{B}_{\Lambda^{\text{fin}}}^\omega(G)$$

¹²Here G_P is the graph with one trivalent vertex.

where P runs over the pants decomposition of Σ . Balanced branes in Σ can be obtained by gluing together local balanced branes in the pants making up the decomposition. This gives us the fully-faithful embedding ι . However in principle ι could fail to be essentially surjective because being balanced is a global property: in the limit $\varinjlim \mathcal{C}(P, \theta)$ we will have branes which restrict to balanced branes on every pair-of-pants P , but which might fail to be globally balanced.

However one can see that ι is in fact an equivalence. Choose balanced generators of $\text{Fuk}(\Sigma)$ which we denote L_1, \dots, L_N . Let M_i the corresponding generators of $\mathcal{B}_\Lambda^\omega(G)$ under the equivalence $\text{Fuk}(\Sigma) \simeq \mathcal{B}_\Lambda^\omega(G)$. The L_i are geometric balanced branes, which can be defined in the Fukaya category independently of the ground ring; thus the M_i will also be constructed in terms of the geometry of the singularity categories making up $\mathcal{B}_\Lambda^\omega(G)$ in a way that does not depend on the ground ring. In particular we can lift them to generators of $\mathcal{B}_{\Lambda^{fin}}^\omega$, which we still denote M_i . At the same time, by construction, the L_i can be lifted to objects in $\mathcal{C}(\Sigma, \theta)$, and clearly under ι the L_i are sent to the M_i

$$L_i \in \mathcal{C}(\Sigma, \theta) \mapsto \iota(L_i) \simeq M_i \in \mathcal{B}_{\Lambda^{fin}}^\omega.$$

So ι is a fully-faithful embedding such that its image contains a collections of generators of $\mathcal{B}_{\Lambda^{fin}}^\omega$. Thus it has to be an equivalence, which is what we needed to show. \square

We record a remarkable consequence of Proposition 6.16 and its proof, namely that all objects in the Fukaya category of a closed Riemann surface of genus $g \geq 2$ are geometric. Corollary 6.18 provides an answer to the second open Problem in Section 7 of [HKK17].

The geometric idea is simple. The objects in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$ can be viewed as O_π -invariant branes in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})$ equipped with an equivariant structure. In particular, by the geometricity of $\tilde{\Sigma}$ they can be viewed as immersed Lagrangians in $\tilde{\Sigma}$. It is easy to see that $(\lambda^{O_\pi})^{-1}$ has a very simple definition on geometric objects: it is just the push-forward of the immersed Lagrangian along $\pi : \tilde{\Sigma} \rightarrow \Sigma$; then we use the equivariant structure to equip the resulting immersed Lagrangian in Σ with a brane structure. This shows that, using Proposition 6.16, geometricity for $\tilde{\Sigma}$ readily implies geometricity for Σ .

Corollary 6.18. *Let S a compact Riemann surface of genus $g \geq 2$. Then every object in $\text{Fuk}(S)$ is geometric.*

Proof. Let us give a more formal argument based on the proof of Proposition 6.16. There we proved that the functor $\lambda^{O_\pi} : \text{Fuk}(\Sigma) \rightarrow \text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$ is essentially surjective. In fact we showed a stronger result: namely for every object L in $\text{Fuk}^{\text{lf}}(\tilde{\Sigma})^{O_\pi}$ we constructed a geometric object L' in $\text{Fuk}(\Sigma)$ such that $\lambda^{O_\pi}(L') \simeq L$. Now since λ^{O_π} is an equivalence, this immediately implies that every object in $\text{Fuk}(\Sigma)$ is equivalent to a geometric object, which is what we wanted to show. \square

7. HMS FOR COMPACT SURFACES

Our description of the Fukaya category of a compact Riemann surface of genus $g \geq 2$ has significant applications to HMS. In particular, we recover Seidel and Efimov's HMS results for curves of genus $g \geq 2$ [Sei11] [Efi12]. Our method of proof however is different, and can help clarify why these mirror constructions actually work. Surfaces have actually many different geometrically meaningful mirrors. The techniques developed in this paper allow us to easily check that these different models indeed give rise to an HMS equivalence, alternative to Seidel's and Efimov's. We focus on two constructions:

- (1) Hori–Vafa mirror symmetry matches a toric CY 3-folds X_Σ , equipped with a superpotential W_Σ to a punctured surface Σ equipped with a pants decomposition. We show that the Hori–Vafa picture holds in complete generality for all toric 3-folds, regardless of the CY assumption.

- (2) We show that the Mumford degeneration of the Jacobian provides a natural mirror LG model for Riemann surfaces of genus two. The same construction appears, for the other direction of HMS, in beautiful recent work of Cannizzo [Can20].

One prefatory remark about the results of this section is in order. We consider the Fukaya category either over $\kappa = \Lambda$, or the balanced Fukaya category over $\kappa = k$; in either case it is considered as a 2-periodic, that is $\kappa((u))$ -linear, category. The equivalences described below can be interpreted either as κ -linear equivalences, or they can be upgraded to $\kappa((u))$ -linear equivalences with the caveat that the $\kappa((u))$ -linear structure on the B-side is only determined up to a rescaling of u . We do not pin down this rescaling factor precisely, though in principle this can be accomplished with a closer analysis of the categories of matrix factorizations. We also remark that this scaling factor can be trivialized in the case of open surfaces and their mirrors.

Let G be a finite graph with no loops, and with vertices of valency 1 or 3. Let G° be the graph obtained from G by removing the 1-valent vertices. Up to symplectomorphism, there is a unique Riemann surface Σ_G (possibly with boundary) such that G° is the dual intersection complex of a pants decomposition of Σ . The 1-valent vertices of G correspond naturally to a subset of the components of $\partial\Sigma_G$: we decorate each of these components with a stop; let $S_G \subset \partial\Sigma_G$ be the collection of stops obtained in this way.

Theorem 7.1. *Let T be a smooth variety of dimension 3 and let $X \subset T$ be a simple normal crossing divisor of the form $X = f^{-1}(0)$ for some morphism $f : T \rightarrow \mathbb{A}^1$. Assume that X has graph-like singular locus, and that the dual intersection complex of X is orientable. Then there is an equivalence of categories*

$$\mathrm{Fuk}(\Sigma_{G(X)}, S_{G(X)}) \simeq \mathrm{DSing}(X)$$

Proof. Let us first make a comment of notation: if $\Sigma_{G(X)}$ has a boundary, by $\mathrm{Fuk}(\Sigma_{G(X)}, S_{G(X)})$ we mean the (partially) wrapped Fukaya category of $\Sigma_{G(X)}$, which is an exact symplectic surface. If $\Sigma_{G(X)}$ is a compact Riemann surface without boundary, then $S_{G(X)} = \emptyset$ and $\mathrm{Fuk}(\Sigma_{G(X)}, S_{G(X)})$ stands for the balanced Fukaya category of $\Sigma_{G(X)}$. The theorem is an immediate consequence of the results we obtained in Section 5 and 6, and in our previous article [PS19]. Indeed, the claim is obtained by composing the equivalences

$$\mathrm{DSing}(X) \stackrel{(*)}{\simeq} \mathcal{B}^\omega(G(X)) \stackrel{(**)}{\simeq} \mathrm{Fuk}(\Sigma_{G(X)}, S_{G(X)})$$

Equivalence $(*)$ is given by Theorem 5.6, after taking compact objects. Let us make some comments on equivalence $(**)$:

- If $G(X)$ is not compact, or is compact but has 1-valent vertices, then $\Sigma_{G(X)}$ is Riemann surface with boundary, and therefore we are in the exact setting. In this case equivalence $(**)$ follows from Theorem 8.3 in [PS19].
- If $G(X)$ is a compact and 3-valent graph, then $\Sigma_{G(X)}$ is a compact Riemann surface without boundary. In this setting, equivalence $(**)$ was obtained in Corollary 6.17.

□

Let Σ_g be a compact Riemann surface of genus $g \geq 2$ without boundary. Let us denote by (Y_g, W_g) the mirror LG model of Σ_g proposed by Seidel (for $g = 2$) and Efimov (for $g > 2$) in [Sei11] [Efi12]. We refer the reader to the original papers for the explicit construction of (Y_g, W_g) .

Corollary 7.2 (Theorem 1.1 [Sei11], Theorem 1.1 [Efi12]). *There is an equivalence of categories*

$$\mathrm{Fuk}(\Sigma_g) \simeq \mathrm{MF}(Y_g, W_g)$$

Proof. The equivalence follows from Theorem 7.1. We only need to check that $X_g := W_g^{-1}(0)$ has graph-like singular locus, and that $G(X_g)$ is isomorphic to the dual intersection complex of a triangulation of Σ_g . For $g = 2$, see Figure 1 in [Sei11]. As for $g > 2$, note that Efimov's construction

extends all steps of Seidel’s proof to higher genus: the singular locus of W_g has an analogous shape to the one of Seidel, except the genus can be arbitrary; in particular, as in Seidel’s proof, $G(X_g)$ encodes the dual intersection complex of a triangulation of Σ_g . \square

Let Y be a smooth toric 3-fold, and let X be its toric boundary divisor. As pointed out in Remark 5.2, X has graph-like singular locus. The graph $G(X)$ is trivalent, and it is compact if Y is compact. However, X is merely the zero locus of a section of a line bundle on Y rather than that zero fiber of a morphism $Y \rightarrow \mathbb{A}^1$. To fix this, choose a pencil that contains X ; then we get a rational map $f : Y \dashrightarrow \mathbb{P}^1$ with $X = f^{-1}(0)$ as a fiber. By resolving the singularities of the closure of the graph of f , we obtain a variety \tilde{Y} that is birational to Y and a morphism $\tilde{f} : \tilde{Y} \rightarrow \mathbb{P}^1$. The resolution process may modify the fiber X by a birational transformation: we call the corresponding fiber $\tilde{X} = \tilde{f}^{-1}(0)$. In many cases it is possible to choose a resolution such that $G(\tilde{X}) \cong G(X)$, that is, the combinatorics of the singular locus is unchanged. As a special case of Theorem 7.1 we obtain the following.

Corollary 7.3. *There is an equivalence of categories*

$$\mathrm{Fuk}(\Sigma_{G(\tilde{X})}) \simeq \mathrm{DSing}(\tilde{X})$$

Remark 7.4. Let us explain why it is worthwhile to highlight, as Corollary 7.3, this specific setting of Theorem 7.1. The statement of Corollary 7.3 is a generalization of Hori–Vafa mirror symmetry, which can be recovered as the special case when Y is Calabi–Yau. Hori–Vafa mirror symmetry is a widely studied problem, with contributions by many mathematicians starting with [AAK16] [Boc16] [Lee16]; we addressed Hori–Vafa mirror symmetry from the perspective of the topological Fukaya category in our previous paper [PS19]. If Y is a smooth toric 3-fold we obtain in particular new mirrors for compact curves of genus $g \geq 3$ (the case $g = 3$ corresponding to $Y = \mathbb{P}^3$). In future work, we will pursue generalizations of this picture to higher dimensions.

7.1. HMS for genus two curves and Mumford degenerations of Abelian surfaces. We conclude this section by explaining how our methods give a proof of a version of HMS for the genus two curve Σ , where the mirror is a Mumford degeneration of abelian surfaces. This type of mirror symmetry was first suggested by Seidel in [Sei12] in the more general contexts of hypersurfaces inside abelian varieties. In [Can20] Cannizzo proves HMS for the genus two curve in this framework, but in the opposite direction: she compares the derived category of Σ with a kind of Fukaya–Seidel category of the mirror Mumford degeneration. We complement Cannizzo’s result by showing that the reverse HMS equivalence also holds. This is an immediate consequence of Theorem 7.1.

We give an informal treatment of the set-up, referring to Section 10 of [AAK16] (see especially Example 10.6) and Section 3 of [Can20] for a more thorough account of the underlying geometry. Let Σ be a Riemann surface of genus two. The starting point of the construction is a choice of embedding of Σ inside an abelian surface A , such as the Jacobian of Σ . We can consider a simultaneous degeneration of Σ inside A over a small disk \mathbb{D}_ε : the central fiber of the degeneration of the ambient abelian surface has two irreducible components, such that their normalizations are both isomorphic to \mathbb{P}^2 ; the curve Σ degenerates to a nodal curve Σ_0 given by two \mathbb{P}^1 which lie in separate components of A_0 , and meet transversely in three distinct points: Σ_0 is sometimes called the banana curve.

The mirror family can be computed via Legendre transform of these degeneration data, and is also a Mumford degeneration of abelian surfaces. We obtain a family $W : Y \rightarrow \mathbb{D}_\varepsilon$ such that the smooth fibers are abelian surfaces, and the normalization of the singular fiber Y_0 is a blow-up of \mathbb{P}^2 at the three torus-fixed points. The LG model (Y, W) gives another mirror of Σ , which is different from Seidel’s original construction.

Corollary 7.5. *There is an equivalence of categories*

$$\mathrm{Fuk}(\Sigma) \simeq \mathrm{MF}(Y, W).$$

Proof. The singular locus of Y_0 is a banana curve, see Example 10.6 in [AAK16]. Thus Y_0 has graph-like singular locus and $\Sigma = \Sigma_{G(Y_0)}$, then the statement follows from Theorem 7.1. \square

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