FLOER COHOMOLOGY IN THE MIRROR OF THE PROJECTIVE PLANE AND A BINODAL CUBIC CURVE

JAMES PASCALEFF

ABSTRACT. We construct a family of Lagrangian submanifolds in the Landau–Ginzburg mirror to the projective plane equipped with a binodal cubic curve as anticanonical divisor. These objects correspond under mirror symmetry to the powers of the twisting sheaf $\mathcal{O}(1)$, and hence their Floer cohomology groups form an algebra isomorphic to the homogeneous coordinate ring. An interesting feature is the presence of a singular torus fibration on the mirror, of which the Lagrangians are sections. This gives rise to a distinguished basis of the Floer cohomology and the homogeneous coordinate ring parametrized by fractional integral points in the singular affine structure on the base of the torus fibration. The algebra structure on the Floer cohomology is computed using the symplectic techniques of Lefschetz fibrations and the TQFT counting sections of such fibrations. We also show that our results agree with the tropical analog proposed by Abouzaid–Gross–Siebert. Extensions to a restricted class of singular affine manifolds and to mirrors of the complements of components of the anticanonical divisor are discussed.

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1. Introduction

This article is concerned with a case of mirror symmetry relating the algebraic geometry of a Fano manifold to the symplectic geometry of a Landau–Ginzburg model. The Fano manifold is the projective plane $\mathbb{CP}^2$ with homogeneous coordinates x, y, z, and additionally we choose as anticanonical divisor $D = C \cup L$, the union of a conic $C = \{xz - y^2\}$ and a line $L = \{y = 0\}$ (a binodal cubic curve). According to Auroux [7], the mirror of this pair $(\mathbb{CP}^2, D)$ is the Landau–Ginzburg model $(X^\vee, W)$

$$X^\vee = \{(u, v) \in \mathbb{C}^2 \mid uv \neq 1\}, \quad W = u + \frac{e^{-\Lambda} v^2}{uv - 1} \tag{1}$$

The function W is known as the superpotential. This example is interesting in the context of the Strominger-Yau-Zaslow picture of mirror symmetry [42] in terms of dual torus fibrations, since the spaces $\mathbb{CP}^2 \setminus D$ and $X^\vee$ admit special Lagrangian torus fibrations with a singularity. The presence of singularities in the torus fibration is known to make the mirror duality vastly more complicated, but in this article we develop techniques to deal with it in the above case (and other cases with similar properties, see §6).

1.1. Summary of results.

The piece of symplectic geometry in the Landau–Ginzburg model $(X^\vee, W)$ we study is the Floer cohomology of Lagrangian submanifolds, while on the mirror side $\mathbb{CP}^2$ we consider the cohomology of the coherent sheaves $\mathcal{O}_{\mathbb{CP}^2}(d)$. We construct a symplectic manifold $X(B)$, that serves as our symplectic model for $X^\vee$, and a collection of Lagrangian submanifolds $\{L(d)\}_{d\in\mathbb{Z}}$ in $X(B)$ that is mirror to the collection $\{\mathcal{O}_{\mathbb{CP}^2}(d)\}_{d\in\mathbb{Z}}$. The constructions of $X(B)$ and $L(d)$ are based on the SYZ picture of mirror symmetry in terms of torus fibrations and affine manifolds, and also use the theory of Lefschetz fibrations. The manifold $X(B)$ carries two fibrations: one is a fibration over a singular affine manifold B whose fibers are Lagrangian tori, while the other is a Lefschetz fibration whose fibers are symplectic submanifolds. See section 3 for the definitions.

To state the main result, let $A_n = H^0(\mathbb{CP}^2, \mathcal{O}_{\mathbb{CP}^2}(n))$, and choose a basis x, y, z of A_1. Thus $A = \bigoplus_{n=0}^{\infty} A_n \cong \mathbb{C}[x, y, z]$ is the homogeneous coordinate ring of $\mathbb{CP}^2$. The Floer cohomology of two Lagrangian submanifolds L, L' is a $\mathbb{Z}$-graded $\mathbb{C}$-vector space denoted $HF^*(L, L')$.

Theorem 1.1. *For each $d \in \mathbb{Z}$ and $n \geq 0$, there is an isomorphism*

$$\psi_{d,n} : HF^0(L(d), L(d+n)) \to A_n \tag{2}$$

carrying the basis of intersection points $L(d) \cap L(d+n)$ to the basis of A_n consisting of the polynomials of the form

$$\{x^a(xz - y^2)^i y^{n-a-2i}\} \cup \{z^a(xz - y^2)^i y^{n-a-2i}\} \tag{3}$$

(where we require $a \geq 0, i \geq 0, n - a - 2i \geq 0$). The system of isomorphisms $\psi_{d,n}$ intertwines the Floer triangle product

(4)

$$\mu^2 : HF^0(L(d+n), L(d+n+m)) \otimes HF^0(L(d), L(d+n)) \to HF^0(L(d), L(d+n+m))$$

and the product of polynomials $A_m \otimes A_n \to A_{n+m}$.

Let us remark on the formulation of the theorem. From the construction of the Lagrangians $L(d)$ it is easy to count the number of intersection points and show that the map $\psi_{d,n}$ exists as an isomorphism of vector spaces. Such a map exists for any choice of basis of A_n. The last statement equating Floer triangle products and the products of polynomials is the nontrivial bit that ties the choices together and shows that we have chosen the right basis for A_n.

This basis we obtain is related to the choice of divisor $D = C \cup L$, as it consists of monomials in the defining section $p = xz - y^2$ of C, the defining section y of L, and the homogeneous variables x or z (but not both x and z in the same monomial). Another point of view has to do with the fact that the ring of functions on $\mathbb{CP}^2 \setminus D$ is a cluster algebra [12]. There are two triples of homogeneous forms on $\mathbb{CP}^2$, (x, p, y) and (z, p, y), which are related by the so-called exchange relation $xz = y^2 + p$, and our basis consists of sections that are monomials in either triple.

For the proof of this theorem, the majority of our technical efforts are aimed at computing the Floer triangle product. This occupies section 4. At this point in the argument, the picture of $X(B)$ as a Lefschetz fibration is the focus. The holomorphic triangles we need to find can be represented as sections of this Lefschetz fibration. The problem of counting sections of Lefschetz fibrations has a TQFT-type structure, developed by Seidel. This structure, where one breaks a given problem into simpler ones by degenerating the base of the Lefschetz fibration, provides the basis of our technique.

There are several variations on Theorem 1.1 that are accessible using the same holomorphic curve analysis. We consider the complement of (some components of) the anticanonical divisor in $\mathbb{CP}^2$, and on the Landau–Ginzburg side, we must change the superpotential W and consider a certain form of wrapped Floer cohomology. Our techniques allow us to treat three cases:

1. the mirror of $(\mathbb{CP}^2 \setminus L, C \setminus (C \cap L))$,
2. the mirror of $(\mathbb{CP}^2 \setminus C, L \setminus (L \cap C))$, and
3. the mirror of $(\mathbb{CP}^2 \setminus (C \cup L), \emptyset)$.

In each case the mirror space is the same manifold $X^\vee$, and the Lagrangians mirror to line bundles are the same $L(d)$, but in each case there is a different prescription for wrapping the Lagrangian submanifolds. The first two cases require "partially wrapped" Floer cohomology, while the third uses the more standard "fully wrapped" Floer cohomology. In each case we denote wrapped Floer cohomology by

$HW^*(L_1, L_2)$. Let $U = \mathbb{CP}^2 \setminus L$, $\mathbb{CP}^2 \setminus C$, or $\mathbb{CP}^2 \setminus (C \cup L)$, and let $A_n(U) = H^0(U, \mathcal{O}_U(n))$. The space $A_n(U)$ consists of rational functions in the variables x, y, z that are regular on U and have degree n. The wrapped version of Theorem 1.1 is as follows (see §7)

Theorem 1.2. *For $d \in \mathbb{Z}$ and $n \geq 0$, there is an isomorphism*

$$\psi_{d,n} : HW^0(L(d), L(d+n)) \to A_n(U) \tag{5}$$

carrying a certain distinguished basis of $HW^0(L(d), L(d+n))$ to the basis of $A_n(U)$ consisting of rational functions of the form

$$\{x^a(xz - y^2)^i y^{n-a-2i}\} \cup \{z^a(xz - y^2)^i y^{n-a-2i}\} \tag{6}$$

where the exponents a and i are restricted to those values which actually give elements of $A_n(U)$. The system of isomorphisms $\psi_{d,n}$ intertwines the Floer triangle product on wrapped Floer cohomology and the product of rational functions $A_m(U) \otimes A_n(U) \to A_{n+m}(U)$.

While our holomorphic triangle counts use the structure of a Lefschetz fibration on $X(B)$, we can also study the geometry of the $X(B)$ as a Lagrangian torus fibration over the base affine manifold B. One of the expectations of SYZ philosophy is that much of the geometry of the space $X(B)$ can be seen tropically, in terms of the geometry of the affine base B.

In fact, this is how we arrive at the construction of the manifold $X(B)$. In section 2, we construct a tropicalization of the fiber of the superpotential $W^{-1}(c)$ over a large positive real value, with respect to a torus fibration on the manifold $X^\vee$ with a single singularity. This gives a tropical curve in the base of our torus fibration. It bounds a compact region B in the base, which agrees with the affine base of the torus fibration on $\mathbb{CP}^2 \setminus D$. The purpose of this section is to motivate the use of the singular affine manifold B as the basis for our main symplectic constructions in section 3.

We also are able to verify a conjectural tropical description of Floer cohomology in the cases we study. This description comes from a recent proposal of Abouzaid, Gross and Siebert for a tropical Fukaya category associated to a singular affine manifold. The Lagrangian submanifolds are taken to be sections of the torus fibration, but all we see of them tropically are their intersection points, which map to fractional integral points of the affine manifold B. The Floer triangle product corresponds to a "tropical triangle product" counting certain tropical curves in B joining the fractional integral points. Since we do not say anything about degenerating holomorphic polygons to tropical ones, we merely verify the equivalence by matching bases and computing on both sides. See section 5 for the precise definitions of the terms.

Theorem 1.3. *There is a bijection between the basis of intersection points $L(d) \cap L(d+n)$ for $HF^0(L(d), L(d+n))$ and the set of $(\frac{1}{n})$-integral points of the affine manifold B. Under this bijection, the counts of pseudo-holomorphic triangles contributing to the Floer triangle product are equal to counts of tropical curves in B joining $(\frac{1}{n})$-integral points.*

The techniques developed in this article apply to a larger but still restricted class of 2–dimensional singular affine manifolds, where the main restriction is that all singularities have *parallel monodromy-invariant directions.* The generalization to these types of manifolds is discussed in 6.

1.2. Context and related work.

1.2.1. *Manifolds with effective anticanonical divisor and Landau–Ginzburg models.* The class of spaces originally considered in mirror symmetry were Calabi–Yau manifolds. Roughly speaking, the mirror to a compact Calabi–Yau manifold X is another compact Calabi–Yau manifold $X^\vee$ of the same dimension. For example, it is in this context that we have the equivalence, discovered by Candelas–de la Ossa–Green–Parkes [10] and proven mathematically by Givental [15] and Lian–Liu–Yau [33], between the Gromov–Witten theory of the quintic threefold $V_5 \subset \mathbb{P}^4$ and the theory of period integrals on a family of Calabi–Yau threefolds known as "mirror quintics." However, mirror symmetry can be considered for other classes of manifolds such as manifolds of general type (Ω^n_X ample), for which a proposal has recently been made by Kapustin–Katzarkov–Orlov–Yotov [27], and (our present concern) manifolds with an effective anticanonical divisor. In both of these latter cases the mirror is not a manifold of the same class.

Let X be an n–dimensional Kähler manifold with an effective anticanonical divisor D. We regard D as part of the data, and write (X, D) for the pair. We also choose a meromorphic $(n, 0)$–form Ω with no zeros and polar locus equal to D. According to Hori–Vafa [25] and Givental, the mirror to (X, D) is a *Landau–Ginzburg model* $(X^\vee, W)$, consisting of a Kähler manifold $X^\vee$, together with a holomorphic function $W : X^\vee \to \mathbb{C}$, called the superpotential.

The mirrors of toric Fano manifolds were derived by Hori–Vafa [25, §5.3] based on physical considerations. Let X be an n–dimensional toric Fano manifold, and let D be the complement of the open torus orbit. The mirror is then $X^\vee = (\mathbb{C}^*)^n$ with a superpotential W given by a sum of monomials corresponding to the one-dimensional cones in the fan for X. Choose a polarization $\mathcal{O}_X(1)$ with corresponding moment polytope P, a lattice polytope in $\mathbb{R}^n$. For each facet F of P, let $\nu(F)$ to be the primitive integer inward-pointing normal vector, and let $\alpha(F)$ be such that $\langle \nu(F), x\rangle + \alpha(F) = 0$ is the equation for the hyperplane containing F. Then mirror Landau-Ginzburg model is given by

$$X^\vee = (\mathbb{C}^*)^n, \quad W = \sum_{F \text{ facet}} e^{-2\pi\alpha(F)} z^{\nu(F)}, \tag{7}$$

where $z^{\nu(F)}$ is a monomial in multi-index notation.

In the case where X is toric but not necessarily Fano, a similar formula for the mirror superpotential is expected to hold, which differs by the addition of "higher order" terms [14, Theorems 4.5, 4.6].

The Hori–Vafa formula contains the case of the projective plane $\mathbb{CP}^2$ with the toric boundary as anticanonical divisor. If x, y, z denote homogeneous coordinates, then D_{toric} can be taken to be $\{xyz = 0\}$, the union of the coordinate lines. We then have

$$X^\vee_{\text{toric}} = (\mathbb{C}^*)^2, \quad W_{\text{toric}} = z_1 + z_2 + \frac{e^{-\Lambda}}{z_1 z_2} \tag{8}$$

where Λ is a parameter that measures the cohomology class of the Kähler form ω on $\mathbb{CP}^2$.

The example with which we are primarily concerned in this paper is also $\mathbb{CP}^2$, but with respect to a different, non-toric boundary divisor. Consider the meromorphic

$(2,0)$–form $\Omega = dx \wedge dz/(xz-1)$, whose polar locus is the binodal cubic curve $D = \{xyz - y^3 = 0\}$. Thus $D = L \cup C$ is the union of a conic $C = \{xz - y^2 = 0\}$ and a line $\{y = 0\}$. The construction of the mirror to this pair $(\mathbb{CP}^2, D)$ is due to Auroux [7], and we have

$$X^\vee = \{(u,v) \in \mathbb{C}^2 \mid uv \neq 1\}, \quad W = u + \frac{e^{-\Lambda}v^2}{uv-1} \tag{9}$$

A direct computation shows that both superpotentials W_{toric} and W have the same critical values

$$\{3e^{-\Lambda/3}e^{-2\pi i(n/3)} \mid n = 0,1,2\}. \tag{10}$$

It is also easy to see that any regular fiber of $W_{\text{toric}}^{-1}(c) \subset X_{\text{toric}}^\vee$ is a three-times-punctured elliptic curve, while any regular fiber $W^{-1}(c) \subset X^\vee$ is a twice-punctured elliptic curve. This is an example of the general expectation that partially smoothing the anticanonical divisor corresponds to partially compactifying the total space of the Landau–Ginzburg model.

1.2.2. *Torus fibrations and affine manifolds.* One justification that (8)–(9) are appropriate mirrors is found in the Strominger–Yau–Zaslow proposal, which expresses mirror symmetry geometrically in terms of dual torus fibrations, a relationship also known as T–duality. Ideally, one would expect that $X \setminus D$ and $X^\vee$ are dual special Lagrangian torus fibrations over the same base B. When this holds true, the mirror $X^\vee$ can be constructed as the *complexified moduli space* of special Lagrangian fibers of $X \setminus D$ [35], [23],[7, §2]. The superpotential W can be expressed as a function on this moduli space counting Maslov index two disks with boundary on the Lagrangian fibers of $X \setminus D$, weighted by symplectic area and the holonomy of a local system [7].

For our purposes, a Lagrangian submanifold L of a Kähler manifold X with meromorphic $(n,0)$–form Ω is called *special* of phase ϕ if $\mathrm{Im}(e^{-i\phi}\Omega)|L = 0$. Obviously this only makes sense in the complement of the polar locus D. The infinitesimal deformations of a special Lagrangian submanifold are given by $H^1(L;\mathbb{R})$, and are unobstructed [35]. If $L \cong T^n$ is a torus, $H^1(L;\mathbb{R})$ is an n–dimensional space, and in good cases the special Lagrangian deformations of L are all disjoint, and form the fibers of a fibration $\pi : X \setminus D \to B$, where B is the global parameter space for the deformations of L.

Assuming this, define the *complexified moduli space* of deformations of L to be the space $\mathcal{M}_L$ consisting of pairs $(L_b, \mathcal{E}_b)$, where $L_b = \pi^{-1}(b)$ is a special Lagrangian deformation of L, and $\mathcal{E}_b$ is a $U(1)$ local system on L_b. There is an obvious projection $\pi^\vee : \mathcal{M}_L \to B$ given by forgetting the local system. The fiber $(\pi^\vee)^{-1}(b)$ is the space of $U(1)$ local systems on the given torus L_b, which is precisely the dual torus $L_b^\vee$. In this sense, the fibrations π and $\pi^\vee$ are dual torus fibrations, and the SYZ proposal can be taken to mean that the mirror $X^\vee$ is precisely this complexified moduli space: $X^\vee = \mathcal{M}_L$. The picture is completed by showing that $\mathcal{M}_L$ naturally admits a complex structure $J^\vee$, a Kähler form $\omega^\vee$, and a holomorphic $(n,0)$–form $\Omega^\vee$. One finds that $\Omega^\vee$ is constructed from ω, while $\omega^\vee$ is constructed from Ω, thus expressing the interchange of symplectic and complex structures between the two sides of the mirror pair. For details we refer the reader to [23],[7, §2].

However, this picture of mirror symmetry cannot be correct as stated, as it quickly hits upon a major stumbling block: the presence of singular fibers in the original fibration $\pi : X \setminus D \to B$. These singularities make it impossible to obtain the mirror manifold by a fiberwise dualization, and generate "quantum corrections" that complicate the T-duality prescription. Attempts to overcome this difficulty led to the work of Kontsevich and Soibelman [30, 31] and Gross and Siebert [19, 20, 21] that implements the SYZ program in an algebro-geometric context. In the case of K3 surfaces, the work of Fukaya–Oh–Ohta–Ono [13] relates this problem to the wall–crossing of holomorphic disk moduli spaces. It is also this difficulty which motivates us to consider the case of $\mathbb{CP}^2$ relative to a binodal cubic curve, where the simplest type of singularity arises.

In the case of X with effective anticanonical divisor D, we can see these corrections in action if we include the superpotential W into the SYZ picture. As W is to be a function on $X^\vee$, which is naively $\mathcal{M}_L$, W assigns a complex number to each pair $(L_b, \mathcal{E}_b)$. This number is a count of holomorphic disks with boundary on L_b, of Maslov index 2, weighted by symplectic area and the holonomy of $\mathcal{E}_b$:

$$W(L_b, \mathcal{E}_b) = \sum_{\beta \in \pi_2(X, L_b), \mu(\beta)=2} n_\beta(L_b) \exp(-\int_\beta \omega) \mathrm{hol}(\mathcal{E}_b, \partial\beta) \tag{11}$$

where $n_\beta(L_b)$ is the count of holomorphic disks in the class β passing through a general point of L_b.

In the toric case, $X \setminus D \cong (\mathbb{C}^*)^n$, and we the special Lagrangian torus fibration is simply the map $\mathrm{Log} : X \setminus D \to \mathbb{R}^n$, $\mathrm{Log}(z_1, \dots, z_n) = (\log|z_1|, \dots, \log|z_n|)$. This fibration has no singularities, and the above prescriptions work as stated. In the toric Fano case, we recover the Hori–Vafa superpotential (7).

In the case of $\mathbb{CP}^2$ with the non-toric divisor D, the torus fibration has one singular fiber, which is a pinched torus (a focus-focus singularity). The above prescription breaks down: one finds that the superpotential counting disks is not a continuous function on the moduli space of special Lagrangians. This leads one to redefine $X^\vee$ by breaking it into pieces and re-gluing so as to make W continuous, leading to (9) [7]. We find that $X^\vee$ also admits a special Lagrangian torus fibration with one singular fiber.

In general, the structure of a special Lagrangian torus fibration $\pi : X \to B$ yields the structure of an *tropical affine manifold* on the base B. This is a manifold with a distinguished collection of affine coordinate charts, such that the transition maps between affine coordinate charts lie in $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) = \mathrm{GL}(n, \mathbb{Z}) \rtimes \mathbb{R}^n$, the group of affine linear transformations with integral linear part. When singular fibers are present in the torus fibration, we simply regard the affine structure as being undefined at the singular fibers and call the resulting structure on the base a *singular tropical affine manifold*. In fact, the base B inherits two affine structures, one from the symplectic form ω, and one from the holomorphic $(n, 0)$–form Ω. The former is called the *symplectic affine structure*, and the latter is called the *complex affine structure*.

Let us recall briefly how the local affine coordinates are defined. For the symplectic affine structure, we choose a collection of loops $\gamma_1, \dots, \gamma_n$ that form a basis of $H_1(L_b; \mathbb{Z})$. Let $X \in T_bB$ be a tangent vector to the base, and take $\tilde{X}$ be any vector field along L_b which lifts it. Then

$$\alpha_i(X) = \int_0^{2\pi} \omega_{\gamma_i(t)}(\dot{\gamma}_i(t), \tilde{X}(\gamma_i(t)))\, dt \tag{12}$$

defines a 1-form on B: since L_b is Lagrangian, the integrand is independent of the lift $\tilde{X}$, and α_i only depends on the class of γ_i in homology. In fact, the collection $(\alpha_i)_{i=1}^n$ forms a basis of T_b^*B, and there is a coordinate system $(y_i)_{i=1}^n$ such that $dy_i = \alpha_i$; these are the affine coordinates. This definition actually shows us that there is a canonical isomorphism $T_b^*B \cong H_1(L_b;\mathbb{R})$. This isomorphism induces an integral structure on T_b^*B: $(T_b^*B)_{\mathbb{Z}} \cong H_1(L_b;\mathbb{Z})$, which is preserved by all transition functions between coordinate charts. Thus, when an affine manifold arises as the base of a torus fibration in this way, the structural group is reduced to $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) = \mathrm{GL}(n,\mathbb{Z}) \rtimes \mathbb{R}^n$, the group of affine linear transformations with integral linear part.

The complex affine structure follows exactly the same pattern, only that we take $\Gamma_1, \dots, \Gamma_n$ to be $(n-1)$–cycles forming a basis of $H_{n-1}(L_b;\mathbb{Z})$, and in place of ω we use $\mathrm{Im}(e^{-i\phi}\Omega)$. Now we have an isomorphism $T_b^*B \cong H_{n-1}(L_b;\mathbb{R})$, or equivalently $T_bB \cong H_1(L_b;\mathbb{R})$, which induces the integral structure.

The affine manifolds we consider in this article satisfy a stronger integrality condition, which requires the translational part of each transition function to be integral as well. We use the term *integral affine manifold* to denote an affine manifold whose structural group has been reduced to $\mathrm{Aff}(\mathbb{Z}^n) = \mathrm{GL}(n,\mathbb{Z}) \rtimes \mathbb{Z}^n$. Such affine manifolds are "defined over $\mathbb{Z}$" and have an intrinsically defined lattice of integral points.

For an integral affine manifold, it makes sense to speak of tropical subvarieties. These are certain piecewise linear complexes contained in B, which in some way correspond to holomorphic or Lagrangian submanifolds of the total space of the torus fibration. Tropical geometry has played a role in much work on mirror symmetry, particularly in the program of Gross and Siebert, and closer to this paper, in Abouzaid's work on mirror symmetry for toric varieties [1, 3]. See [26] for a general introduction to tropical geometry. Though most of the methods in this paper are explicitly symplectic, tropical geometry appears in section 2, where we compute the tropicalization of the fiber of the superpotential, and in section 5, where a class of tropical curves corresponding to holomorphic polygons is considered.

1.2.3. *Homological mirror symmetry.* The results on Floer cohomology that we prove fall under the heading homological mirror symmetry (HMS) [28], which holds that mirror symmetry can be interpreted as an equivalence of categories associated to the complex or algebraic geometry of X, and the symplectic geometry of $X^\vee$, and vice–versa. The categories which are appropriate depend somewhat on the situation, so let us focus on the case of the a manifold X with anticanonical divisor D, and its mirror Landau–Ginzburg model $(X^\vee, W)$.

Associated to (X, D), we take the derived category of coherent sheaves $D^b(\mathrm{Coh}\, X)$, while to $(X^\vee, W)$ we associate a Fukaya-type A_∞-category $\mathcal{F}(X^\vee, W)$ whose objects are certain Lagrangian submanifolds of $X^\vee$, morphism spaces are generated by intersection points, and the A_∞ product structures are defined by counting pseudoholomorphic polygons with boundary on a collection of Lagrangian submanifolds. Our main reference for Floer cohomology and Fukaya categories is the book of Seidel [41].

The superpotential W enters the definition of $\mathcal{F}(X^\vee, W)$ by restricting the class of objects to what are termed *admissible Lagrangian submanifolds*. In this paper, it will mean that we allow Lagrangian submanifolds that are not closed but which have boundary on a reducible hypersurface whose components are defined by setting one term of the superpotential equal to a constant. Historically, there have been several attempts to formulate the notion of admissibility. Originally, Kontsevich [29] and Hori–Iqbal–Vafa [24] considered those Lagrangian submanifolds L, not necessarily compact, which outside of a compact subset are invariant with respect to the gradient flow of $\mathrm{Re}(W)$. An alternative formulation, due to Abouzaid [1, 3], trades the non-compact end for a boundary on a fiber $\{W = c\}$ of W, together with the condition that, near the boundary, the L maps by W to a curve in $\mathbb{C}$. A further reformulation, which is more directly related to the SYZ picture, replaces the fiber $\{W = c\}$ with the union of hypersurfaces $\bigcup_\beta \{z_\beta = c\}$, where z_β is the term in the superpotential (11) corresponding to the class $\beta \in \pi_2(X, \pi^{-1}(b))$, and admissibility means that near $\{z_\beta = c\}$, L maps by z_β to a curve in $\mathbb{C}$. The admissibility condition we use in this paper is closest to this last formulation.

With these definitions, homological mirror symmetry amounts to an equivalence of categories $D^\pi\mathcal{F}(X^\vee, W) \to D^b(\mathrm{Coh}\, X)$, where D^π denotes the split-closed derived category of the A_∞–category. This piece of mirror symmetry has been addressed many times [9, 8, 37, 1, 3, 11], including results for the projective plane and its toric mirror.

However, in this article, we emphasize less the equivalences of categories themselves, and focus more on geometric structures which arise from a combination of the HMS equivalence with the SYZ picture. When dual torus fibrations are present on the manifolds in a mirror pair, one expects the correspondence between coherent sheaves and Lagrangian submanifolds to be expressible in terms of a Fourier–Mukai transform with respect to the torus fibration [32]. In particular, Lagrangian submanifolds $L \subset X^\vee$ that are sections of the torus fibration correspond to line bundles on X, and the Lagrangians $L(d)$ we consider are of this type.

In this context, our Theorem 1.1 can be interpreted as yielding an embedding (at the cohomology level) of the subcategory of $\mathcal{F}(X(B), W)$ containing the Lagrangians $L(d)$ into $D^b(\mathrm{Coh}\, \mathbb{CP}^2)$.

1.2.4. *Distinguished bases.* The combination of SYZ and HMS also gives rise to the expectation that, at least in favorable situations, the spaces of sections of coherent sheaves on X can be equipped with distinguished bases. Suppose that $F : \mathcal{F}(X^\vee) \to D^b(X)$ is a functor implementing the HMS equivalence. Let $L_1, L_2 \in \mathrm{Ob}(\mathcal{F}(X^\vee))$ be two objects of the Fukaya category supported by transversely intersecting Lagrangian submanifolds. Then

$$HF(L_1, L_2) \cong \mathrm{RHom}(F(L_1), F(L_2)). \tag{13}$$

Suppose furthermore that the differential on the Floer cochain complex $CF(L_1, L_2)$ vanishes, so that $HF(L_1, L_2) \cong CF(L_1, L_2)$. As $CF(L_1, L_2)$ is defined to have a basis in bijection with the set of intersection points $L_1 \cap L_2$, one obtains a basis of $\mathrm{RHom}(F(L_1), F(L_2))$ parametrized by the same set via the above isomorphisms. If $\mathcal{F}$ is some sheaf of interest, and by convenient choice of L_1 and L_2 we can ensure $F(L_1) \cong \mathcal{O}_X$ and $F(L_2) \cong \mathcal{F}$, then we will obtain a basis for $H^i(X, \mathcal{F})$.

When E and $E^\vee$ are mirror dual elliptic curves, this phenomenon is evident in the work of Polishchuk–Zaslow [36] and especially M. Gross [6, Ch. 8]. Both E and $E^\vee$ may be written as special Lagrangian S^1-fibrations over the same base $B \cong S^1$. The base has an integral affine structure as $\mathbb{R}/\mathbb{Z}$. The Lagrangians $L(d) \subset E^\vee$ are sections of this torus fibration with slope d, and their intersection points project precisely to the fractional integral points of the base B.

$$L(0) \cap L(d) \leftrightarrow B\left(\frac{1}{d}\mathbb{Z}\right) := \frac{1}{d}\text{-integral points of } B \tag{14}$$

Under HMS, we obtain $F(L(0)) = \mathcal{O}_E$ and $F(L(d)) = \mathcal{O}_E(d)$. the basis of intersection points $L(0) \cap L(d)$ corresponds to a basis of $\Gamma(E, \mathcal{O}_E(d))$ consisting of theta functions.

Another illustration is the case of toric varieties and their mirror Landau-Ginzburg models [1, 3]. In this case, Abouzaid constructs a family of Lagrangian submanifolds $L(d)$ mirror to the powers of the ample line bundle $\mathcal{O}_X(d)$. These Lagrangian submanifolds are topologically discs with boundary on a level set of the superpotential, $W^{-1}(c)$ for some c. For $d > 0$, the Floer complex $CF^*(L(0), L(d))$ is concentrated in degree zero. Hence

$$CF^0(L(0), L(d)) = HF^0(L(0), L(d)) = H^0(X, \mathcal{O}_X(d)). \tag{15}$$

The basis of intersection points $L(0) \cap L(d)$ corresponds to the basis of characters of the algebraic torus $T = (\mathbb{C}^*)^n$ which appear in the T-module $H^0(X, \mathcal{O}_X(d))$. The same formula (14) is valid in the case of toric varieties, where the base B is the moment polytope P of the toric variety X. Abouzaid interprets P as a subset of the base of the torus fibration on $X^\vee = (\mathbb{C}^*)^n$ (the fibration given by the Clifford tori), which moreover appears as a chamber bounded by a tropical variety corresponding to a level set $W^{-1}(c)$ of the superpotential.

As is explained in section 2, these features are also present in our case. The base B is identified with the base of the torus fibration on $\mathbb{CP}^2 \setminus D$, with its symplectic affine structure, and also identified as a subset of base of the torus fibration on $X^\vee$ with its complex affine structure, bounded by the tropicalization of a level set of the superpotential. We once again find a bijection between the intersection points of our Lagrangians $L(d)$ and the fractional integral points of the affine base.

Ongoing work of Gross–Hacking–Keel [17] seeks to extend these constructions to other manifolds, such as K3 surfaces, using a purely algebraic and tropical framework. In this paper we are concerned with extensions to cases that are tractable from the point of view of symplectic geometry, although the tropical analog of our results is described in section 5.

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2. The fiber of W and its tropicalization

2.1. **Torus fibrations on $\mathbb{CP}^2 \setminus D$ and its mirror.** Let

$$D = \{xyz - y^3 = 0\} \subset \mathbb{CP}^2 \tag{16}$$

be a binodal cubic curve. We equip $\mathbb{CP}^2$ and $\mathbb{CP}^2 \setminus D$ with standard Fubini-Study symplectic forms. Both $\mathbb{CP}^2 \setminus D$ and its mirror $X^\vee = \{(u, v) \in \mathbb{C}^2 \mid uv \neq 1\}$ admit special Lagrangian torus fibrations. In fact, these spaces are diffeomorphic, each being $\mathbb{C}^2$ minus a conic. The torus fibrations are essentially the same on both sides, but we are interested in the *symplectic* affine structure associated to the fibration on $\mathbb{CP}^2 \setminus D$ and the *complex* affine structure associated to $X^\vee$.

The construction is of the torus fibrations is taken from [7, §5]. We have that $\mathbb{CP}^2 \setminus D$ is an affine algebraic variety with coordinates x and z, where $xz \neq 1$. Hence we can define a map

$$f : \mathbb{CP}^2 \setminus D \to \mathbb{C}^* \quad f(x, z) = xz - 1 \tag{17}$$

This map is a Lefschetz fibration with critical point $(0, 0)$ and critical value -1. The fibers are affine conics, and the map is invariant under the S^1 action $e^{i\theta}(x, z) = (e^{i\theta}x, e^{-i\theta}z)$ that rotates the fibers. Each fiber contains a distinguished S^1–orbit, namely the vanishing cycle $\{|x| = |z|\}$. We can parametrize the other S^1–orbits by the function $\delta(x, z)$ which denotes the signed symplectic area between the vanishing cycle and the orbit through (x, z). The function δ is a moment map for the S^1-action. Symplectic parallel transport in every direction preserves the circle at level $\delta = \lambda$, and so by choosing any loop $\gamma \subset \mathbb{C}^*$, and $\lambda \in (-\Lambda, \Lambda)$ (where $\Lambda = \int_{\mathbb{CP}^1} \omega$ is the area of a line), we obtain a Lagrangian torus $T_{\gamma,\lambda} \subset \mathbb{CP}^2 \setminus D$. If we let $T_{R,\lambda}$ denote the torus at level λ over the circle of radius R centered at the origin in $\mathbb{C}^*$, we find that $T_{R,\lambda}$ is special Lagrangian with respect to the form

$$\Omega = dx \wedge dz/(xz - 1). \tag{18}$$

The torus fibration on $X^\vee$ is essentially the same, except that the coordinates (x, z) are changed to (u, v). For the rest of the paper, we denote by

$$w = uv - 1 \tag{19}$$

the quantity to which we project in order to obtain the Lagrangian tori $T_{R,\lambda}$ (and later the Lagrangian sections $L(d)$) as fibering over paths. For the time being, and in order to enable the explicit computations in section 2.3, we will equip $X^\vee$ with the standard symplectic form in the (u, v)–coordinates, so the quantity

$$\delta(u, v) = |u|^2 - |v|^2 \tag{20}$$

is the standard moment map. In summary, for $X^\vee$, we have

$$T_{R,\lambda} = \{(u, v) \mid |w| = |uv - 1| = R, |u|^2 - |v|^2 = \lambda\}. \tag{21}$$

Each torus fibration has a unique singular fiber $T_{1,0}$, which is a pinched torus.

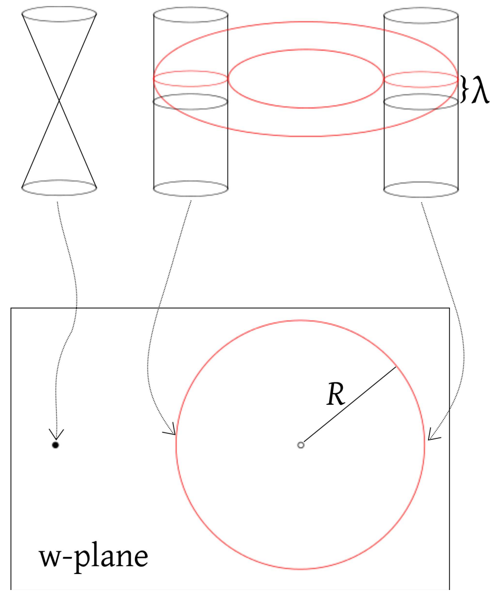

FIGURE 1. The Lefschetz fibration with a torus that maps to a circle.

Figure 1 shows several fibers of the Lefschetz fibration, with a Lagrangian torus that maps to a circle in the base. The two marked points in the base represent a Lefschetz critical value (filled-in circle), and a puncture (open circle).

For the symplectic affine structure, the affine coordinates are obtained by integrating the symplectic two-form ω over one-cycles in the torus fibers to obtain one-forms on the base, which may be integrated to functions. For the complex affine structure, we instead integrate the imaginary part of the holomorphic volume form $\operatorname{Im}\Omega$ over $(n-1)$ cycles in the fiber to obtain one-forms on the base, where in our case $n = 2$. See [16, §2] for an explanation of this construction.

We begin by describing the symplectic affine structure on the base B of the torus fibration on $\mathbb{CP}^2 \setminus D$. Recall that an integral affine structure has a canonically defined local system of integral tangent vectors. In the case when the affine manifold has singularities, this local system may have monodromy around the singular locus. The following result can be extracted from the analysis of [7, §5.2].

Proposition 2.1. *The integral affine manifold B is topologically a disk. It has one singular point, with monodromy conjugate to* $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. *In affine coordinates, the boundary of B consists of two line segments that are straight with respect to the affine structure. The corners are locally equivalent, by an affine linear transformation, to the standard quadrant $\mathbb{R}^2_{\geq 0} \subset \mathbb{R}^2$.*

Proof. The symplectic affine coordinates are the symplectic areas of disks in $\mathbb{CP}^2$ with boundary on $T_{R,\lambda}$. Let H denote the class of a line. We consider the cases $R > 1$ and $R < 1$.

On the $R > 1$ side, we take $\beta_1, \beta_2 \in H_2(\mathbb{CP}^2, T_{R,\lambda})$ to be the classes of two sections over the disk bounded by the circle of radius R in the base, where β_1 intersects the

z-axis and β_2 the x-axis. Then the torus fiber collapses onto line $\{y = 0\}$ when

$$\langle [\omega], H - \beta_1 - \beta_2 \rangle = 0. \tag{22}$$

On the $R < 1$ side, we take $\alpha, \beta \in H_2(\mathbb{CP}^2, T_{R,\lambda})$, where β is now the unique class of sections over the disk bounded by the circle of radius R, and α is the class of a disk connecting an S^1-orbit to the vanishing cycle within the conic fiber and capping off with the thimble. The torus fiber collapses onto the conic $\{xz - y^2 = 0\}$ when

$$\langle [\omega], \beta \rangle = 0. \tag{23}$$

The two sides $R > 1$ and $R < 1$ are glued together along the wall at $R = 1$, but the gluing is different for $\lambda > 0$ than for $\lambda < 0$, leading to the monodromy. Let us take $\eta = \langle [\omega], \alpha \rangle$ and $\xi = \langle [\omega], \beta \rangle$ as affine coordinates in the $R < 1$ region. We continue these across the $\lambda > 0$ part of the wall using correspondence between homology classes:

$$\begin{aligned} \alpha &\leftrightarrow \beta_1 - \beta_2 \\ \beta &\leftrightarrow \beta_2 \\ H - 2\beta - \alpha &\leftrightarrow H - \beta_1 - \beta_2 \end{aligned} \tag{24}$$

Thus, in the $\lambda > 0$ part of the base, the conic appears as $\xi = 0$, while the line appears as

$$0 = \langle [\omega], H - 2\beta - \alpha \rangle = \Lambda - 2\xi - \eta \tag{25}$$

which is a line of slope of $-1/2$ with respect to the coordinates (η, ξ). The pair of functions ξ and $\Lambda - 2\xi - \eta$ also form an affine coordinate system, and in this system the corner appears as a standard quadrant.

In the $\lambda < 0$ part of the base, we instead use

$$\begin{aligned} \alpha &\leftrightarrow \beta_1 - \beta_2 \\ \beta &\leftrightarrow \beta_1 \\ H - 2\beta + \alpha &\leftrightarrow H - \beta_1 - \beta_2 \end{aligned} \tag{26}$$

Hence in this region the conic appears as $\xi = 0$ again, while the line appears as

$$0 = \langle [\omega], H - 2\beta + \alpha \rangle = \Lambda - 2\xi + \eta \tag{27}$$

which is a line of slope $1/2$ with respect to the coordinates (η, ξ). The pair of functions ξ and $\Lambda - 2\xi + \eta$ also form an affine coordinate system, and in this system the corner appears as a standard quadrant.

The discrepancy between the two gluings represents the monodromy. As we pass from $\{R > 1, \lambda > 0\} \to \{R < 1, \lambda > 0\} \to \{R < 1, \lambda < 0\} \to \{R > 1, \lambda < 0\} \to \{R > 1, \lambda > 0\}$, the coordinates (η, ξ) under go the transformation $(\eta, \xi) \to (\eta, \xi - \eta)$, which is indeed a simple shear.

□

Figure 2 shows the affine manifold B. The marked point is a singularity of the affine structure, and the dotted line is a cut in the affine coordinates. The affine coordinates η, ξ, ψ are indicated. The function ξ is undefined on the cut below the singularity, while ψ is undefined at points directly above the singularity. Going around the singularity counterclockwise, the monodromy of the local system of integral tangent

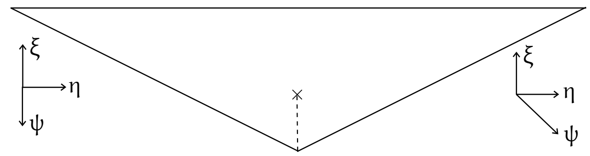

FIGURE 2. The affine manifold B.

vectors $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, which also serves as the gluing map along the cut. The edges of the picture are boundaries. The upper edge corresponds to points where the torus collapses onto the line $\{y = 0\}$. It has slope zero in this picture. The lower edge corresponds to points where the torus collapses onto the conic $\{xz - y^2 = 0\}$. On the left portion the lower edge has slope $-1/2$, while on the right portion it has slope $1/2$. The lower edge is actually straight with respect to the affine structure, and the nontrivial gluing is what compensates for the apparent bend.

On the mirror side, we compute the complex affine structure (determined by the holomorphic volume form) on the base of the torus fibration on $X^\vee$. The holomorphic volume form is

$$\Omega = \frac{du \wedge dv}{uv - 1} = \frac{du \wedge dv}{w}. \tag{28}$$

Differentiating the defining equation $uv = 1 + w$ and substituting gives the other formulas

$$\Omega = \frac{du}{u} \wedge \frac{dw}{w}, \quad \text{when } u \neq 0, \tag{29}$$

$$\Omega = -\frac{dv}{v} \wedge \frac{dw}{w}, \quad \text{when } v \neq 0. \tag{30}$$

The special Lagrangian fibration on $X^\vee$ to consider is constructed in [7]. The fibers are the tori

(31)

$$T_{R,\lambda} = \{(u, v) \in X^\vee \mid |uv - 1| = R, |u|^2 - |v|^2 = \lambda\}, \quad (R, \lambda) \in (0, \infty) \times (-\infty, \infty),$$

and the fiber $T_{1,0}$ is a pinched torus. Thus (R, λ) are coordinates on the base of this fibration. They are not affine coordinates, which must be computed using $\operatorname{Im} \Omega$. Due to the simple algebraic form of this fibration, it is possible to find an integral representation of the complex affine coordinates explicitly.

Proposition 2.2. *The following functions are affine linear with respect to affine structure induced by* $\operatorname{Im}\Omega$.

$$\begin{aligned}
\eta &= \log|w| = \log R \\
\xi &= \frac{1}{2\pi}\int_{T_{R,\lambda}\cap\{u\in\mathbb{R}_+\}} \log|u|\, d\arg(w) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\log\left(\frac{\lambda+\sqrt{\lambda^2+4\cdot|1+Re^{i\theta}|^2}}{2}\right) d\theta \\
\psi &= \frac{1}{2\pi}\int_{T_{R,\lambda}\cap\{v\in\mathbb{R}_+\}} \log|v|\, d\arg(w) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\log\left(\frac{-\lambda+\sqrt{\lambda^2+4\cdot|1+Re^{i\theta}|^2}}{2}\right) d\theta
\end{aligned} \tag{32}$$

(η *is defined everywhere,* ξ *is defined where* $u \neq 0$*, and* ψ *is defined where* $v \neq 0$*). They satisfy the following relations on their common domain of definition.*

1. *At every point,* η *is independent from* ξ *and from* ψ*.*
2. *In the subset where* $R < 1$*, the relation* $\xi + \psi = 0$ *holds.*
3. *In the subset where* $R > 1$*, the relation* $\xi + \psi = \eta$ *holds.*

Proof. This is essentially straightforward so we only give an example of the computation.

The general procedure for computing affine coordinates from the flux of the holomorphic volume form is as follows: we choose, over a local chart on the base, a collection of $(2n-1)$–manifolds $\{\Gamma_i\}_{i=1}^n$ in the total space X such that the torus fibers T_b intersect each Γ_i in an $(n-1)$–cycle, and such that these $(n-1)$–cycles $T_b \cap \Gamma_i$ form a basis of $H_{n-1}(T_b;\mathbb{Z})$. The affine coordinates $(y_i)_{i=1}^n$ are defined up to constant shift by the property that

$$y_i(b') - y_i(b) = \frac{1}{2\pi}\int_{\Gamma_i\cap\pi^{-1}(\gamma)} \operatorname{Im}\Omega \tag{33}$$

where γ is any path in the local chart on the base connecting b to b'. Because Ω is closed, this integral does not depend on the choice of γ.

To get the coordinate η, start with the submanifold

$$\Gamma_1 = \{w \in \mathbb{R}_+\} \tag{34}$$

The intersection $\Gamma_1 \cap T_{R,\lambda}$ is a loop on $T_{R,\lambda}$. The function $\arg(u)$ gives a coordinate on this loop (briefly, $w \in \mathbb{R}_+$ and $|w| = R$ determine uv, along with $|u|^2 - |v|^2 = \lambda$ this determines the $|u|$ and $|v|$; the only parameter left is $\arg(u)$ since $\arg(v) = -\arg(u)$), and we declare the loop to be oriented so that $-d\arg(u)$ restricts to a positive volume form on it. Using (29) we see

$$\operatorname{Im}\Omega = d\arg(u)\wedge d\log|w| + d\log|u|\wedge d\arg(w) \tag{35}$$

Using the fact that $\arg(w)$ is constant on Γ_1, we see that for any path γ in the subset of the base where $R < 1$ connecting $b = (R,\lambda)$ to $b' = (R',\lambda')$, we have

$$\int_{\Gamma_1\cap\pi^{-1}(\gamma)} \operatorname{Im}\Omega = \int_{\Gamma_1\cap\pi^{-1}(\gamma)} d\arg(u)\wedge d\log|w|. \tag{36}$$

But $d\arg(u) \wedge d\log|w| = d(-\log|w|\,d\arg(u)))$, so the integral above equals

$$\int_{\Gamma_1\cap T_{b'}} -\log|w|\,d\arg(u) - \int_{\Gamma_1\cap T_b} -\log|w|\,d\arg(u) = 2\pi(\log R' - \log R) \tag{37}$$

(the minus signs within the integrals are absorbed by the orientation convention for $\Gamma_1 \cap T_b$). Thus $\eta = \log R$ is the affine coordinate corresponding to Γ_1.

The same idea applied to $\Gamma_2 = \{u \in \mathbb{R}_+\}$ yields the affine coordinate

$$\xi = \frac{1}{2\pi}\int_{\Gamma_2\cap T_b} \log|u|\,d\arg(w) \tag{38}$$

To arrive at the second formula for ξ, we must solve for $|u|$ in terms of R, λ, and $\theta = \arg(w)$. The equations $uv = 1 + Re^{i\theta}$ and $|u|^2 - |v|^2 = \lambda$ imply $|u|^4 - \lambda|u|^2 = |1 + Re^{i\theta}|^2$. Solving for $|u|^2$ by the quadratic formula and taking logarithms gives the result.

The formulas for ψ are obtained by applying the same method to $\Gamma_2' = \{v \in \mathbb{R}_+\}$.

To prove the linear relations, we find that $\xi + \psi$ reduces by the law of logarithms and the Cauchy integral formula to

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1 + Re^{i\theta}|\,d\theta = \begin{cases} 0, & R < 1 \\ \log R, & R > 1 \end{cases} \tag{39}$$

□

Proposition 2.2 determines the monodromy around the singular point (at $\eta = \xi = \psi = 0$) of the base, and show that the affine structure is in fact integral. Once again, the monodromy is a shear.

2.2. **The topology of the map W.**

A direct computation shows that the superpotential W given by (9) has three critical points

$$\mathrm{Crit}(W) = \{(v = e^{\Lambda/3}e^{2\pi i(n/3)}, w = 1) \mid n = 0, 1, 2\}, \tag{40}$$

and corresponding critical values

$$\mathrm{Critv}(W) = \{3e^{-\Lambda/3}e^{-2\pi i(n/3)} \mid n = 0, 1, 2\}. \tag{41}$$

As expected, $\mathrm{Critv}(W)$ is the set of eigenvalues of quantum multiplication by $c_1(T\mathbb{CP}^2)$ in $QH^*(\mathbb{CP}^2)$, that is, multiplication by $3h$ in the ring $\mathbb{C}[h]/\langle h^3 = e^{-\Lambda}\rangle$.

Proposition 2.3. *Any regular fiber $W^{-1}(c) \subset X^\vee$ is a twice-punctured elliptic curve.*

Proof. In the (u, v) coordinates, $W^{-1}(c)$ is defined by the equation

$$u + \frac{e^{-\Lambda}v^2}{uv - 1} = c, \tag{42}$$

$$u(uv - 1) + e^{-\Lambda}v^2 = c(uv - 1). \tag{43}$$

This is an affine cubic plane curve, and it is disjoint from the affine conic $\{uv - 1 = 0\}$. It is smooth as long as c is a regular value.

The projective closure of $W^{-1}(c)$ in (u, v) coordinates is given by the homogeneous equation (with ξ as the third coordinate)

$$u(uv - \xi^2) + e^{-\Lambda}v^2\xi = c\xi(uv - \xi^2). \tag{44}$$

This is a projective cubic plane curve, hence elliptic, and it intersects the line at infinity $\{\xi = 0\}$ when $u^2 v = 0$. So it is tangent to the line at infinity at $(u : v : \xi) = (0 : 1 : 0)$ and intersects it transversely at $(u : v : \xi) = (1 : 0 : 0)$. Hence the affine curve is the projective curve minus these two points. □

Remark 1. The function W above is to be compared to the "standard" superpotential for $\mathbb{CP}^2$, namely,

$$W = x + y + \frac{e^{-\Lambda}}{xy} \tag{45}$$

corresponding to the choice of the toric boundary divisor, a union of three lines, as anticanonical divisor. This W has the same critical values, and its regular fibers are all thrice-punctured elliptic curves. Hence smoothing the anticanonical divisor to the union of a conic and a line corresponds to compactifying one of the punctures of $W^{-1}(c)$.

2.3. **Tropicalization in a singular affine structure.** Our goal is to find the same affine manifold B (that comes from symplectic structure of $\mathbb{CP}^2 \setminus D$) embedded in the base of the torus fibration on $X^\vee$, equipped with the *complex* affine structure. We shall see that it is obtained as a bounded chamber inside a particular tropical curve, the tropicalization of the fiber of W.

For some real number $\epsilon > 0$, consider the curve $W^{-1}(e^{\epsilon\Lambda})$:

$$W = u + \frac{e^{-\Lambda} v^2}{w} = e^{\epsilon\Lambda} \tag{46}$$

The tropicalization corresponds to the limit $\Lambda \to \infty$, or $t := e^{-\Lambda} \to 0$. The amoeba of $A(W^{-1}(t^{-\epsilon}))$ is the image of the curve in the affine base of the torus fibration on $X^\vee$. We want to produce a tropical curve in the base that reflects the asymptotic geometry of these amoebas $A(W^{-1}(t^{-\epsilon}))$ as $t \to 0$. In the standard situation, this is done by rescaling the amoebas by $\log t$ and taking the Hausdorff limit. Our situation is not standard because the affine manifold in which our tropical curve is to live has a singularity. Thus, none of the standard tropicalization techniques [26] apply directly. Since we do not know of any general theory of tropicalization when the affine structure is singular, we will here content ourselves with an ad hoc method involving explicit computation in two coordinate charts, and checking that the results fit together in the singular affine structure.

In the standard picture of tropicalization, one considers a family of subvarieties of an algebraic torus $V_t \subset (\mathbb{C}^*)^n$. The map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ given by $\mathrm{Log}(z_1, \dots, z_n) = (\log|z_1|, \dots, \log|z_n|)$ projects these varieties to their amoebas $\mathrm{Log}(V_t)$, and the rescaled limit of these amoebas is the tropicalization of the family V_t. The tropicalization is also given as the non-archimedean amoeba of the defining equation of V_t.

The map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ is special Lagrangian fibration. Its fibers are the tori $\{|z_1| = r_1, \dots, |z_n| = r_n\}$. These tori are Lagrangian with respect to the standard symplectic form, and they are special with respect to the holomorphic volume form

$$\Omega_{\text{toric}} = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}, \tag{47}$$

which has logarithmic poles along the coordinate hyperplanes in $\mathbb{C}^n$. The complex affine coordinates are $\log|z_i|$.

The torus fibration on $X^\vee$ is approximated by this standard structure as follows. Equation (29) shows that in the (w, u) coordinates (where $u \neq 0$), the holomorphic volume form is standard. If the special Lagrangian fibration were also standard, the affine coordinates would be $(\log|w|, \log|u|)$. Proposition 2.2 shows that, while $\eta = \log|w|$ is still an affine coordinate (reflecting the fact that there is still an S^1-symmetry), the other affine coordinate ξ is the average value of $\log|u|$ along a loop in the fiber. We also see that as $|\lambda|$ becomes large, the approximation $\xi \approx \log|u|$ holds with increasing accuracy. Similarly, in the (w, v) coordinates (when $v \neq 0$), the holomorphic volume form is standard, and $\eta = \log|w|$ and $\psi \approx \log|v|$ form affine coordinates.

We shall see that as $t = e^{-\Lambda} \to 0$, the amoebas $A(W^{-1}(t^{-\epsilon}))$ move farther away from the singularity, where the approximations $\xi \approx \log|u|$ and $\psi \approx \log|v|$ hold with increasing accuracy, while $\eta = \log|w|$ holds exactly everywhere.

The approximations $\xi \approx \log|u|$ and $\psi \approx \log|v|$ and the identity $\eta = \log|w|$ suggest the approach to finding the tropicalization. We represent the equation $W = u + e^{-\Lambda}v^2/w = e^{\epsilon\Lambda}$ as a polynomial equation in the coordinates $(w, u) \in (\mathbb{C}^*)^2$. We then compute the corresponding tropical curve, using the standard procedures [26], and we plot the result in the affine plane whose coordinates are $(\log|w|, \log|u|)$. This is shown in Figure 3(a). We repeat the process in the coordinate system (w, v), and plot the resulting tropical curve in the affine plane whose coordinates are $(\log|w|, \log|v|)$. This is shown in Figure 3(b). We then transfer these tropical curves into the singular affine manifold by simply identifying ξ with $\log|u|$ and ψ with $\log|v|$.

We now observe that the two curves actually match up, at least away from the vertical line passing through the singularity, and we claim that the result is as depicted in Figure 3(c). This is evident in comparing the left-hand portions of parts (a) and (b) of Figure 3. On the right-hand portion, we must take into account that the coordinate $\psi \approx \log|v|$ has been affected by a shear in passing from (b) to (c).

There is apparently a problem along the vertical line passing through the singularity, since the curves depicted in (a) and (b) have vertical legs there, which we claim do not appear in (c). Our reasoning is this: the coordinate system $(w, u) \in (\mathbb{C}^*)^2$ only covers the locus where $u \neq 0$, which is also where the function ξ is well defined. Thus we cannot expect the curve in (a) to be valid near the vertical going down from the singularity, it is valid above the singularity. Conversely, the figure in (b) is only valid in the region below the singularity. Thus the extra vertical legs are illusory.

We call the resulting tropical curve T_ϵ.

Proposition 2.4. *For $\epsilon > 0$, T_ϵ is a trivalent graph with two vertices, a cycle of two finite edges, and two infinite edges.*

Proposition 2.5. *For $\epsilon > 0$, the complement of T_ϵ has a bounded component that is an integral affine manifold with singularities that is isomorphic, after rescaling, to the base B of the special Lagrangian fibration on $\mathbb{CP}^2 \setminus D$ with the affine structure coming from the symplectic form.*

Remark 2. The topology of the tropical curve T_ϵ corresponds to that of a twice-punctured elliptic curve.

Remark 3. We can apply the same patching procedure for ϵ in the range $(-1/3) < \epsilon < 0$. This changes Figure 3 so that the triangles in (a) and (b) become smaller and

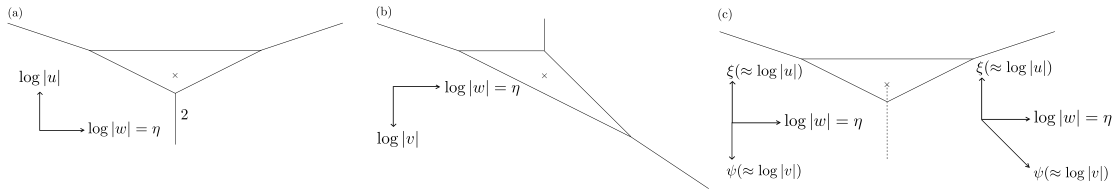

FIGURE 3. The tropical fiber of W for $\epsilon > 0$. (a) (w, u) coordinates. (b) (w, v) coordinates. (c) The tropical fiber T_ϵ.

they lie entirely below point marked with $\times$. Thus, assuming that (a) is valid above the singularity, while (b) is valid below the singularity, we find that T_ϵ is essentially the same as what appears in (b), but that the vertical leg must terminate at the singularity, in order to be consistent with the fact that in (a) no part of the curve appears above the singularity. Thus we find that T_ϵ is a trivalent graph with three vertices, a cycle of three finite edges, two infinite edges and one edge connecting a vertex to the singular point of the affine structure.

We note that the value $\epsilon = -1/3$ corresponds to the critical values of W.

Remark 4. This proposition is another case of the phenomenon, described in Abouzaid's paper [1], that for toric varieties, the bounded chamber of the fiber of the superpotential is isomorphic to the moment polytope. In the general case of a manifold X with effective anticanonical divisor D, the boundary of the symplectic affine base of the torus fibration on $X \setminus D$ corresponds to a torus fiber collapsing onto D, a particular class of holomorphic disks having vanishing area, and the corresponding term of the superpotential having unit norm. On the other hand, the tropicalization of the fiber of the superpotential has some parts corresponding to one of the terms having unit norm, and it is expected that these bound a chamber which is isomorphic to the base of the original torus fibration.

3. Symplectic constructions

Let B the affine manifold which is the bounded chamber of the tropicalization of the fiber of W, constructed in the previous section. In this section we construct a symplectic manifold $X(B)$, which is a torus fibration over B, together with a Lefschetz fibration $w : X(B) \to X(I)$, where $X(I)$ is an annulus. We also construct a collection $\{L(d)\}_{d\in\mathbb{Z}}$ of Lagrangian submanifolds. These submanifolds are sections of the torus fibration $X(B) \to B$, and they fiber over paths with respect to the Lefschetz fibration $w : X(B) \to X(I)$. The Lagrangians have an admissibility property governing their behavior at the boundary of $X(B)$. Corresponding to the two sides of B, and hence to the two terms of $W = u + e^{-\Lambda}v^2/(uv-1)$, we have horizontal boundary faces $\partial^h X(B)$, along each of which the symplectic connection for the Lefschetz fibration defines a foliation. Choosing a leaf of the foliation on each face defines a boundary condition (corresponding to the fiber of W) for our Lagrangian submanifolds.

The motivation for these constructions is existence of the map $w = uv - 1 : X^\vee \to \mathbb{C}^*$, which is a Lefschetz fibration with general fiber an affine conic and a single critical value. The tori in the SYZ fibration considered in section 2 fiber over loops in this projection, so it is natural to attempt to use it to understand as much of the geometry as possible. In particular it will allow us to apply the techniques of [41], [38], [37].

3.1. **Symplectic monodromy associated to a Hessian metric.** Let B be a two-dimensional affine manifold with affine coordinates (η, ξ) that embed B as a subset of $\mathbb{R}^2$. In this subsection B does not have any singularities. Suppose that $\eta : B \to \mathbb{R}$ is a submersion over some interval $I \subset \mathbb{R}$, and that the fibers of this map are connected intervals. For our purposes, we consider the case where B is a quadrilateral, bounded on two opposite sides by line segments of constant η (the *vertical* boundary $\partial^v B$), and on the other two sides by line segments that are transverse to the projection to η (the *horizontal* boundary $\partial^h B$).

Associated to B and I, we define complex manifolds $X(B)$ and $X(I)$. The space $X(B)$ is the subset of the complex torus $(\mathbb{C}^*)^2$ with coordinates w and z such that $(\eta, \xi) = (\log|w|, \log|z|)$ lies in B, and $X(I)$ is the subset of $(\mathbb{C}^*)$ with coordinate w such that $\eta = \log|w|$ lies in I. Thus $X(I)$ is an annulus, and we have a map $w : X(B) \to X(I)$, which is a non-singular fibration with fibers isomorphic to annuli. Philosophically speaking, the map $\eta : B \to I$ is a tropical model of the map $w : X(B) \to X(I)$.

In this situation, the most natural way to prescribe a Kähler structure on $X(B)$ is through a Hessian metric on the base B. This is a metric g such that locally $g = \operatorname{Hess} K$ for some function $K : B \to \mathbb{R}$, where the Hessian is computed with respect to an affine coordinate system. If $\pi : X(B) \to B$ denotes the projection, then $\phi = K \circ \pi$ is a Kähler potential on $X(B)$, and the positivity of $g = \operatorname{Hess} K$ corresponds to the positivity of the real closed $(1,1)$-form $\omega = -dd^c\phi$. Explicitly, if $y_1, \dots, y_n$ are affine coordinates corresponding to complex coordinates $z_1, \dots, z_n$ on $(\mathbb{C}^*)^n$, then

$$g = \sum_{i,j=1}^{n} \frac{\partial^2 K}{\partial y_i \partial y_j} dy_i dy_j \tag{48}$$

$$\omega = -dd^c\phi = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} \frac{\partial^2 K}{\partial y_i \partial y_j} \frac{dz_i}{z_i} \wedge \frac{d\bar{z}_j}{\bar{z}_j} \tag{49}$$

Any such Kähler structure is invariant under the S^1–action $e^{i\theta}(z, w) = (e^{i\theta}z, w)$ that rotates the fibers of the map $w : X(B) \to X(I)$.

We now have a Kähler structure on $X(B)$ such that the fibration $w : X(B) \to X(I)$ has symplectic fibers. Thus there is a symplectic connection on this fibration whose horizontal subspaces are the symplectic orthogonal spaces to the fibers. This connection defines a notion of symplectic parallel transport along paths in the base $X(I)$, and symplectic monodromy around loops in the base. Since our fibers have boundary we must show that the symplectic parallel transport preserves the boundary. Given that, symplectic parallel transport defines symplectomorphisms between the fibers.

We now compute the symplectic connection. Let $X \in T_{(z,w)}X(B)$ denote a tangent vector. Let $Y \in \ker dw$ denote the general vertical vector. The relation defining the horizontal distribution is $\omega(X,Y)=0$, or,

(50)

$$\begin{aligned}0 &= \left\{K_{\eta\eta}\frac{dw}{w}\wedge\frac{d\bar w}{\bar w}+K_{\eta\xi}\left(\frac{dw}{w}\wedge\frac{d\bar z}{\bar z}+\frac{dz}{z}\wedge\frac{d\bar w}{\bar w}\right)+K_{\xi\xi}\frac{dz}{z}\wedge\frac{d\bar z}{\bar z}\right\}(X,Y)\\ &= K_{\eta\xi}\left(\frac{dw(X)}{w}\frac{d\bar z(Y)}{\bar z}-\frac{d\bar w(X)}{\bar w}\frac{dz(Y)}{z}\right)+K_{\xi\xi}\left(\frac{dz(X)}{z}\frac{d\bar z(Y)}{\bar z}-\frac{dz(Y)}{z}\frac{d\bar z(X)}{\bar z}\right)\\ &= \left(K_{\eta\xi}\frac{dw(X)}{w}+K_{\xi\xi}\frac{dz(X)}{z}\right)\frac{d\bar z(Y)}{\bar z}-\text{ complex conjugate}\end{aligned}$$

Since $dz(Y)$ can have any phase, this shows that the quantity in parentheses on the last line must vanish:

$$d\log z(X)=-\frac{K_{\eta\xi}}{K_{\xi\xi}}d\log w(X) \tag{51}$$

Tropically, this formula has the following interpretation: In the (η,ξ) coordinates, the vertical tangent space is spanned by the vector $(0,1)$. The g-orthogonal to this space is spanned by the vector $(K_{\xi\xi},-K_{\eta\xi})$, whose slope with respect to the affine coordinates is the factor $-K_{\eta\xi}/K_{\xi\xi}$ appearing in the formula for the connection.

Lemma 3.1. *Let K be such that $\partial^h B$ is g-orthogonal to the fibers of the map $\eta : B\to I$, and assume that the slope σ_F of each boundary face F of B is rational. Then*

(1) *parallel transport around the loop $\{|w|=R\}$ acts on the fiber over $w=R$ by rotating each circle of constant $|z|$ through a phase $2\pi(-K_{\eta\xi}/K_{\xi\xi})$, and*
(2) *for each boundary face F, we have $2\pi(-K_{\eta\xi}/K_{\xi\xi})=2\pi\sigma_F$, the part of $\partial^h X(B)$ lying over F is foliated by multi-sections of the fibration that are horizontal with respect to the symplectic connection.*

Proof. Consider the parallel transport of the connection around the loop $\{|w|=R\}$, which is a generator of $\pi_1(X(I))$. As w traverses the path $R\exp(it)$, the initial condition $(z,w)=(r\exp(i\theta),R)$ generates the solution $(r\exp(i\theta+(-K_{\eta\xi}/K_{\xi\xi})it),R\exp(it))$, where the expression $-K_{\eta\xi}/K_{\xi\xi}$ is constant along the solution curve. As a self-map of the fiber over $w=R$, this monodromy transformation maps circles of constant $|z|$ to themselves, but rotates each by the phase $2\pi(-K_{\eta\xi}/K_{\xi\xi})$.

We now consider the behavior of the symplectic connection near the horizontal boundary $\partial^h X(B)$. Let F be a component of $\partial^h B$. Since F is a straight line segment g-orthogonal to the fibers of η, the function $-K_{\eta\xi}/K_{\xi\xi}$ is constant on F and equal to its slope, which we denote $\sigma=\sigma_F$. This slope is rational by assumption. The part of $X(B)$ lying over F is defined by the condition $\log|z|=\sigma\log|w|+C$. Let $w=w_0\exp(\rho(t)+i\phi(t))$ describe an arbitrary curve in the base annulus $X(I)$. If (z_0,w_0) is an initial point that lies over F, then

$$z=z_0\exp\{\sigma(\rho(t)+i\phi(t))\},w=w_0\exp(\rho(t)+i\phi(t)) \tag{52}$$

is a path in $X(B)$ that lies entirely over F, and which by virtue of this fact also solves the symplectic parallel transport equation. Thus the part of $\partial^h X(B)$ that lies over F is foliated by horizontal sections of the fibration, namely $(w/w_0)^\sigma=(z/z_0)$ where $\pi(z_0,w_0)\in F$. Take note that σ is merely rational, so these horizontal sections may actually be multi-sections. □

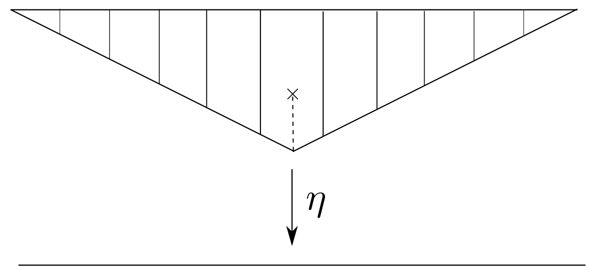

FIGURE 4. The fibration $B \to I$.

Examples of the Hessian metrics such that $\partial^h B$ is g-orthogonal to the fibers of η may be constructed by starting with the function

$$F(x,y) = x^2 + \frac{y^2}{x} \tag{53}$$

$$\operatorname{Hess} F = \begin{pmatrix} 2 + 2\frac{y^2}{x^3} & -2\frac{y}{x^2} \\ -2\frac{y}{x^2} & 2\frac{1}{x} \end{pmatrix} \tag{54}$$

$$\frac{-F_{xy}}{F_{yy}} = \frac{y}{x} \tag{55}$$

Thus the families of lines $x = c$ and $y = \sigma x$ form an orthogonal net for $\operatorname{Hess} F$. By taking x and y to be shifts of the affine coordinates η and ξ on B, we can obtain a Hessian metric on B such that the vertical boundary consists of lines of the form $x = c$, while the horizontal boundary consists of lines of the form $y = \sigma x$.

3.2. **Focus-focus singularities and Lefschetz singularities.** Now we consider the case where the affine structure on B contains a focus-focus singularity, that is, a singularity with monodromy conjugate to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and the monodromy invariant direction of this singularity is parallel to the fibers of the map $\eta : B \to I$. The goal is to construct a symplectic manifold $X(B)$, along with a Lefschetz fibration $w : X(B) \to X(I)$. Topologically $X(B)$ is a torus fibration over B with a single pinched torus fiber over the singular point.

Let B be an affine manifold with a single focus-focus singularity, and $\eta : B \to I$ a globally defined affine coordinate. Suppose B has vertical boundary consisting of fibers of η, as before, and suppose that the horizontal boundary consists of line segments of rational slope. If we draw the singular affine structure with a branch cut, one side will appear straight while the other appears bent, though the bending is compensated by the monodromy of the focus-focus singularity. Figure 4 shows the projection $B \to I$, with the fibers drawn as vertical lines.

Proposition 3.2. *The manifold $X(B)$ admits an exact symplectic structure making $w : X(B) \to X(I)$ a symplectic Lefschetz fibration, with a single Lefschetz critical point. This critical point coincides with focus-focus singularity of the torus fibration $\pi : X(B) \to B$. There is a neighborhood $U \subset B$ of the fiber of η through the singularity such that on the complement of $\pi^{-1}(U)$, the symplectic form is locally given by a Hessian metric of the form considered in Lemma 3.1.*

Proof. Suppose for convenience that the singularity occurs at $\eta = 0$. For the first step, divide the base B into regions $B_{-\epsilon} = \eta^{-1}(-\infty, -\epsilon]$ and $B_{+\epsilon} = \eta^{-1}[\epsilon, \infty)$. On these affine manifolds we may take the Hessian metrics and associated Kähler forms considered in Lemma 3.1, using functions K_- and K_+ derived from the example following that Lemma. Hence we get a fibration with symplectic connection over the disjoint union of two annuli: $w : X(B_{-\epsilon} \coprod B_{+\epsilon}) \to X(I_{-\epsilon} \coprod I_{+\epsilon})$

The second step is extend the Lefschetz fibration over a band connecting the two annuli. Consider the two annuli $X(I_{-\epsilon}) \coprod X(I_{+\epsilon}) \subset \mathbb{C}^*$. Choose a path connecting these two annuli, along the positive real axis, say. By identifying the fibers over the end points, the fibration extends over this path. By thickening the path up to a band and filling in the fibers over the band, we get a Lefschetz fibration over a surface which is topologically a pair of pants. Since the fibration over the band is trivial, the gluing can be done in such a way that the horizontal boundary components of the total space are still foliated by multi-sections of the symplectic connection.

The third step is to fill in the rest of the region between the two annuli with the standard model of a Lefschetz singularity. In order for this to make sense, we need the symplectic monodromy around the loop in the base being filled in to be a Dehn twist. This can be seen by comparing the symplectic monodromy transformations around the loops in $X(I_{-\epsilon})$ and $X(I_{+\epsilon})$. Let z_- and z_+ denote complex coordinates on $X(B_{-\epsilon})$ and $X(B_{+\epsilon})$ corresponding to a direction transverse to the fibration $\eta : B \to I$, so that z_- and z_+ give coordinates on the fibers of $w : X(B) \to X(I)$. Assume these coordinates match up in one of the bands connecting $B_{-\epsilon}$ to $B_{+\epsilon}$. Due to the monodromy of the affine manifold they do not coincide in the other band, where we put the branch cut. Let F_1 and F_0 denote the top and bottom faces of $\partial^h B$ respectively, and suppose that F_0 is split into two parts F_{0+} and F_{0-} by the branch cut. Associated to each of these we have a slope σ_F.

Now we compute the monodromy obtained when we traverse a loop in $X(I_{-\epsilon})$ in the negative sense followed by a loop in $X(I_{+\epsilon})$ in the positive sense, connecting these paths through the band connecting the two annuli. We use Lemma 3.1 to measure the difference between the amounts of phase rotation in the z_- and z_+ coordinates along the top and bottom horizontal boundaries under symplectic parallel transport, encoding this as an overall twisting. By Lemma 3.1, as we transport around the negative loop in $X(I_{-\epsilon})$, the z_- coordinate on $\pi^{-1}(F_1)$ rotates through an angle $-2\pi\sigma_{F_1}$ and the z_- coordinate on $\pi^{-1}(F_{0-})$ rotates through an angle $-2\pi\sigma_{F_{0-}}$. Thus the relative twist of the two ends of the fiber is $-(\sigma_{F_1} - \sigma_{F_{0-}})$ turns. By the same token, on the other side the z_+ coordinate on $\pi^{-1}(F_1)$ rotates through an angle $2\pi\sigma_{F_1}$, while the z_+ coordinate on $\pi^{-1}(F_{0+})$ rotates through $2\pi\sigma_{F_{0+}}$, and the overall relative twisting is $(\sigma_{F_1} - \sigma_{F_{0+}})$ turns. Composing these monodromy transformations, we find that the fiber undergoes a twisting of $\sigma_{F_{0-}} - \sigma_{F_{0+}}$ turns. Because the monodromy of the affine structure is a shear, this difference of slopes equals -1. Thus the fiber undergoes a right-handed Dehn twist (whose local model is obtained by taking the annulus $\{1 \leq |z| \leq 2\}$ and rotating the outer boundary by one *clockwise* turn).

This allows us to fill in the fibration with a standard fibration with a single Lefschetz singularity whose vanishing cycle is the equatorial circle on the cylinder fiber. Because the top and bottom boundaries are fixed under the monodromy transformation, the foliation of the horizontal boundary extends over the gluing. Since this local model is symmetric under the S^1–action which rotates the fibers, choosing an S^1–invariant gluing allows us to define a symplectic S^1–action on $X(B)$ which rotates the fibers of $w : X(B) \to X(I)$.

Since the total space is S^1–symmetric, we can construct the Lagrangian tori as in section 2.1, by taking circles of constant $|w|$ in the base and S^1–orbits in the fiber. These actually coincide with the tori found in $X(B_{-\epsilon} \coprod B_{+\epsilon})$ as fibers of the projection to B, so this construction extends the torus fibrations on $X(B_{-\epsilon} \coprod B_{+\epsilon})$ to all of $X(B)$. □

Remark 5. Since this construction is local on the base $X(I)$, the construction extends in an obvious way to the situation where several focus-focus singularities with parallel monodromy-invariant directions are present. See Section 6 for further discussion.

Remark 6. There is a more direct way to obtain the sort of symplectic structure we desire, based on a version of the mapping-torus construction. Start with a Lefschetz fibration over an rectangle, with a critical value corresponding to the singular point of B. Trivialize the fibration over the top and bottom sides of the base rectangle, and glue the top and bottom together, identifying the fibers using a symplectomorphism, yielding a fibration over an annulus that we identify with $X(I)$. Choosing the gluing symplectomorphism appropriately, we can ensure that the monodromies around the sub-annuli $X(I_{\pm\epsilon}) \subset X(I)$ agree with what is obtained from the Hessian metric construction. Our preference for the construction described above is that it proceeds more naturally from the affine geometry of B.

Remark 7. In the main example of this paper, the affine manifold does not have vertical boundary, but rather corners at the extreme values of η. To define the symplectic structure, we deal with this by cutting off arbitrarily small pieces of B at the corners so that the result has vertical boundaries. However, in terms of the correspondence to tropical geometry, we still consider the unmodified manifold B.

Definition 1. Let B be the affine manifold appearing in the mirror of $(\mathbb{CP}^2, D)$. We denote by $X(B)$ be the symplectic manifold obtained from Proposition 3.2, after cutting off the corners of B as in Remark 7.

Recall that the horizontal boundary $\partial^h X(B)$ is the union of two faces $(\partial^h X(B))_1$ and $(\partial^h X(B))_0$ corresponding to F_1 and F_0, the top and bottom faces of B. Likewise we speak of the top and bottom boundary circles of the fibers of w. The following proposition summarizes key properties of $X(B)$. It follows from the analysis of the monodromies in the proof of Proposition 3.2 and Lemma 3.1.

Proposition 3.3. *The Lefschetz fibration $w : X(B) \to X(I)$ has the following properties:*

1. *The monodromy given by traversing the inner boundary loop* clockwise *fixes the top boundary circle of the fiber, and rotates the bottom boundary circle through a half turn, in such a way that the square of this operation is isotopic to a* right-handed *Dehn twist.*
2. *The monodromy given by traversing the outer boundary loop* clockwise *fixes the top boundary circle and acts on the bottom boundary circle by a half-turn, except that the square of this operation is isotopic to a* left-handed *Dehn twist.*

(3) *As the top face F_1 has integral slope (zero in our diagrams), the leaves of the foliation on $(\partial^h X(B))_1$ are single-valued sections of the w-fibration.*
(4) *As the bottom face F_2 has half-integral slope ($\pm 1/2$ on either side of the cut in our diagrams), the leaves of the foliation on $(\partial^h X(B))_0$ are two-valued sections of the w-fibration.*

Remark 8. The foliation of the boundary faces by horizontal sections is related to the superpotential $W = u + e^{-\Lambda}v^2/(uv-1)$ studied in the previous Section 2. The general fiber $W =$ constant is a twice-punctured torus. On the other hand, we can consider the curves defined by the individual terms of W. The equation $u =$ constant defines a curve such that the projection $w = uv - 1 : X^\vee \to \mathbb{C}$ is one-to-one, so this curve is a section of w. The curves defined by the second term $v^2/(uv-1) =$ constant are likewise two-valued sections of $w : X^\vee \to \mathbb{C}$. Our perspective is that the leaves of the foliations on the boundary components of $X(B)$ correspond roughly to these curves defined by the individual terms of the superpotential.

3.3. **Lagrangians fibered over paths.** The base of the Lefschetz fibration is the annulus $X(I) = \{R^{-1} \le |w| \le R\}$ with a critical value at $w = -1$. The symplectic structures constructed in 3.2 have the property that the symplectic connection is flat throughout the annuli $X(I_{-\epsilon}) = \{R^{-1} \le |w| \le e^{-\epsilon}\}$, $X(I_{+\epsilon}) = \{e^{\epsilon} \le |w| \le R\}$, as well as through a band along the positive real axis joining these annuli.

We call the Lagrangians $L(d)$ constructed here *sections*, because they will turn out to be sections of the torus fibration on $X(B)$. At the same time, they are *not* sections of the Lefschetz fibration, but rather fiber over paths in $X(I)$.

3.3.1. *The zero-section.* The first step is to construct the Lagrangian submanifold $L(0) \subset X(B)$, which we will use as a zero-section of the torus fibration and reference point through out the paper.

Definition 2. Let $\ell(0) \subset X(I)$ be the path that runs along the positive real axis. Trivializing the fibration along that path, take a path in the fiber cylinder connecting the top and bottom boundary circles. The Lagrangian tori intersect the fibers of the Lefschetz fibration in circles, and we choose our fiber path so that it intersects each such circle once. We denote by $L(0) \subset X(B)$ the product Lagrangian submanifold.

If we want to be specific, we could take the factor in the fiber to be the positive real locus of the coordinates z_- or z_+. By construction, $L(0)$ intersects each Lagrangian torus once.

Once $L(0)$ is chosen, it selects leaves of the foliations by horizontal sections on each boundary face, namely those leaves containing the fiberwise boundary of $L(0)$. Call these leaves Σ_0 and Σ_1 (bottom and top respectively). Clearly we could have chosen these leaves first and then constructed $L(0)$ accordingly.

3.3.2. *The degree d section.* We can now use $L(0)$ as a reference to construct the other Lagrangians $L(d)$.

Definition 3. Let $\ell(d)$ be a base path, with the same end points and midpoint as $\ell(0)$, and which winds d times (relative to $\ell(0)$) in $X(I_{-\epsilon})$ and also d times in $X(I_{+\epsilon})$. The winding of $\ell(d)$ is clockwise as we go from smaller to larger radius. The Lagrangian $L(d)$ is defined to coincide with $L(0)$ in the fiber over the common endpoint of $\ell(0)$

and $\ell(d)$ on the inner boundary circle of $X(I)$, and to be defined elsewhere as the submanifold swept out by symplectic parallel transport along the path $\ell(d)$.

Recall that we cut off the corners of B so that we actually have fibers at the endpoints of $\ell(0)$ and $\ell(d)$. Because the boundary leaves Σ_0 and Σ_1 are preserved by symplectic parallel transport, the boundary of $L(d)$ is contained in these same leaves, which as in Remark 8 correspond to hypersurfaces defined by the individual terms of the superpotential.

The behavior of $L(d)$ in the fiber direction is determined with the aid of Proposition 3.3.

Proposition 3.4. *As we parallel transport clockwise along $\ell(d)$ in inner annulus $X(I_{-\epsilon})$, the fiber component of $L(d)$ is transformed by half-right-handed Dehn twists, while as we parallel transport clockwise along $\ell(d)$ in the outer annulus $X(I_{+\epsilon})$, we pick up half-left-handed Dehn twists. In the end, all of the twisting cancels out, so that $L(d)$ and $L(0)$ again coincide over the common endpoint of $\ell(0)$ and $\ell(d)$ on the outer boundary circle of $X(I)$.*

The proposition is illustrated in Figure 5, which depicts $L(0)$, $L(1)$, and $L(2)$. The lower portion of the figure shows the base: the straight line is $\ell(0)$, while the spirals are $\ell(1)$ and $\ell(2)$. The marked point is the Lefschetz critical value. The upper portion of the figure shows the five fibers where $\ell(0)$ and $\ell(2)$ intersect. The behavior of $L(i)$ for $i = 0, 1, 2$ in these fibers is depicted.

The Lagrangian submanifold $L(d)$ is indeed a section of the torus fibration. If $T_{R,\lambda}$ is the torus over the circle $\{|w| = R\}$ at height λ, then since $\ell(d)$ intersects $\{|w| = R\}$ at one point, there is exactly one fiber of $w : X(B) \to X(I)$, where $L(d)$ and $T_{R,\lambda}$ intersect. Since $L(d)$ intersects each S^1–orbit in that fiber once, we find that $L(d)$ and $T_{R,\lambda}$ indeed intersect once.

3.3.3. *Admissibility.* We now explain in what sense these Lagrangians are admissible. The relevant notion of admissibility is the one found in [7, §7.2], where admissibility with respect to a reducible hypersurface whose components correspond to the terms of the superpotential is discussed. In our case, we have two components Σ_0 and Σ_1, and the admissibility condition is that, near Σ_i, the holomorphic function z_i such that $\Sigma_i = \{z_i = 1\}$ satisfies $z_i|L \in \mathbb{R}$. The Lagrangian $L(d)$ will have this property if the symplectic structure is chosen so that the symplectic monodromy near Σ_1 is actually trivial, while the symplectic monodromy near Σ_0 is a rigid rotation by π. Otherwise, we can only say that the phase of z_i varies within a small range near Σ_i. In the computations in section 4 we actually perturb these Lagrangians in a prescribed manner near the boundary.

The notion of admissibility also prescribes the behavior over the endpoints of the base path $\ell(d)$. Recall that we cut off the corners of B in order to define the symplectic structure. Our notion of admissibility remembers that there was a corner at this point, by requiring the $L(d)$ to coincide with $L(0)$ over the endpoints of $\ell(d)$. The justification for this is that, if we try to extend the geometry all the way to the corner of B, we see that the two components Σ_0 and Σ_1 would intersect, and we need the Lagrangian to behave tamely at this intersection. A local model for this sort of situation is the two complex curves $\Sigma_0 = \{x = 0\}$, $\Sigma_1 = \{y = 0\}$ intersecting at

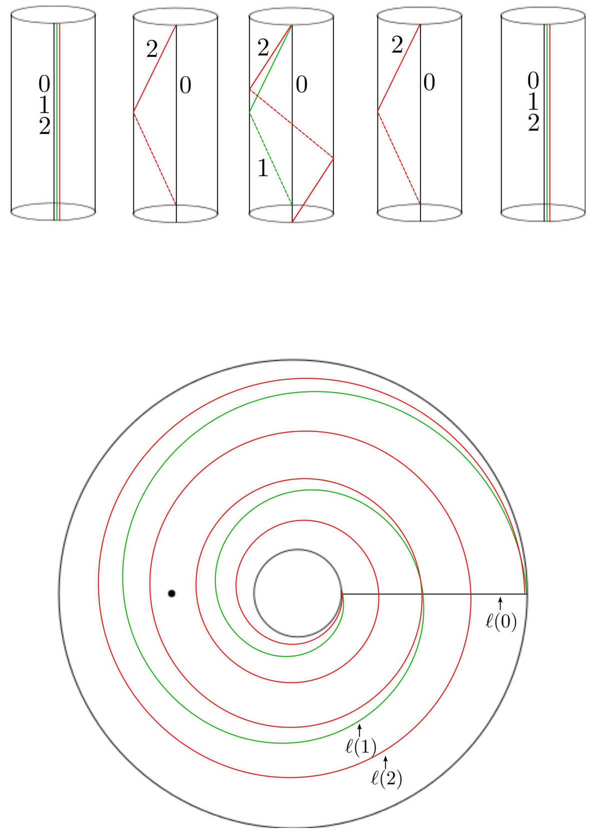

FIGURE 5. The Lagrangians $L(0)$, $L(1)$, and $L(2)$. Parts of the figure corresponding to $L(i)$ are labeled by i.

$(0,0)$ in $\mathbb{C}^2$, with the Lagrangian being $L = (\mathbb{R}_{\geq 0})^2$, and with the fibration being $w = x + y : \mathbb{C}^2 \to \mathbb{C}$.

3.3.4. *Positive perturbations.* The Lagrangians $L(d)$ constructed above intersect each other on the boundary of $X(B)$, and in particular it is not clear whether such intersection points are supposed count toward the Floer cohomology. There is a canonical way to perturb the Lagrangians so as to push all intersection points which should count toward Floer cohomology into the interior of $X(B)$, also used by Abouzaid [1]. This construction is not symmetric with respect to switching the Lagrangians.

Let K and L be two of the Lagrangians $L(d)$. The perturbation process we use in computing morphisms $CF^*(K, L)$ from K to L has two steps. First we perturb the Lagrangians near the boundary, staying within the class of Lagrangians fibering over paths, so that they intersect at the boundary in such way that the tangent space to the base path of L is a small *counterclockwise* rotation of the tangent space to base path of K, and similarly in the fiber. If $K = L(d_1)$ and $L = L(d_2)$ with $d_1 < d_2$, this will create new intersection points in the interior, while if $d_1 > d_2$ the pair K, L is already in the desired position.

The second step is that we perturb K and L at the boundary intersection so as to destroy the intersection there. Alternatively, we could just forget about the boundary intersection points in our computations, but actually removing them is more convenient in the technical part of the paper. The pair of Lagrangians obtained from

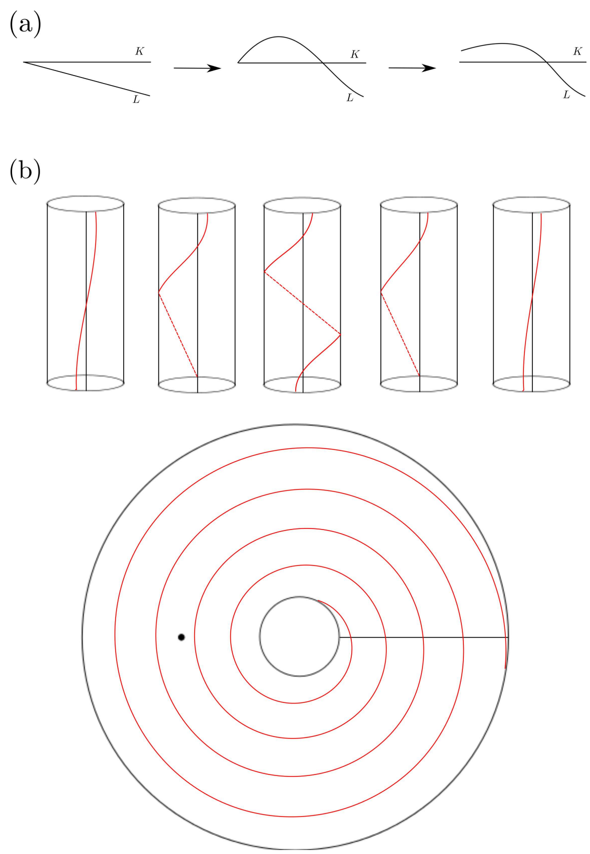

FIGURE 6. (a) The local picture of positive perturbation. (b) The end result when $K = L(0)$ and $L = L(2)$.

this process is called *positively perturbed*. This has the effect that one of Lagrangians in the pair may not admissible in the sense above, since it does not have boundary on Σ_0 and Σ_1. In section 4, the Lagrangians we work with are positive perturbations of admissible Lagrangians.

Please see figure 6 for a local picture of the positive perturbation, and the end result in the case of $K = L(0)$ and $L = L(2)$.

Remark 9. The asymmetry in the perturbations is essential for the mirror symmetry statement that we wish to prove. Mirror symmetry predicts that $HF^*(L(d_1), L(d_2))$ is isomorphic to $\mathrm{Ext}^*(\mathcal{O}_{\mathbb{CP}^2}(d_1), \mathcal{O}_{\mathbb{CP}^2}(d_2)) \cong H^*(\mathbb{CP}^2, \mathcal{O}_{\mathbb{CP}^2}(d_2 - d_1))$. (The main purpose of the paper is to verify that this is true for $d_1 < d_2$ in a way compatible with the product structures.) Even at the level of total dimension, this space is not symmetric with respect to swapping d_1 and d_2, so it makes sense that we must treat intersection points differently depending on whether we consider them to be morphisms from $L(d_1)$ to $L(d_2)$ or vice versa.

While we only discuss the product μ^2 on Floer cohomology in this paper, it would be possible to use this sort of perturbation in the definition of higher A_∞ operations, and to define a Fukaya category, by adapting the analogous constructions in Abouzaid's work [1, 3].

3.3.5. *Gradings and degrees.* We now explain how the degrees of the intersection points between the positively perturbed Lagrangians $L(d)$ is determined. Our conventions follow Seidel [41, §11]. We choose a smooth trivialization of the canonical bundle of $X(B)$ given by a complex volume form Ω. Our $X(B)$ is diffeomorphic to the manifold $X^\vee$ considered in section 2, which carries the volume form

$$\Omega_{X^\vee} = \frac{du \wedge dv}{w} = \frac{du}{u} \wedge \frac{dw}{w} \tag{56}$$

having the convenient property that, away from the singular fiber of the Lefschetz fibration, $\Omega_{X^\vee}$ decomposes as a wedge product of one-forms with respect to the local product structure determined by the coordinates (u, w). We can transfer this form over to $X(B)$, and deform it to a complex volume form Ω that is compatible with the fibration structure, in the sense that, when Ω is restricted to the tangent space of a point on a smooth fiber, it decomposes into a wedge product of a one-form on the vertical tangent space and a one-form on the horizontal space.

We can also describe the trivialization determined by Ω in terms of the foliation by the tori $T_{R,\lambda}$. The tangent space to $X(B)$ at any point is the complexification of the tangent space to the torus $T_{R,\lambda}$ passing through that point, so an orientation on $T_{R,\lambda}$ trivializes the canonical bundle of $X(B)$ along that torus. Since all of these tori can be oriented in a consistent manner, we obtain a trivialization of the canonical bundle over the complement of the critical point in $X(B)$. This trivialization is homotopic to the one determined by Ω for the following reason. The holomorphic volume form $\Omega_{X^\vee}$ on $X^\vee$ admits a family of special Lagrangian tori, which means precisely that $\Omega_{X^\vee}$ restricted to each torus is a real volume form times a constant phase factor. When we transfer $\Omega_{X^\vee}$ over to $X(B)$, this foliation by special Lagrangian tori goes over to a foliation by tori that is isotopic to the foliation by the tori $T_{R,\lambda}$. (The fact that the tori are Lagrangian is not important here, only that they are totally real.) The foliation by the tori $T_{R,\lambda}$ is also compatible with the fibration structure, as each torus fibers over a circle in the base $X(I)$ and intersects each fiber in a circle. Thus we obtain foliations by circles on the base and the fiber.

The point of these choices is that we will be able to compute the degrees of our intersection points in terms of amounts of rotation with respect to the circle foliations on the base and the fiber. Because the Lagrangian $L(d)$ satisfies $H^1(L(d)) = 0$, it admits gradings. Moreover, because it fibers over a path in the base, and consists of a path in each fiber, it may be graded separately in the base and fiber directions. To say what we mean by this, recall the notion of a *squared phase map* from [41]. This is the map

$$\alpha_{L(d)} : L(d) \to S^1, \quad \alpha_{L(d)}(p) = \frac{\Omega(v_1 \wedge v_2)^2}{|\Omega(v_1 \wedge v_2)|^2} \tag{57}$$

where $\{v_1, v_2\}$ is a basis of $T_pL(d)$. A grading is then a real-valued lift $\tilde{\alpha}_{L(d)} : L(d) \to \mathbb{R}$ such that $\exp(2\pi i \tilde{\alpha}_{L(d)}) = \alpha_{L(d)}$. In our situation, we can decompose this squared phase map into base and fiber components

$$\alpha^b_{L(d)},\ \alpha^f_{L(d)} : L(d) \to S^1 \tag{58}$$

such that $\alpha_{L(d)} = \alpha^b_{L(d)} \cdot \alpha^f_{L(d)}$, such that $\alpha^b_{L(d)}$ is the squared phase with respect to the foliation by circles on the base, and $\alpha^f_{L(d)}$ is the squared phase with respect to the foliation by circles on the fiber. Normalizing and homotoping the volume form Ω if necessary, we may assume that the squared phase of the torus $T_{R,\lambda}$ is $1 \in S^1$ for all three versions, and so that the squared phase of $L(0)$ is $-1 \in S^1$ for both α^b and α^f (and hence its total phase is $\alpha = 1$).

Now, if we set $d > 0$, and we consider $L(d)$, we find that the tangent space of $L(d)$, in either the base or fiber, is obtained from that of $T_{R,\lambda}$ by a small counterclockwise rotation. This means that $\alpha^b_{L(d)}$ and $\alpha^f_{L(d)}$ lie on the upper half circle between 1 and -1 in S^1.

Next, choose lifts $\tilde{\alpha}^b$ and $\tilde{\alpha}^f$ for the base and fiber squared phase maps of the manifolds $L(d)$ for $d \geq 0$, preferring values the interval $[0, 1]$. Note that $L(0)$ has

$$\tilde{\alpha}^b_{L(0)} = \tilde{\alpha}^f_{L(0)} = \frac{1}{2} \tag{59}$$

For the other $L(d)$ with $d > 0$ the value of $\tilde{\alpha}^b_{L(d)}$ will always lie in the interval $(0, \frac{1}{2})$, while the value of $\tilde{\alpha}^f$ lies in the interval $(0, 1)$, since the intersection of $L(d)$ with any fiber is always a path that is transverse to the foliation by circles.

Finally, with the gradings in place, we can compute the degrees of the intersection points between $L(0)$ and $L(d)$ for $d > 0$.

Proposition 3.5. *Suppose $d_1 < d_2$. The the Floer complex $CF^*(L(d_1), L(d_2))$ of the positively perturbed pair $L(d_1), L(d_2)$ is concentrated in degree zero. The Floer complex $CF^*(L(d_2)), L(d_1))$ for the positively perturbed pair $L(d_2), L(d_1)$ is concentrated in degree two.*

Proof. Let us first consider the case $d_1 = 0$ and $d_2 = d > 0$. Using the local splitting given by the fibration, this degree is a sum of contributions from the base and fiber. Since the base and fiber are complex one-dimensional, the base and fiber contributions are for $p \in L(0) \cap L(d)$ using [41, Eq. 11.35],

$$i^b(L(0), L(d), p) = [\tilde{\alpha}^b_{L(d)} - \tilde{\alpha}^b_{L(0)}] + 1 \tag{60}$$

$$i^f(L(0), L(d), p) = [\tilde{\alpha}^f_{L(d)} - \tilde{\alpha}^f_{L(0)}] + 1 \tag{61}$$

d where $[\cdot]$ denotes the greatest integer function. Look at the diagram shows that, at any intersection point p, the quantity inside the $[\cdot]$ is negative, as the tangent space to $L(d)$ is a small *clockwise* rotation of the tangent space to $L(0)$. Thus both $i^b(L(0), L(d), p)$ and $i^f(L(0), L(d), p)$ are zero, and the total degree is zero.

If we keep $d > 0$, but swap the roles of $L(0)$ and $L(d)$, we find that

$$i^b(L(d), L(0), p) = [\tilde{\alpha}^b_{L(0)} - \tilde{\alpha}^b_{L(d)}] + 1 = 1 \tag{62}$$

$$i^f(L(d), L(0), p) = [\tilde{\alpha}^f_{L(0)} - \tilde{\alpha}^f_{L(d)}] + 1 = 1 \tag{63}$$

so that the intersection point p, regarded as a morphism from $L(d)$ to $L(0)$, has degree 2.

A similar analysis shows that the degrees of intersections $L(d_1) \cap L(d_2)$, regarded as a morphism from $L(d_1)$ to $L(d_2)$, depends on $d_2 - d_1$. If $d_2 - d_1$ is positive, they have degree 0, while if $d_2 - d_1$ is negative, they have degree 2. □

Since, in each case, the complex is concentrated in a single degree, the differential vanishes, and we may identify the complex $CF^*(L(d_1), L(d_2))$ with its cohomology $HF^*(L(d_1), L(d_2))$.

3.3.6. *Intersection points and integral points.* With the above prescription for perturbing the Lagrangians $L(d)$, we now work out the bijection between the intersection points of $L(0)$ and $L(d)$ for $d > 0$, regarded as morphisms from $L(0)$ to $L(d)$, and the $(1/d)$-integral points of B.

Proposition 3.6. *Let B be scaled so that the top face has length 2. Take $d > 0$. Then the intersection points $L(0) \cap L(d)$ giving a basis of $CF^0(L(0), L(d))$ correspond to $(1/d)$-integral points of B, including those on the boundary of B. For $d < 0$, the intersection points giving a basis of $CF^2(L(0), L(d))$ correspond the same set but with points on the boundary of B excluded. For the pair $L(d_1), L(d_2)$, the same conclusion holds with $d = d_2 - d_1$.*

This proposition is proved by explicitly indexing all of the various points. We start at the intersection point of $\ell(0)$ and $\ell(d)$ near the inner radius of the annulus $X(I)$. This intersection point survives the perturbation. In the fiber over this point there is one intersection point that survives after perturbation. As we transport around the inner part of the annulus, we pick up half-twists in the fiber (Proposition 3.4), which increases the number of intersection points by one after every two turns in the base. This pattern continues until $\ell(d)$ reaches the middle radius and starts winding around the other side of the Lefschetz singularity, where the pattern reverses. (See Figure 5.)

Assign the rational numbers

$$[-1, 1] \cap (1/d)\mathbb{Z} = \{-1, -(d-1)/d, \ldots, -1/d, 0, 1/d, \ldots, 1\}$$

to the intersection points of $\ell(0)$ and $\ell(d)$, then over the point indexed by a/d the number of intersection points in the fiber is $1 + \left\lfloor \frac{d-|a|}{2} \right\rfloor$. We index the points in a given fiber using the index $i \in \{0, 1, \ldots, \left\lfloor \frac{d-|a|}{2} \right\rfloor\}$, starting at the top of the fiber.

If we scale B so that the top face has affine length 2, the $1/d$ integral points of B are also organized by the projection $\eta : B \to I$ into columns indexed by $[-1, 1] \cap (1/d)\mathbb{Z}$. By inspection, the column over a/d has $1 + \left\lfloor \frac{d-|a|}{2} \right\rfloor$ points. These are also indexed by $i \in \{0, 1, \ldots, \left\lfloor \frac{d-|a|}{2} \right\rfloor\}$, starting at the top of the fiber.

Under this bijection the $(1/d)$-integral points on the boundary of B correspond to intersections of the unperturbed $L(0)$ and $L(d)$ that lie on the boundary of $X(B)$. According to our definition of positive perturbation, these points do (do not) contribute when $d > 0$ ($d < 0$).

Definition 4. Take $d > 0$. For $a \in \{-d, \ldots, d\}$, and $i \in \{0, 1, \ldots, \left\lfloor \frac{d-|a|}{2} \right\rfloor\}$, let $q_{a,i}(n) \in L(n) \cap L(n+d)$ be the intersection which lies in the column indexed by a/d, and which is the ith from the top of the fiber.

We can already verify mirror symmetry at the level of the Hilbert polynomial:

$$|L(0) \cap L(d)| = \left| B\left(\frac{1}{d}\mathbb{Z}\right) \right| = \frac{(d+2)(d+1)}{2} = \dim H^0(\mathbb{CP}^2, \mathcal{O}_{\mathbb{CP}^2}(d)) \tag{64}$$

Figure 7 shows the points of $B(\frac{1}{4}\mathbb{Z})$ representing the basis of morphisms $L(d) \to L(d+4)$.

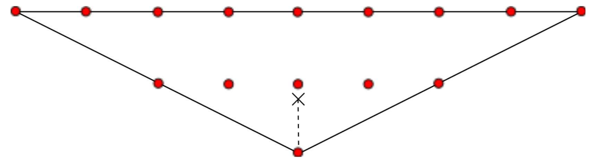

FIGURE 7. The 1/4–integral points of B.

3.3.7. *Hamiltonian isotopies.* There is an alternative way to express the relationship between $L(d)$ and $L(0)$, which is by a Hamiltonian isotopy. There is a Hamiltonian function H on $X(B)$ such that the time-d flow of H takes $L(0)$ to $L(d)$. During the intermediate times of this isotopy the Lagrangian will not be admissible (or even close to it), but at the end of the isotopy admissibility is restored. This observation is used when we consider wrapped Floer cohomology in section 7.

This isotopy also allows us to identify the intersection points $L(n) \cap L(n+d)$ for different values of n. For this reason, we will write $q_{a,i}(n)$ as simply $q_{a,i}$.

4. A degeneration of holomorphic triangles

Since we have set up our Lagrangians as fibered over paths, a holomorphic triangle with boundary on the Lagrangians composed with the projection is a holomorphic triangle in the base, which is an annulus, with boundary along the corresponding paths. The triangles that are most interesting are those that pass over the critical value $w = -1$ (possibly several times). In general, such triangles are immersed in the annulus, and, after passing the the universal cover of the annulus, are embedded. Hence, we can regard such triangles as sections over a triangle in the base of a Lefschetz fibration having as base a strip with a $\mathbb{Z}$–family of critical values. Once this is done, we can apply a TQFT for counting sections of Lefschetz fibrations developed by Seidel [41, 39].

We consider the deformation of the Lefschetz fibration over the triangle where the critical values bubble out along one of the sides. At the end of this degeneration, we count sections of a trivial fibration over a $(k+3)$–gon, along with sections of k identical fibrations, each having a disk with one critical value and one boundary marked point. Each of these fibrations is equipped with a Lagrangian boundary condition given by following the degeneration of the original Lagrangian submanifolds. The sections of the trivial fibration over the $(k+3)$–gon can be reduced to counts in the fiber, while the counts of the k other parts are identical, and can be computed directly using the techniques of [39].

4.1. **Triangles as sections.** Let $q_1 \in HF^0(L(0), L(n))$ and $q_2 \in HF^0(L(n), L(n+m))$ be two degree zero morphisms whose Floer product $\mu^2(q_2, q_1)$ we wish to compute. Suppose that $p \in HF^0(L(0), L(n+m))$ contributes to this product. Let S denote a disk with three boundary punctures, and a complex structure that is allowed to vary. To find the coefficient of p in $\mu^2(q_2, q_1)$, we count pseudo-holomorphic triangles, that is, pseudo-holomorphic maps $u : S \to X(B)$ that send the punctures to q_1, q_2, p and the boundary components to $L(0), L(n), L(n+m)$. Various authors have studied the construction of such invariants; we follow the theory as developed by Seidel in [41, 39].

Because we want to make contact with the theory of pseudo-holomorphic sections of Lefschetz fibrations, we need to import the setup from [39, §2.1]. Let $\pi : E \to B$ be a exact symplectic Lefschetz fibration. We choose primitive θ for the symplectic form ω such that the Liouville vector field Z defined on the fiber M by $\iota_Z(\omega|_M) = \theta|_M$ points outward along the boundary of M. The flow of Z defines a collar $[-\epsilon, 0] \times M \to M$, and we let σ be the function on a neighborhood of ∂M given by projection to the $[-\epsilon, 0]$ factor. We pick an almost complex structure j on the base of the fibration, we always consider an almost complex structure J on the total space which is *compatible relative to* j, meaning

(1) J is integrable in a neighborhood of each critical point,
(2) π is a (J, j)-holomorphic map,
(3) For each fiber E_z, the form $\omega(\cdot, J\cdot)|TE_z$ is symmetric and positive definite
(4) on a neighborhood of the horizontal boundary of E, J satisfies $\theta \circ J = d(e^\sigma)$.

Later on, in Section 4.4, we will also need to consider J which are *horizontal*, meaning that J preserves the horizontal subspaces of the symplectic connection. Once we extend the Lagrangian boundary condition Section 4.2 so that our Lagrangians are compact in each fiber, these conditions imply that the curves we wish to count lie in a compact subset disjoint from the horizontal boundary [39, Lemma 2.2]. The second condition is particularly useful because it allows us to study pseudo-holomorphic curves by projecting them to the base of the fibration.

Consider the projection of such a triangle to the base by $w : X(B) \to X(I)$. This yields a 2-chain on the base with boundary on the corresponding base paths $\ell(0), \ell(n), \ell(n+m)$ with corners at the points $w(q_1), w(q_2), w(p)$. This projection is not necessarily embedded, so the next step is to pass to the universal cover of the base. Let $\tilde{X}(I)$ denote infinite strip which is the universal cover of the annulus, and let $\tilde{w} : \tilde{X}(B) \to \tilde{X}(I)$ denote the induced fibration. We denote by $\pi : \tilde{X}(B) \to X(B)$ the covering map. When drawing pictures in the base $\tilde{X}(I)$, we can represent it as $[-1, 1] \times \mathbb{R}$, with the infinite direction drawn vertically. With this convention, the path $\ell(d)$ lifts to a $\mathbb{Z}$-family of paths which have slope $-d$.

Figure 8 shows the universal cover of $X(I)$, with the base paths for $L(0)$, $L(1)$, and $L(2)$.

The choice of lift of q_1 determines a lift of $L(0)$ and $L(n)$, which then determines a lift of q_2 and of $L(n+m)$, which in turn determines where the lift of any p must lie. By looking at the slopes of the base paths $\ell(0), \ell(n), \ell(n+m)$ involved, we obtain the following proposition:

Proposition 4.1. *In the terminology of Definition 4, Suppose that* $q_1 = q_{a,i}$ *lies in the fiber indexed by* a/n*, and that* $q_2 = q_{b,j}$ *lies in the fiber indexed by* b/m*. Then any* p *contributing to the product is* $q_{a+b,h}$ *for some* h*, that is, it lies in the fiber indexed by* $(a+b)/(n+m)$

We can rephrase this proposition as saying that we can introduce a second grading on $HF^*(L(d), L(d+n))$ where $HF^{*,a}(L(d), L(d+n))$ is generated by $q_{a,i}$ for $i \in \{0, 1, \dots \left\lfloor \frac{n-|a|}{2} \right\rfloor\}$, and that μ^2 respects this grading.

We must now consider two cases: either the input generators q_1 and q_2 lie in different fibers of w, or they lie in the same fiber. The rest of this section is devoted to case where q_1 and q_2 lie in different fibers, which is the more difficulty one. In the case

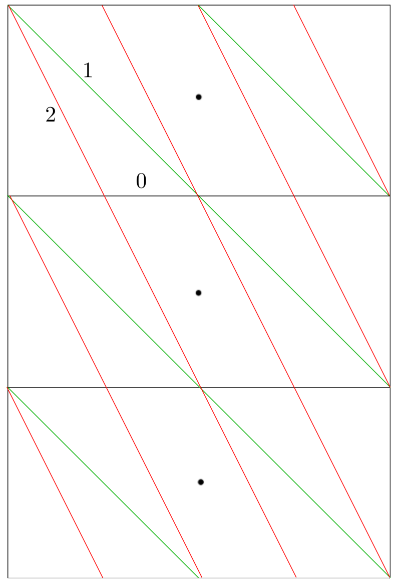

FIGURE 8. The universal cover of $X(I)$.

where q_1 and q_2 lie in the same fiber, the preceding proposition says that the output p must also lie in the same fiber. This means that we can compute the contribution of p to $\mu^2(q_2, q_1)$ by counting triangles that lie entirely in this fiber, since the projection of such a triangle to the base must be constant. We will employ an alternative strategy where we perturb one of the base paths, say $\ell(0)$, by a small amount so that there are no points where all three base paths intersect. This forces the triangle in the fiber to spread out over a small triangle in the base, so that in particular it cannot be constant. With this proviso, the arguments in this section apply without exception.

Now we show that any triangles contributing to the product of interest are sections of the Lefschetz fibration $\tilde{w} : \tilde{X}(B) \to \tilde{X}(I)$:

Proposition 4.2. *Let $u : S \to X(B)$ be a pseudo-holomorphic triangle contributing to the component of p in $\mu^2(q_2, q_1)$, such that q_1 and q_2 do not lie in the same fiber of w. Then there is a triangle T in $\tilde{X}(I)$ bounded by appropriate lifts of $\ell(0), \ell(n)$ and $\ell(n+m)$, a holomorphic isomorphism $\tau : S \to T$, and a pseudo-holomorphic section $s : T \to \tilde{X}(B)$, such that $u = \pi \circ s \circ \tau$.*

Conversely, any pseudo-holomorphic section $s : T \to \tilde{X}(B)$ with boundary on $L(0), L(n), L(n+m)$ which maps the corners to q_2, q_1, p contributes to the coefficient of p in $\mu^2(q_2, q_1)$.

Proof. Since $u : S \to X(B)$ is a map from a simply-connected domain, there is a lift $\tilde{u} : S \to \tilde{X}(B)$. Since q_1 and q_2 are not in the same fiber, the image of u cannot be contained entirely within a fiber. The triangle T and the lifts of the $\ell(d)$ are determined by the considerations from the previous proposition. Clearly $\tilde{w} \circ \tilde{u}$ defines a 2-chain in $\tilde{X}(I)$, which by maximum principle is supported on T. By positivity of intersection with the fibers of $\tilde{w}$, all components of this 2-chain are positive, and the map $\tilde{w} \circ \tilde{u} : S \to T$ is a ramified covering. However, if the degree were greater than one, then ∂S would have to wind around $\ell(0), \ell(n), \ell(n+m)$ more than once, contradicting the boundary conditions we placed on the map u.

Since the projection $\tilde{w} : \tilde{X}(B) \to \tilde{X}(I)$ is holomorphic, the composition $\tilde{w} \circ \tilde{u} : S \to T$ is a holomorphic map which sends the boundary to the boundary and the punctures to the punctures, so it is a holomorphic isomorphism, and we let τ be its inverse.

For the converse, uniformization for the disk with three boundary punctures yields a complex structure on S and a map $\tau : S \to T$, such the composition $\tilde{u} = s \circ \tau$ is the desired triangle in $\tilde{X}(B)$. Composing this with $\pi : \tilde{X}(B) \to X(B)$ yields the triangle in $X(B)$. □

Proposition 4.3. *Suppose $q_1 = q_{a,i}$ lies in the fiber indexed by a/n, and $q_2 = q_{b,j}$ in the fiber indexed by b/m. Then the sections in Proposition 4.2 cover the critical values of the Lefschetz fibration k times, where*

- *$k = 0$ if a and b are both non-negative or both non-positive.*
- *$k = \min(|a|, |b|)$ if a and b have different signs.*

Proof. We identify $\tilde{X}(I)$ with $[-1, 1] \times \mathbb{R}$. The critical values lie at the points $\{0\} \times (\mathbb{Z} + \frac{1}{2})$.

If a and b are both non-negative or both non-positive, then the triangle T is entirely to one side of the vertical line $\{0\} \times \mathbb{R}$ where all the critical values lie.

Suppose a and b have opposite signs and $|a| \leq |b|$. Then the output point lies at $(a + b)/(n + m)$, which has the same sign as b. The side of T corresponding to $\ell(n)$ crosses the line $\{0\} \times \mathbb{R}$ at $(0, a)$, while the side corresponding to $\ell(0)$ crosses at $(0, 0)$, so the distance is $|a|$, and in fact the set $\{0\} \times (\mathbb{Z} + \frac{1}{2})$ contains $|a|$ points in this interval.

If $|b| \leq |a|$, the output point at $(a + b)/(n + m)$ has the same sign as a, and so we need to look at where $\ell(n+m)$ intersects the line $\{0\} \times \mathbb{R}$. This happens at $(0, a+b)$, and the distance to $(0, a)$ is $|b|$. □

With the notation introduced so far, we can state the main result of our computation for $\mu^2(q_{b,j}, q_{a,i})$.

Proposition 4.4. *Suppose that $q_{a,i} \in HF^0(L(0), L(n))$ and $q_{b,j} \in HF^0(L(n), L(n + m))$ as in Definition 4, and let k be as in Proposition 4.3. Then*

$$\mu^2(q_{b,j}, q_{a,i}) = \sum_{s=0}^{k} \binom{k}{s} q_{a+b,i+j+s} \tag{65}$$

Proof. This proposition is the combination of Propositions 4.8, 4.10, 4.12, 4.13, and 4.24. □

The proof of Proposition 4.4 is where most of the technical work of this paper is spent, and it will occupy Sections 4.2–4.6. Sections 4.2 and 4.3 present a technique for degenerating the total space of the Lefschetz fibration to break these counts into simpler pieces. Sections 4.4 and 4.5 compute these pieces, and Section 4.6 determines the signs.

We shall now reformulate Proposition 4.4 in algebro-geometric terms. Let

$$A = \bigoplus_{d=0}^{\infty} A_d = \bigoplus_{d=0}^{\infty} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong \mathbb{K}[x, y, z] \tag{66}$$

be the homogeneous coordinate ring of $\mathbb{P}^2$. We write elements of $A_d = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ as degree d homogeneous polynomials in the variables x, y, z. Define

$$p = xz - y^2, \tag{67}$$

and set, for $a \in \{-d, \dots, d\}$, $i \in \{0, \dots, \left\lfloor \frac{d-|a|}{2} \right\rfloor\}$,

$$Q_{a,i} = \left\{ \begin{array}{ll} x^{-a}p^i y^{d+a-2i} & \text{if } a \leq 0 \\ z^a p^i y^{d-a-2i} & \text{if } a > 0 \end{array} \right\} \in A_d. \tag{68}$$

Proposition 4.5. *Take $Q_{a,i} \in A_n$ and $Q_{b,j} \in A_m$, then in A,*

$$Q_{a,i}Q_{b,j} = \sum_{s=0}^{k} \binom{k}{s} Q_{a+b,i+j+s} \tag{69}$$

where $k = \min(|a|, |b|)$ if a and b have different signs, and $k = 0$ otherwise.

Proof. The case where a and b have the same sign is obvious.

Suppose that $a \leq 0$ and $b \geq 0$, and suppose that $|a| \leq |b|$. Then we have $a + b \geq 0$, and $k = -a$.

$$Q_{a,i}Q_{b,j} = x^{-a}p^i y^{n+a-2i} z^b p^j y^{m-b-2j} = z^{a+b}(xz)^k p^{i+j} y^{n+m+a-b-2(i+j)} \tag{70}$$

Since $xz = p + y^2$, we have

$$(xz)^k = \sum_{s=0}^{k} \binom{k}{s} p^s y^{2(k-s)} \tag{71}$$

$$Q_{a,i}Q_{b,j} = \sum_{s=0}^{k} \binom{k}{s} z^{a+b} p^{i+j+s} y^{(n+m)-(a+b)-2(i+j+s)} \tag{72}$$

and the monomial on the right is just $Q_{a+b,i+j+s}$. The other cases are similar. □

The operation μ^2 determines a product, $q_2 \cdot q_1 = (-1)^{|q_1|}\mu^2(q_2, q_1)$. The sign is present to connect the conventions for an A_∞-category at those of a dg-category; in the case all morphisms have degree zero the sign is trivial. The following proposition states how our Lagrangian intersections give rise to a distinguished basis of the homogeneous coordinate ring A. From the preceding propositions, we can deduce Theorem 1.1.

Proposition 4.6. *The map $\psi_{d,n} : HF^0(L(d), L(d+n)) \to A_n$ defined by*

$$\psi_{d,n} : q_{a,i} \mapsto Q_{a,i} \tag{73}$$

is an isomorphism. We have

$$\psi_{d,n+m}(q_1 \cdot q_2) = \psi_{d,n}(q_1) \cdot \psi_{d+n,m}(q_2) \tag{74}$$

Proof. That $\psi_{d,n}$ is an isomorphism is because it maps a basis to a basis. The other statement is the combination of Propositions 4.4 and 4.5. □

Thus the proof of Theorem 1.1 will be complete once we prove Proposition 4.4, which in turn depends on Propositions 4.8, 4.10, 4.12, 4.13, and 4.24. The rest of this section is devoted to the proofs of these propositions.

4.2. **Extending the fiber.** One problem with our Lagrangian boundary conditions $L(d)$ is that they intersect the horizontal boundary $\partial^h X(B)$. This raises the possibility that as we degenerate the fibration, part of a pseudo-holomorphic section can escape through $\partial^h X(B)$.

We now describe a technical trick that, by attaching bands to the fiber, allows us to close up the Lagrangians for the purpose of a particular computation, and thereby use only the results in the literature on sections with Lagrangian boundary conditions disjoint from the boundaries of the fibers. Proposition 4.7 states that this attachment does not change the spaces of holomorphic curves that we wish to count, because they do not enter the bands. On the other hand, when we degenerate the fibration, we find that some of these curves degenerate into curves that do enter the bands.

The starting point for this construction is, given inputs a_1 and a_2, to consider the portion of the fibration $\tilde{X}(B)|T \to T$ lying over the triangle T in the base. We recall the assumption from section 3.2 that the symplectic connection is actually flat near the horizontal boundary. After passing to the universal cover of the base, the symplectic monodromies around the boundaries of $X(I)$ can be trivialized, and the fibration is actually symplectically trivial near the horizontal boundary. We also assume that the boundary intersections of our Lagrangians have been positively perturbed as in Section 3.3.4, so that they do not intersect at the boundary. After trivializing the fibration near the horizontal boundary, we find that in each fiber F_z of $\tilde{X}(B)|T \to T$, there are six points on ∂F_z (three on either component), arising as the symplectic parallel transport of the boundary points of $L(0)$, $L(n)$, and $L(n+m)$. These are the points where the Lagrangians $L(0), L(n), L(n+m)$ are allowed to intersect ∂F_z (though $L(d) \cap \partial F_z$ is only nonempty if $z \in \ell(d)$ is on the appropriate boundary component of the triangle T). The two sets of three points on each component of ∂F_z are matched according to which Lagrangian they come from, and we extend the fiber F_z to $\hat{F}_z$ by attaching three bands running between the two components of F_z according to this matching. We call the resulting fibration $\hat{X}_T \to T$. We have an embedding $\iota : \tilde{X}(B)|T \to \hat{X}_T$.

Over the component of ∂T where the Lagrangian boundary condition $L(0)$ lies, we extend $L(0)$ to $\hat{L}(0)$, closing it up fiberwise to a circle by letting it run through the corresponding band in $\hat{F}_z$. Similarly we close up $L(n)$ and $L(n+m)$ to their hat-versions, each passing through a different band.

Figure 9 shows the cylinder with a band attached. The actual extended fiber has three such bands.

It is immediate from the construction that $\hat{X}_T \backslash \mathrm{image}(\iota)$ is symplectically a product. Let us use a complex structure which is also a product in this region. Transversality can be achieved using such structures because all intersection points lie in image(ι), and hence any pseudo-holomorphic section must also pass through image(ι), where we are free to perturb J as usual. We have the following proposition:

Proposition 4.7. *Any pseudo-holomorphic section $s : T \to \hat{X}_T$ with boundary conditions $\hat{L}(d)$, $d = 0, n, n+m$ lies within* image(ι).

Proof. Let $p : \hat{X}_T \setminus \mathrm{image}(\iota) \to \hat{F}$ denote the projection to the fiber, whose image consist of the bands. Let us consider one band containing the Lagrangian L. Introduce

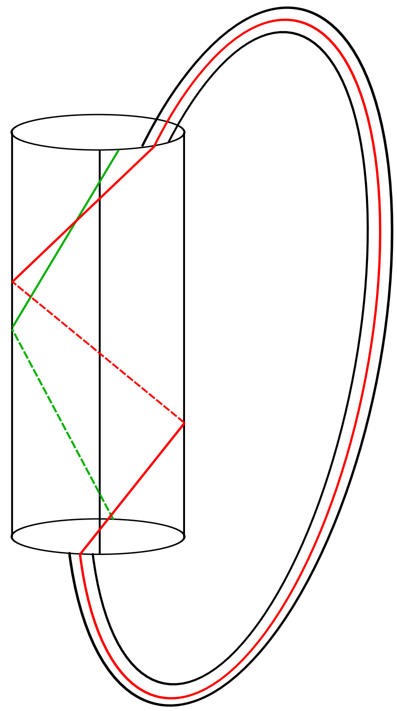

FIGURE 9. Attaching a band to close up one of the Lagrangians in the fiber.

coordinates $(s,t) \in [0,a] \times [-1,1]$ on the band such that the part of the Lagrangian within the band is $L = [0,a] \times \{0\}$.

Let $V \subset T$ be the preimage of the open band $(0,a) \times [-1,1]$ under $p \circ s$. The set V is relatively open. Let $V^\circ = V \setminus \partial T$. As the map $p \circ s : V^\circ \to [0,a] \times [-1,1]$ is actually a holomorphic map between Riemann surfaces, the open mapping theorem implies that the image $W = (p \circ s)(V^\circ)$ is an open subset of $(0,a) \times (-1,1)$. Also, the boundary of W is contained in the union of the Lagrangian $[0,a] \times \{0\}$ (where the boundary of the domain is sent) and the ends of the band $\{0,a\} \times (-1,1)$ (where image of V connects to the rest of the holomorphic curve). The only such W is the empty set. □

4.3. **Degenerating the fibration.** By propositions 4.2 and 4.7, in order to compute the Floer product between two morphisms $q_1 \in HF^0(L(0), L(n))$ and $q_2 \in HF^0(L(n), L(n+m))$, it is just as good to count sections of the fibration $\hat{X}_T \to T$ with Lagrangian boundary conditions $\hat{L}(0)$, $\hat{L}(n)$, $\hat{L}(n+m)$.

In order to obtain these counts, we will consider a degeneration of the Lefschetz fibration. We consider a one-parameter family of Lefschetz fibrations $\hat{X}^r \to T^r$, which for $r = 1$ is simply the one we started with. As r goes to zero, the fibration deforms so that all of the k critical values contained within T move toward the side of T corresponding to $\ell(n)$. In the limit, a disc bubble appears around each critical value, and at $r = 0$, the base T has broken into a $(k+3)$-gon, with k "new" vertices along the side corresponding to $\ell(n)$, each of which joins to a disk with a single critical value. We can equip each fibration with Lagrangian boundary conditions varying continuously with r, and degenerating to a collection of Lagrangian boundary conditions for each of the component fibrations at $r = 0$.

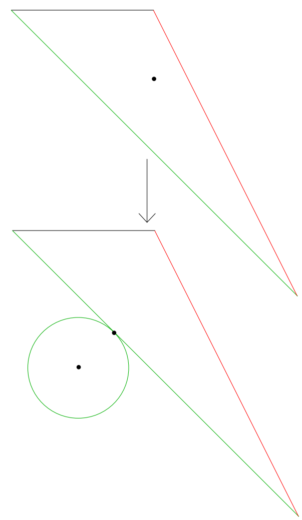

FIGURE 10. Degenerating the fibration.

The base of the Lefschetz fibration undergoes the degeneration shown in figure 10. This figure shows specifically the case for the product of $x \in CF^*(L(0), L(1))$ and $z \in CF^*(L(1), L(2))$. The marked point on the upper portion is the Lefschetz critical value, while the marked points on the lower portion are the Lefschetz critical value and a node.

When performing this construction carefully, it is better to describe this family as a smoothing of the degenerate $r = 0$ end. Let T^0 be a disk with $(k+3)$ boundary punctures. We emphasize that the conformal structure of T^0 is fixed. Let $\hat{X}^0 \to T^0$ be a symplectically trivial Lefschetz fibration. Let $D_1, \dots, D_k$ denote the k disks with one boundary puncture that are our "bubbles". For each i let $E_i \to D_i$ denote a Lefschetz fibration with a single critical value, and which is trivial near the puncture. Let the symplectic monodromy around ∂D_i be denoted τ_i.

We will equip each of the components of this fibration with a Lagrangian boundary condition as follows.

- The base T^0 has one boundary component corresponding to $\ell(0)$, one boundary component corresponding to $\ell(n+m)$, and $(k+1)$ boundary components corresponding to $\ell(n)$. Since the fibration $\hat{X}^0 \to T^0$ is trivial, it suffices to describe each boundary condition in the fiber. Over the point where $\ell(0)$ and $\ell(n+m)$ come together, we identify the fiber with that of $\hat{X}^1 \to T^1$, and take $\hat{L}(0)^0$ and $\hat{L}(n+m)^0$ to be the corresponding Lagrangians.
- Over the $k+1$ boundary components of T^0 corresponding to $\ell(n)$, we construct a sequence $\hat{L}(n)^0_i$ of Lagrangians. At the puncture where $\ell(0)$ and $\ell(n)$ come together, we take $\hat{L}(n)^0_0$ to have the same position relative to $\hat{L}(0)^0$ (already constructed) that $\hat{L}(n)$ has relative to $\hat{L}(0)$ in the original fibration. As we pass each of the new punctures where the disks are attached, the monodromies τ_i must be applied. So we let

$$\hat{L}(n)^0_i = \tau_i(\hat{L}(n)^0_{i-1}) \tag{75}$$

This can be done so that, over the puncture where $\ell(n)$ and $\ell(n+m)$ come together, $\hat{L}(n)^0_k$ and $\hat{L}(n+m)^0$ intersect as the original $\hat{L}(n)$ and $\hat{L}(n+m)$ do.

- Over the k disks D_i, we take a Lagrangian boundary condition which interpolates between $\hat{L}(n)^0_{i-1}$ and $\hat{L}(n)^0_i$. This is possible because each D_i contains a single Lefschetz critical value.

Note that at this stage we only care about the twists τ_i up to Hamiltonian isotopy, but we will make a particular choice in §4.4, as required by the technical considerations there. This gives us τ_i such that $\hat{L}(n)^0_{i-1}$ and $\hat{L}(n)^0_i$ have one intersection point over i-th new puncture of T^0.

Since the boundary conditions agree over the corresponding punctures, we can glue the components over T^0 and $D_1, \ldots, D_k$ with large gluing length in order to obtain $\hat{X}^\epsilon \to T^\epsilon$ for small $\epsilon > 0$, which has three boundary components and k critical values all near the $\ell(n)$ boundary. There is a family of Lefschetz fibrations interpolating between $\hat{X}^\epsilon \to T^\epsilon$ back to our original $\hat{X}^1 \to T^1$, along which the critical values move back to their original points.

We equip T^0 and $D_1, \ldots, D_k$ with complex structures, and equip T^ϵ with a family of complex structures j^ϵ converging to those in the limit. All total spaces are equipped with relatively compatible almost complex structures.

Now we appeal to the gluing theorem [39, Proposition 2.2] telling us how the curve counts behave under the degeneration.

Proposition 4.8. *The count of pseudo-holomorphic sections of $\hat{X}^1 \to T^1$, with Lagrangian boundary conditions $\hat{L}(d)$, $d = 0, n, n+m$ is obtained from the counts of sections of $\hat{X}^0 \to T^0$ and $E_i \to D_i$ by gluing together sections whose values over the punctures match.*

Proof. Considering pseudo-holomorphic sections of the fibrations $\hat{X}^\epsilon \to T^\epsilon$, as long as the gluing length is large, the sections for $r = \epsilon$ will be obtained from sections over T^0 and $D_1, \ldots, D_k$ by gluing sections with matching values at the punctures [39, Proposition 2.2]. During the deformation from $r = \epsilon$ to $r = 1$, no Floer strip breaking can occur because our Lagrangians $\hat{L}(d)$, $d = 0, n, n+m$ do not bound any strips, even topologically, so the count remains the same at $r = 1$. □

4.4. **Horizontal sections over a disk with one critical value.** We now determine the counts of pseudo-holomorphic sections of the Lefschetz fibrations $E_i \to D_i$ using the theory of horizontal sections developed by Seidel in [39]. In particular we will apply results from section 2.5 of that paper, so we adopt its notation. This theory involves analysis of the symplectic connection, whose horizontal subspaces are the symplectic orthogonal subspaces to the fibers, and the symplectic parallel transport defined by horizontal lifting of paths in the base of the fibration.

In order for this technique to work, we need to set up carefully a model Lefschetz fibration over a base S, the disk with one end, in order to ensure that all the sections we need to count are in fact horizontal. The key properties are:

- Transversality of the boundary conditions over the strip-like end. This means that we cannot use a standard model Dehn twist fibration, but need to introduce a perturbation somewhere.

- Non-negative curvature. The curvature of the symplectic connection is a two-form on the base with values in functions on the fibers (the Lie algebra of the group of Hamiltonian diffeomorphisms), and we require that this two-form evaluated on a positive basis of the tangent space to the base yields a non-negative function on the fiber. The standard model Dehn twist fibration has non-negative curvature, but requiring the perturbation to have non-negative curvature imposes a constraint.
- Vanishing action of horizontal sections. This imposes a further constraint on the perturbation.

As we progress through the construction, the definitions of all of these terms will be recalled.

The first step is to construct a fibration which is flat away from the critical point, following section 3.3 of [39]. Let S be the Riemann surface $\{\operatorname{Re} z \le 0, |\operatorname{Im} z| \le 1\} \cup \{|z| \le 1\}$ (a negative half-strip which has been rounded off with a half-disk).

Let M denote the fiber, which is a cylinder with three bands attached. The vanishing cycle V is the equator of the cylinder. The circle running through the core of one of these bands is L. Equip M with a symplectic form $\omega = d\theta$ such that L is exact. Over $S^- = (-\infty, -1] \times [-1, 1]$, we let $\pi : E^- \to S^-$ be a trivial fibration and equip E with 1-form Θ and 2-form $\Omega = d\Theta$ which are pulled back from the fiber (to get a symplectic form we add the pullback a positive 2-form $\nu \in \Omega^2(S)$, but this does not affect the symplectic connection). Following the pasting procedure described in [39, §3.3] (though with the cut on the right rather than the left), we can complete this fibration $\pi : E \to S$ to one where the monodromy around the boundary is τ_V, a standard model Dehn twist supported near the vanishing cycle. The result has non-negative curvature, is actually trivial on the part of the fiber away from the support of the Dehn twist, and is flat over the part of the base away from the critical value.

For this fibration, there is a Lagrangian boundary condition which over the end corresponds to the pair $(L, \tau_V(L))$. Of course, these are not transverse since τ_V is identity in the band. Hence we will perturb the symplectic form by a term which depends on a Hamiltonian. We will introduce the perturbation in a neighborhood of the edge $(-\infty, -1] \times \{1\}$. Let β be a cutoff function which

- is supported in $U = \{s + it \mid -2 \le s \le -1, 1 - \epsilon \le t \le 1\}$,
- vanishes along the bottom, left and right sides of U, and
- has $\partial\beta/\partial t \ge 0$.

Figure 11 shows the base of the fibration; the region where β has support is shaded.

Let H be a Hamiltonian function on the fiber M, and let X_H be its vector field defined by $\omega(\cdot, X_H) = dH$. Then over S^-, where our fibration was originally trivial with $\Omega = d\Theta$ pulled back from the fiber, consider

$$\Theta' = \Theta + H\beta ds \tag{76}$$

$$\Omega' = d\Theta' = \Omega + \beta dH \wedge ds - H(\partial\beta/\partial t) ds \wedge dt \tag{77}$$

This modifies the symplectic connection as follows: Let $Y \in TE^v$ denote the general vertical vector. Then if $Z \in TS$, and Z^h is its horizontal lift

$$0 = \Omega'(Y, Z^h) = \Omega(Y, Z^h) + \beta dH(Y) ds(Z) \tag{78}$$

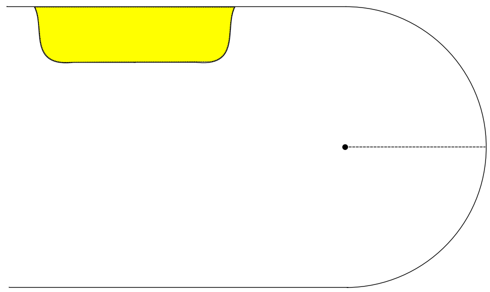

FIGURE 11. The base of the fibration with the region of perturbation shaded.

If we denote by Z again the horizontal lift with respect to the trivial connection, we have $Z^h = Z - \beta ds(Z) X_H$. Since $\beta \geq 0$, this means that as we parallel transport in the *negative* s-direction through the region U, we pick up a bit of the Hamiltonian flow of H, compared with the trivial connection. By adjusting the function β, we can ensure that the parallel transport along the boundary in the positive sense picks up ϕ_H, the time 1 flow of H.

We must compute the curvature of this connection. This is the 2-form on the base with values in functions on the fibers given by $\Omega'(Z_1^h, Z_2^h)$. It will suffice to compute for $Z_1 = \partial/\partial s$ and $Z_2 = \partial/\partial t$, a positive basis. We have $Z_1^h = \partial/\partial s - \beta X_H$ and $Z_2 = \partial/\partial t$.

$$\begin{aligned} &\Omega'(Z_1^h, Z_2^h) = \\ &\Omega(Z_1^h, Z_2^h) + \beta dH(Z_1^h) ds(Z_2^h) - \beta dH(Z_2^h) ds(Z_1^h) - H(\partial\beta/\partial t) ds \wedge dt(Z_1^h, Z_2^h) \end{aligned} \tag{79}$$

The first term vanishes because Z_2^h is horizontal for the trivial connection, the second term because $ds(Z_2^h) = 0$, and the third because $dH(Z_2^h) = 0$. This leaves $-H(\partial\beta/\partial t) ds \wedge dt(Z_1^h, Z_2^h) = -H(\partial\beta/\partial t)$. By our assumptions on β, this is nonnegative as long as $H \leq 0$.

We equip the deformed fibration with a Lagrangian boundary condition Q given by parallel transport of L around the boundary. This picks up a Dehn twist and the time 1 flow of H, so over the end we have the pair $(L, \phi_H(\tau_V(L)))$.

A *horizontal section* is a map $u : S \to E$ such that $Du(TS) = TE^h$. The importance of such sections is that, while they are determined by the symplectic connection, they are pseudo-holomorphic for any *horizontal complex structure* J, meaning an almost complex structure that preserves TE^h and for which the projection $E \to S$ is holomorphic.

The *action* $A(u)$ of a section u is defined to be $\int_S u^*\Omega'$. The symplectic area of u is then $A(u) + \int_S \nu$. The identity relating action to energy is ([39], eq. 2.31)

$$\frac{1}{2}\int_S ||(Du)^v||^2 + \int_S f(u)\nu = A(u) + \int_S ||\bar{\partial}_J u||^2 \tag{80}$$

for any horizontal complex structure J. Here $Du = (Du)^h + (Du)^v$ is the splitting induced by the connection, and f is the function determined by the curvature as $f(\pi^*\nu | TE^h) = \Omega' | TE^h$. In our example f is supported near the critical point and in $\operatorname{supp}\beta$, where $f = -H(\partial\beta/\partial t)$.

A direct consequence of (80) is the following:

Proposition 4.9. *If the curvature of* $\pi : E \to S$ *is non-negative, and if* u *is a* J*-holomorphic section with* $A(u) = 0$*, then* u *is horizontal and the curvature function* f *vanishes on the image of* u*.*

With all this in mind, we choose our Hamiltonian $H : M \to \mathbb{R}$ as follows:

- $H \leq 0$ and $H = 0$ near ∂M. This ensures that the fibration is still trivial near the horizontal boundary and that the curvature is non-negative within $\operatorname{supp} \beta$.
- H achieves its global maximum of 0 near ∂M and on the "cocore" of the band in M. It achieves its minimum along the vanishing cycle, and has no other critical points in the cylinder or lying on L (which intersects the vanishing cycle and the cocore once). The first condition implies that horizontal sections passing through the cocore do not pick up any curvature, while the second is there in order to ensure that $(L, \phi_H(\tau_V(L)))$ is a transverse pair.
- The flow ϕ_H has the property that $L \cap \phi_H(\tau_V(L))$ consists of one point x lying on the cocore of the band. At this point, the tangent space to $\phi_H(\tau_V(L))$ is a small clockwise rotation of the tangent space to L.

  This implies that this intersection point has degree zero. To see this, recall from Section 3.3.5 that the fiber has foliation by circles. We extend this foliation into the attached bands, foliating each band by intervals. The Dehn twist τ_V preserves this foliation, so it admits a grading in the sense of [41, (12i)], namely an $\mathbb{R}$-valued lift of the S^1-valued function $\alpha(D\tau_V(\Lambda))/\alpha(\Lambda)$ defined on the Lagrangian Grassmannian bundle. We can choose the lift to be zero in the bands, where τ_V is the identity. The Lagrangians L and $\phi_H(\tau_V(L))$ do not rotate with this foliation, so they admit gradings. Thus, if we equip L with some grading and $\phi_H(\tau_V(L))$ with the induced grading, the $\mathbb{R}$-valued grading functions for L and $\phi_H(\tau_V(L))$ differ by a small amount. Since tangent space to $\phi_H(\tau_V(L))$ is a small clockwise rotation of the tangent space to L, the intersection point x has degree zero.

After this setup, we come to the problem of determining the set $\mathcal{M}_J(x)$ of J-holomorphic sections $u : S \to E$ satisfying $u(\partial S) \subset Q$ which are asymptotic to $x \in L \cap \phi_H(\tau_V(L))$ over the end.

Proposition 4.10. *Let* J *be a horizontal complex structure. Then* $\mathcal{M}_J(x)$ *consists of precisely one section. It is horizontal, has* $A(u) = 0$*, and is regular.*

Proof. The first step is to construct a horizontal section. Any horizontal section, if it exists, is determined by parallel transport of the point $x \in L \cap \phi_H(\tau_V(L))$ throughout E. Consider the section over S^- given by $u^- : (s,t) \mapsto (s,t,x)$. This is horizontal outside $\operatorname{supp} \beta$, since the fibration is trivial over $S^- \setminus \operatorname{supp} \beta$. In $\operatorname{supp} \beta$, the fact that x lies at a critical point of H means that $(TE^h)_{(s,t,x)}$ is the same as for the trivial connection, so the section is horizontal there as well. Near the singularity, the fibration is trivial in the band where x lies, so u^- extends to a horizontal section $u : S \to E$.

For this section, we compute $A(u) = \int_S u^*\Omega'$. Since u lies in the region where τ_V is trivial, $\int_{S \setminus \operatorname{supp} \beta} u^*\Omega' = 0$. Since u passes through the point x where $H(x) = 0$, the contribution $\int_{\operatorname{supp} \beta} u^*\Omega' = \int_{\operatorname{supp} \beta} -H(x)(\partial\beta/\partial t)\nu$ vanishes.

Since $A(u) = 0$, any $u' \in \mathcal{M}_J(X)$ must also have action 0. To see this, note that Θ' is exact on each fiber of the boundary Lagrangian Q. This implies that $\Theta'|_Q = \pi^*\kappa_Q + dK_Q$, where κ_Q is a one-form on ∂S, and K_Q is a function on Q. Thus

$$A(u) = \int_S u^*\Omega' = \int_{\partial S} u^*\Theta' = \int_{\partial S} \kappa_Q + \int_{\partial S} dK_Q \tag{81}$$

The first term on the right-hand side does not depend on u, and the second term, which is the net change in K_Q along $u(\partial S)$, depends on the asymptotic data for u, which is the same for all $u \in \mathcal{M}_J(X)$.

Because the curvature of $\pi : E \to S$ is non-negative, proposition 4.9 implies that u' is horizontal. Since u' and u are both asymptotic to x, they are equal. Hence $\mathcal{M}_J(x) = \{u\}$.

It remains to show that u is regular. The linearization of parallel transport along u trivializes $u^*(TE^v)$ such that the boundary condition $u^*(TQ)$ is mapped to a family of Lagrangian subspaces which, as we traverse ∂S in the positive sense, tilt clockwise by a small amount. That $\operatorname{ind} D_{u,J} = 0$ follows from Proposition 11.13 of [41]. On the other hand, Lemma 2.27 of [39] applies to the section u, implying $\ker D_{u,J} = 0$. We could also appeal to Lemma 11.5 of [41] (with $\mu(\rho_1) = 0$ and $|\Sigma^-| = 1$) to see that $\ker D_{u,J} = 0$. Hence $\operatorname{coker} D_{u,J} = 0$ as well. ☐

4.5. **Polygons with fixed conformal structure.** Recall from section 4.3 the trivial fibration $\pi : \hat{X}^0 \to T^0$, with fiber M, where T^0 is a a disk with $(k+3)$ boundary punctures. We have $\hat{X}^0 = M \times T^0$ symplectically. We equip $\hat{X}^0$ with a product almost complex structure $J = J_M \times j$, where j is the complex structure on T^0.

Proposition 4.11. *The (j, J)-holomorphic sections $u : T^0 \to \hat{X}^0$ are in one-to-one correspondence with (j, J_M)-holomorphic maps $u_M : T^0 \to M$.*

Proof. Given $u : T^0 \to \hat{X}^0$, write $u = (u_M, u_{T^0})$ with respect to the product splitting. Since J is a product each component is pseudo-holomorphic in the appropriate sense. But u_{T^0} is the identity map because u is a section. This correspondence is clearly invertible. ☐

This reduces the problem of counting sections $u : T^0 \to \hat{X}^0$ to the problem of counting maps $u_M : T^0 \to M$. We emphasize that we are counting maps from a Riemann surface with a fixed conformal structure.

The maps $u_M : T^0 \to M$ are holomorphic maps between Riemann surfaces, and hence their classification is mostly combinatorial. However, because we are in a situation where the conformal structure on the domain is fixed, we are not quite in the situation described, for example, in [41, §13]. The holomorphic curves we are looking for have non-convex corners and hence boundary branch points or "slits," and if the situation is complicated enough they may also have branch points in the interior. However, the condition that the conformal structure of the domain is fixed makes this an index zero problem, which is to say it prevents these slits and branch points from deforming continuously. The question is then, given a combinatorial type of such a curve, what conformal structures (with multiplicity) can be realized by holomorphic representatives?

The $(k+3)$ boundary components of T^0 are equipped with Lagrangian boundary conditions $\hat{L}(0)^0, \hat{L}(n)^0_0, \dots, \hat{L}(n)^0_k, \hat{L}(n+m)^0$. Recall that over the ends where disk bubbles are attached we have $\hat{L}(n)^0_i = \tau_i(\hat{L}(n)^0_{i-1})$, where τ_i is the monodromy around the i-critical point inside the i-th disk bubble. In section 4.4, we refined the construction and made a particular choice for this monodromy:

$$\tau_i = \phi_H \circ \tau_V \tag{82}$$

Since all of these symplectomorphisms are isotopic, we will denote them all by τ for most of this section.

In order to simplify notation, for the rest of this section 4.5 we will drop the hats and superscript zeros and denote by

$$L(0), L(n), \tau L(n), \tau^2 L(n), \dots \tau^k L(n), L(n+m) \subset M \tag{83}$$

the Lagrangians in the fiber M which give rise to the boundary conditions over the $(k+3)$-punctured disk T. Though the monodromies are all denoted by τ, we actually choose the perturbations slightly differently so as to ensure that this collection of Lagrangians is in general position in M; this is necessary for the argument in Lemma 4.20.

Recall that $q_1 = q_{a,i}$ and $q_2 = q_{b,j}$ are the morphisms whose product we wish to compute. We now regard $q_{a,i} \in L(0) \cap L(n)$ and $q_{b,j} \in \tau^k L(n) \cap L(n+m)$. Let $x_i \in \tau^{i-1} L(n) \cap \tau^i L(n)$ denote the unique intersection point.

Recall that the possible output points $q_{a+b,h} \in L(0) \cap L(n+m)$ are indexed by $h \in \{0, 1, \dots, \left\lfloor \frac{(n+m)-|a+b|}{2} \right\rfloor\}$.

We can now state the main results of this section.

Proposition 4.12. *If h is such that $0 \le h - (i+j) \le k$, then there are $\binom{k}{h-(i+j)}$ homotopy classes of maps $u : T \to M$ satisfying the boundary conditions and asymptotic to $q_{a,i}, x_1, \dots, x_k, q_{b,j}, q_{a+b,h}$ at the punctures. For h outside this range there are no holomorphic maps satisfying these conditions.*

Proposition 4.13. *For each feasible homotopy class from Proposition 4.12, and for each complex structure j on T, there is exactly one holomorphic representative $u : T \to M$.*

The strategy of proof is first to prove Proposition 4.12, which is done in §4.5.1. Then we show the existence of holomorphic representatives for some conformal structure (§4.5.2), and show that the number of representatives does not depend on the conformal structure. By degenerating the domain we are able to show uniqueness (§4.5.3).

4.5.1. *Homotopy classes.* The analysis of homotopy classes requires some explicit combinatorics, which we shall now set up. Recall that M is a cylinder with three bands attached, one for each of $L(0), L(n), L(n+m)$. We will classify homotopy classes of maps $u : T \to M$ by their boundary loop ∂u, which must be contractible, traverse $L(0), L(n), \dots, \tau^k L(n), L(n+m)$ in order, and hit the intersection points $q_{a,i}, x_1, \dots, x_k, q_{b,j}, q_{a+b,h}$ in order. That they arise as the boundary of a holomorphic map controls their behavior within the bands.

Let us use $L(n)$ to frame the cylinder, so that winding around the cylinder is computed with respect to $L(n)$. Let $x \in M$ be a basepoint which is located in the band near the intersection points x_r. Let $\alpha \in \pi_1(M, x)$ denote a loop that enters the cylinder from the bottom, veers right, winds around once, and goes back downward to x. We also have a class $\beta \in \pi_1(M)$ that is represented by the loop $L(n)$, oriented upward through the cylinder. Note that α and β generate a free group in $\pi_1(M)$.

Lemma 4.14. *$L(0)$ winds $-(n-|a|)/2$ times relative to $L(n)$. $\tau^r L(n)$ winds r times relative to $L(n)$. $L(n+m)$ winds $(m-|b|)/2+k$ times relative to $L(n)$.*

Proof. We are using the symbols $L(0), \tau^r L(n), L(n+m)$ to represent certain Lagrangians in a single fiber M, but we can compute the windings by comparing our situation to certain fibers of the original fibration. In the original fibration, $L(0)$ and $L(n)$ intersect in the fiber containing $q_{a,i}$. Thus we have that $L(0)$ winds $-(n-|a|)/2$ times relative to $L(n)$. Now $\tau^r L(n)$ winds r times relative to $L(n)$ by construction. Finally, in the original fibration $L(n+m)$ and $L(n)$ intersect in the fiber containing $q_{b,j}$, and the relative winding in that fiber is $(m-|b|)/2$. In the current context, this becomes the winding of $L(n+m)$ relative to $\tau^k L(n)$. Thus the winding of $L(n+m)$ relative to $L(n)$ in current context is $(m-|b|)/2+k$. ☐

The only unknown is how many times the boundary path traverses $L(0)$, $L(n)$, $\tau L(n)$, etc., as we traverse the boundary in the positive sense. The argument from Proposition 4.7 shows that a holomorphic map cannot enter the bands corresponding to $L(0)$ and $L(n+m)$. Hence the portion of our loop along $L(0)$ and $L(n+m)$ lies within the cylinder, and therefore it is determined by the choices of $q_{a,i}$, $q_{b,j}$ and $q_{a+b,h}$.

As for the portion of the loop along $\tau^r L(n)$, this can be represented by a sequence of integers $(\delta_r)_{r=0}^k$, where δ_r represents the number of times we wind around $\tau^r L(n)$.

With these conventions in place, we can compute the winding of a choice of paths satisfying the boundary and asymptotic conditions. We record the parts of the computation:

- Passing from x_k to $q_{b,j}$ by a short upward path on $\tau^k L(n)$ contributes

$$\left(1 - \frac{j}{(m-|b|)/2}\right)(k) \tag{84}$$

- The winding around the cylinder along $L(n+m)$ as we pass from $q_{b,j}$ to $q_{a+b,h}$ is

$$\left[\frac{h}{(n+m-|a+b|)/2} - \frac{j}{(m-|b|)/2}\right](-1)((m-|b|)/2+k) \tag{85}$$

- The winding around the cylinder along $L(0)$ as we pass from $q_{a+b,h}$ to $q_{a,i}$ is

$$\left[\frac{i}{(n-|a|)/2} - \frac{h}{(n+m-|a+b|)/2}\right](n-|a|)/2 \tag{86}$$

- Passing from $q_{a,i}$ to x_1 by a short downward path on $L(n)$ contributes 0 to the winding around the cylinder.

Adding up these contributions and using the fact that $(m-|b|)/2+(n-|a|)/2+k = (n+m-|a+b|)/2$, we get a total of $i+j-h+k$. Thus, if we go up on $\tau^k L(n)$ and down on $L(n)$, we pick up the class $\alpha^{i+j-h+k} \in \pi_1(M,x)$ for the loop from x_k to x_1.

Thus, the class $\alpha^{i+j-h+k}$ corresponds to the choice $\delta_r = 0$ for $0 \leq r \leq k$. The homotopy class of any other path can be computed from this as follows:

- Taking another path on $L(n)$ contributes a factor β^{δ_0} on the right.

- Passing from x_r to x_{r+1} along $\tau^r L(n)$ contributes the word

$$(\alpha^r\beta)^{\delta_r} \tag{87}$$

  where $\delta_r \in \mathbb{Z}$, and this class is added on the right.
- Going down on $\tau^k L(n)$ rather than up contributes the class $(\alpha^k\beta)^{\delta_k}$ on the left, which up to conjugation is the same as adding the class $(\alpha^k\beta)^{\delta_k}$ on the right.

Thus the class in question is

$$\alpha^{i+j-h+k}\prod_{r=0}^{k}(\alpha^r\beta)^{\delta_r} \tag{88}$$

The key condition is that this class is trivial in $\pi_1(M)$. This means in particular that all of the β's must cancel out. Because α and β generate a free group we have the following:

Lemma 4.15. *In the word* (88)*, the β's cancel out if and only if $\delta_r \in \{-1, 0, 1\}$ for $0 \le r \le k$, the first nonzero δ is* 1*, the last nonzero δ is* -1*, and the nonzero δ's alternate in sign.*

Proof. The first thing to see is that $|\delta_r| \le 1$ for $1 \le r \le k$. This is because $(\alpha^r\beta)^2 = \alpha^r\beta\alpha^r\beta$ contains an isolated β, while $(\alpha^r\beta)^{-2}$ contains an isolated β^{-1}. Then we can see that the nonzero δ's must alternate sign, since having two consecutive δ's equal to 1 yields $\alpha^{r_1}\beta\alpha^{r_2}\beta$, which has an isolated beta, while having two consecutive δ's equal to -1 would yield an isolated β^{-1}.

Since the nonzero δ's in the range $1 \le r \le k$ alternate in sign, all the β's coming from the range $1 \le r \le k$ cancel, except for possibly the first or the last. This implies that $|\delta_0| \le 1$ as well, since the only thing that can cancel β^{δ_0} is the first non-identity factor or the last non-identity factor.

Now the β from the first non-identity factor can only cancel the β from the last non-identity factor if all the α's as well as β's in between cancel. This means that

$$\sum_{r=a}^{b} r\delta_r = 0 \tag{89}$$

for the appropriate range of r: $a \le r \le b$. Since the δ's are in $\{-1, 0, 1\}$ and they alternate in sign, the only solution to this equation is when all $\delta = 0$. In this case, the first and the last non-identity factors are in fact consecutive, the first has $\delta = 1$, while the last has $\delta = -1$. This shows that it is impossible to have β^{-1} from the first factor cancel a β from the last factor.

In general, we find that each β is canceled by a β^{-1} from the next non-identity factor, so that the nonzero δ's alternate sign for $0 \le r \le k$, the first nonzero δ is 1, and the last nonzero δ is -1. □

By passing to $H_1(M;\mathbb{Z})$, we obtain the relations

$$\sum_{r=0}^{k} \delta_r = 0 \tag{90}$$

$$i + j - h + k + \sum_{r=0}^{k} r\delta_r = 0 \tag{91}$$

Equation (90) is implied by Lemma 4.15, while (91) determines which h the homotopy class contributes to.

The sequences $(\delta_r)_{r=0}^{k}$ which solve the constraints are in one-to-one correspondence with sequences $(s_r)_{r=0}^{k-1}$ such that $s_r \in \{0, 1\}$. In one direction, we extend the sequence by $s_{-1} = 0 = s_k$, and set

$$\delta_r = s_r - s_{r-1} \tag{92}$$

In the other direction, given δ_r, we can solve for s_r using this equation the initial condition $s_{-1} = 0$. Since the signs of the nonzero δ_r alternate, and the first nonzero term is 1, we will have $s_r \in \{0, 1\}$, and since the final nonzero term is -1, we will have $s_k = 0$, thus inverting the correspondence. This yields 2^k solutions.

Plugging this into the summation in (91), we have

$$\sum_{r=0}^{k} r\delta_r = \sum_{r=0}^{k} r(s_r - s_{r-1}) = \sum_{r=0}^{k-1} (-1)s_r \tag{93}$$

because the summation telescopes. This is simply minus the number of 1's in the sequence s_r. Thus we obtain

$$h - (i + j) = k - \sum_{r=0}^{k-1} s_r \tag{94}$$

The right hand side is always an integer between 0 and k, and it takes the value s for $\binom{k}{s}$ choices of the sequence $(s_r)_{r=0}^{k-1}$. Thus homotopy classes of maps exist for h such that $0 \leq h - (i + j) \leq k$, and there are $\binom{k}{h-(i+j)}$ such classes. This proves Proposition 4.12.

4.5.2. *Existence of holomorphic representatives for some conformal structure.* The first step in characterizing the holomorphic representatives of these homotopy classes is to prove the existence of holomorphic sections for some conformal structure. This is also essentially combinatorial.

We begin with some general concepts that will be useful in the proof.

Definition 5. Let $\gamma : S^1 \to \mathbb{C}$ be a piecewise smooth loop. A *subloop* γ' of γ is the restriction $\gamma' = \gamma| \bigcup_\alpha I_\alpha$ to a collection of intervals $\bigcup_\alpha I_\alpha$. The indexing set inherits a cyclic order from S^1, and we require that for each α, $\gamma(\max I_\alpha) = \gamma(\min I_{\alpha+1})$ is a self-intersection of γ. Thus γ' simply "skips" the portion of γ between $\max I_\alpha$ and $\min I_{\alpha+1}$. A subloop is called *simple* if it is non-self-intersecting.

Note that a subloop is not the same as a loop formed by segments of γ joining self-intersections. Such an object is only a subloop if the segments appear in a cyclic order compatible with γ.

Definition 6. A piecewise smooth loop $\gamma : S^1 \to \mathbb{C}$ is said to have the *(weak) positive winding property* (PWP) if the winding number of γ around any point in $\mathbb{C} \setminus \text{image}(\gamma)$ is non-negative. The loop γ is said to have the *strong positive winding property* (SPWP) if every subloop $\gamma' \subset \gamma$ has the positive winding property.

Lemma 4.16. *A loop $\gamma : S^1 \to \mathbb{C}$ has SPWP if and only if every* simple *subloop has PWP.*

Proof. The "only if" direction is contained in the definition. Suppose that every simple subloop of γ has PWP. If γ' is a subloop that is not simple, then by splitting γ' at a self-intersection, we can write γ' as the composition of two proper subloops. Repeating this inductively, we can write γ' as the composition of simple subloops. By hypothesis, each of these subloops winds positively, and the winding of γ' about a point is the sum of the contributions from each of the simple subloops. □

The next lemma shows that the strong positive winding property is stable under branched covers. Suppose γ is a loop and $y \in \mathbb{C} \setminus \text{image}(\gamma)$ is a point where the winding number of γ is $m > 1$. Taking the $m : 1$ cover branched at y, we can lift γ to a path $\tilde{\gamma}$. The path $\tilde{\gamma}$ is a loop that covers γ.

Lemma 4.17. *Suppose γ has SPWP. Then $\tilde{\gamma}$ has SPWP.*

Proof. Suppose that $\tilde{\gamma}$ does not have SPWP. Then some subloop $\tilde{\gamma}'$ does not have PWP. By Lemma 4.16, we may take $\tilde{\gamma}'$ to be a *simple* subloop. Thus $\tilde{\gamma}'$ winds around some region once clockwise, and we have $\tilde{\gamma}' = \partial\tilde{C}$, where $\tilde{C}$ is the chain consisting of this region with coefficient -1. Pushing $\tilde{\gamma}'$ and $\tilde{C}$ forward under the branched cover, we obtain a subloop $\gamma' \subset \gamma$, and chain C such that $\gamma' = \partial C$. Since $\tilde{C}$ is purely negative, no cancellation can occur when we push forward, and C is purely negative as well. Thus γ' winds negatively about a point in the support of C, which contradicts SPWP for γ. □

Lemma 4.18. *Suppose $\gamma : S^1 \to \mathbb{C}$ is a piecewise smooth loop with SPWP. Then there is a map $u : D^2 \to \mathbb{C}$ holomorphic on the interior of D^2 such that $\partial[u] = \gamma$.*

Proof. The idea of the proof is to iteratively take branched covers of the plane and lift γ so as to reduce the density of winding. So let $y \in \mathbb{C} \setminus \text{image}(\gamma)$ be a point where the winding number is $m > 1$. Taking the $m : 1$ branched cover at y, we obtain as in Lemma 4.17 a lift $\tilde{\gamma}$ that covers γ once and has SPWP. Repeating this process and using Lemma 4.17 to guarantee that the lift always has SPWP, eventually we obtain a *simple* piecewise smooth loop with positive winding. This loop bounds a simply connected region in M, where M is the Riemann surface resulting from the branched covering construction. By the Riemann mapping theorem, there is a map from this region to the unit disk D^2, biholomorphic on the interior and continuous on the closure, that maps each boundary segment to an arc in ∂D^2 and each corner to a point on ∂D^2. Let $\tilde{u} : D^2 \to M$ be the inverse of that map. Composing $\tilde{u}$ with the covering $M \to \mathbb{C}$ yields the desired map u. □

Fix a choice of homotopy class ϕ of polygons $u : T \to M$, which essentially means fixing a choice for the sequence $(\delta_r)_{r=0}^{k}$. Passing to the universal cover $\tilde{M}$ of the fiber, fix a choice of lift $\tilde{u} : T \to \tilde{M}$. Let $y \in \tilde{M}$ be any point. Because the boundary loop $\partial[u]$ is contractible in M, it lifts to a closed loop $\partial[\tilde{u}]$ in $\tilde{M}$. In fact $\partial[\tilde{u}]$ is contained in a domain which is isomorphic to a domain in $\mathbb{C}$, and with this identification, we have the following.

Lemma 4.19. *The boundary loop $\partial[\tilde{u}]$ has SPWP.*

Proof. The key observation is that the slopes of the paths $L(0), L(n), \ldots, \tau^k L(n), L(n+m)$ through the cylinder are positive and monotonically decreasing. If we frame the cylinder using $L(0)$, then

- $L(0)$ has slope ∞
- $L(n)$ has slope $[(n-|a|)/2]^{-1}$
- $\tau^r L(n)$ has slope $[r+(n-|a|)/2]^{-1}$
- $L(n+m)$ has slope $[(n+m-|a+b|)/2]^{-1}$.

Only at the intersection between $L(n+m)$ and $L(0)$ does the slope increase.

Hence as we traverse $\partial[\tilde{u}]$, or any subloop thereof, the slope can only increase at one point. This point is either where the subloop either uses or skips over $L(0)$.

Now we use some elementary plane geometry. Suppose that P is an oriented polygonal path in the plane (possibly self-intersecting), all of whose sides $(S_i)_{i=1}^N$ have positive slope. Suppose that P winds negatively around some point y. We claim the slope has to increase at no fewer than two vertices. We prove this by induction on the number of sides N.

Checking cases proves the claim when $N=3$ and P is a triangle. See figure 12(a).

Suppose for induction that the claim is true for $N<N_0$. Now we prove the claim if P is a *simple* N_0-gon, so assume for a contradiction that P is a simple N_0-gon that winds negatively and has at most one vertex where the slope increases. We assume that $N_0>3$, so there must be a side S_i such that at both ends there is a decrease in slope. We remove the side S_i from the N_0-gon P and extend the two incident sides S_{i-1} and S_{i+1} in order to obtain an (N_0-1)-gon P'. Since P has at most one vertex where the slope increases, so does P': Since the slope decreased at both $S_{i-1}S_i$ and at S_iS_{i+1}, it will decrease at the new vertex $S_{i-1}S_{i+1}$. See Figure 12(b). Since the original simple loop P winds negatively, the new loop P' must also wind negatively around some point. This contradicts the induction hypothesis. Now we have established the claim for simple N_0-gons.

Now suppose P is a self-intersecting N_0-gon, that winds negatively around y. Using Lemma 4.16, we can find a simple subloop P' winding negatively around y. Since P' is a simple N-gon for some $N \leq N_0$, it must have at least two vertices where the slope increases. For each vertex v of P' where the slope of P' increases, there is a vertex in the original polygon P where the slope increases, either at v itself, or at some point in the interval of P that was deleted at v. Thus, since P' has at least two vertices where the slope increases, so does P. □

Having chosen a lift $\partial[\tilde{u}]$ of the boundary loop, define a 2-chain C on $\tilde{M}$ whose multiplicity at y is the winding number of $\partial[\tilde{u}]$ around y. This has $\partial C = \partial[\tilde{u}]$.

Lemma 4.20. *There is a complex structure j on T and a holomorphic map $\tilde{u} : T \to \tilde{M}$ such that $\tilde{u}_*[T] = C$.*

Proof. Lemma 4.19 allows us to apply Lemma 4.18, which yields map $\tilde{u} : T \to \tilde{M}$. More precisely, the complex structure on T is the one obtained from uniformization of the region bounded by the simple lift of $\gamma = \partial[u]$ at the end of the construction in 4.18, as a Riemann surface with boundary and punctures (at the non-smooth points of the loop). □

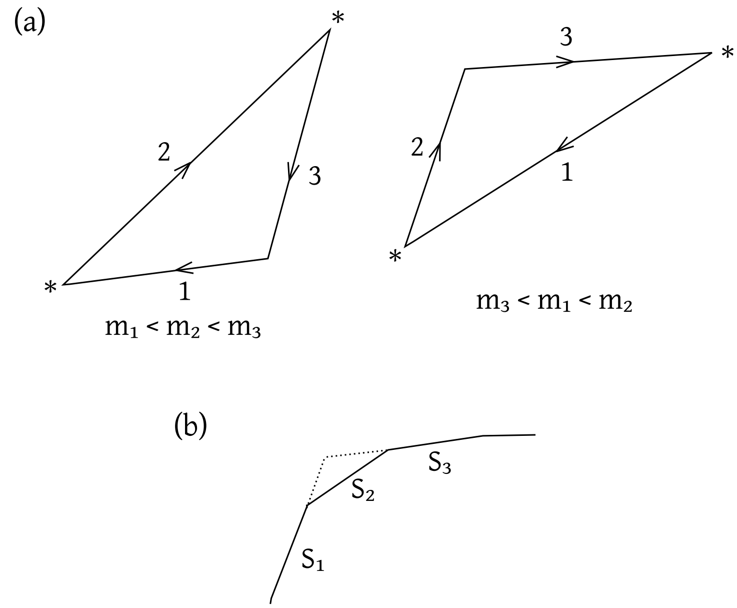

FIGURE 12. (a) Negatively winding triangles with positive slopes have two slope increases (m denotes slope). (b) Removing the side S_2.

Pushing the map $\tilde{u}$ from Lemma 4.20 down to M, we obtain the existence of a holomorphic representative in the homotopy class ϕ, for a particular complex structure on the domain.

4.5.3. *The moduli space of holomorphic representatives with varying conformal structure.* Let $\mathcal{M}(\phi, j)$ denote the moduli space of (j, J_M)-holomorphic maps $u : T \to M$ in the homotopy class ϕ. Let $\mathcal{M}(\phi) = \bigcup_j \mathcal{M}(\phi, j)$ denote the moduli space of such maps with varying conformal structure on the domain. Let $\mathcal{R}^{k+3}$ denote the moduli space of conformal structures on the disk with $(k+3)$ boundary punctures. There is a natural map $\pi : \mathcal{M}(\phi) \to \mathcal{R}^{k+3}$ which forgets the map.

Lemma 4.21. *For any $u \in \mathcal{M}(\phi, j)$, we have* $\operatorname{ind} D_{u,(j,J_M)} = 0$ *and* $\ker D_{u,(j,J_M)} = 0$.

Proof. Let the intersection points $q_{a,i}, x_1, \dots, x_k, q_{b,j}$ be regarded as positive punctures and let $q_{a+b,h}$ be regarded as a negative puncture. Then by the conventions for Maslov index, we have that all of these intersection points have index 0, for as we go $L(0) \to L(n)$, $L(n) \to \tau L(n)$, $\dots$, $\tau^k L(n) \to L(n+m)$, and $L(0) \to L(n+m)$, the Lagrangian tangent space tilts clockwise by a small amount. Then by Proposition 11.13 of [41] (with $|\Sigma^-| = 1$ for the negative puncture), we have $\operatorname{ind} D_{u,(j,J_M)} = 0$.

Furthermore, the operator $D_{u,(j,J_M)}$ is a Cauchy–Riemann operator acting on the line bundle u^*TM, so the results of Section (11d) of [41] apply. The hypotheses of Lemma 11.5 are satisfied with $\mu(\rho_1) = 0$ and $|\Sigma^-| = 1$, so $\ker D_{u,(j,J_M)} = 0$. □

Lemma 4.22. *The map $\pi : \mathcal{M}(\phi) \to \mathcal{R}^{k+3}$ is a proper submersion of relative dimension zero (that is, a finite covering).*

Proof. The index $\operatorname{ind} D_{u,(j,J_M)} = 0$ is the expected dimension of the space of curves with fixed conformal structure. When we allow the conformal structure to vary, this adds $k = \dim \mathcal{R}^{k+3}$ dimensions, so $\mathcal{M}(\phi)$ has expected dimension k. Lemma 4.21 implies coker $D_{u,(j,J_M)} = 0$, so $\mathcal{M}(\phi)$ is a manifold of dimension k.

If $u \in \mathcal{M}(\phi, j) \subset \mathcal{M}(\phi)$ is a point where the map $\pi : \mathcal{M}(\phi) \to \mathcal{R}^{k+3}$ is not a submersion, we must have $\ker D\pi \neq 0$. On the other hand $\ker D\pi$ consists of infinitesimal deformations of the map which do not change the conformal structure on the domain, and so is equal to $\ker D_{u,(j,J_M)}$, which is zero by the previous lemma. Hence π is a submersion.

The properness of π is an instance of Gromov–Floer compactness. The only thing to check is whether, as we vary $j \in \mathcal{R}^{k+3}$, any strips can break off. This is impossible because our boundary conditions do not bound any bigons in M. □

Lemma 4.23. *The map $\pi : \mathcal{M}(\phi) \to \mathcal{R}^{k+3}$ has degree one.*

Proof. For this Lemma we will pass to the Gromov–Floer compactification $\bar{\pi} : \bar{\mathcal{M}}(\phi) \to \bar{\mathcal{R}}^{k+3}$. Because no bigons can break off, this compactification consists entirely of stable disks, and $\bar{\pi}$ is also a proper submersion. Hence to count the degree of π, it will suffice to count the points in the fiber of $\bar{\pi}$ over a corner of $\bar{\mathcal{R}}^{k+3}$, which is to say when the domain is a maximally degenerate stable disk.

A maximally degenerate stable disk $S = (G, (S_\alpha))$ consists of a trivalent graph $G = (V, E^{fin} \cup E^\infty)$ without cycles, with $(k+3)$ infinite edges E^∞, and a disk S_α with three boundary punctures for each $\alpha \in V$. The boundary punctures of S^α are labeled by elements of $E^{fin} \cup E^\infty$. The elements of E^{fin} correspond to nodes of the stable disk, while the elements of E^∞ correspond to boundary punctures of the smooth domains in $\mathcal{R}^{k+3}$. The homotopy class ϕ determines the Lagrangian boundary conditions on each component S_α, and the asymptotic values at the boundary punctures labeled by E^∞. The position of the nodes labeled by E^{fin} is not determined *a priori*.

Looking at the Lagrangians $L(0)$, $L(n)$, $\ldots$, $L(n+m)$ shows that any three of them bound triangles, and that such triangles are determined by two of the corners. Hence by tree-induction starting at the leaves of stable disk (those S_α for which two punctures are labeled by E^∞), the positions of all the nodes are determined by ϕ, or we run into a contradiction because no triangles consistent with the labeling exist.

Furthermore, in each homotopy class of triangles consistent with the labeling of S_α, there is exactly one holomorphic representative.

Hence there is at most one stable map from the stable domain $S = (G, (S_\alpha))$ to M consistent with the homotopy class ϕ.

Thus we have shown that the degree of $\pi : \mathcal{M}(\phi) \to \mathcal{R}^{k+3}$ is either zero or one. On the other hand, Lemma 4.20 shows that $\mathcal{M}(\phi)$ is not empty, so the degree must be one. □

Proposition 4.13 follows immediately from Lemmas 4.22 and 4.23.

4.6. **Signs.** In order to determine the signs appearing in the counts of triangles, we need to specify the brane structures on the Lagrangians $L(d)$. Since $L(d)$ fibers over a curve in the base, and its intersection with each fiber is a curve, the tangent bundle of $L(d)$ is trivial, and we can define a framing of $TL(d)$ using the vertical and horizontal tangent vectors at each point. Using this framing, we can give $L(d)$ a trivial Spin(2) structure, which is induced by product of the trivial Spin(1) structures on the horizontal an vertical tangent bundles.

Although we have not said much about it up until now, strictly speaking the generators of $CF^*(L(d_1), L(d_2))$ are not canonically identified with intersection points $q \in L(d_1) \cap L(d_2)$. Rather, each intersection point q gives rise to an abstract 1-dimensional $\mathbb{R}$–vector space, the orientation line $o(q)$. Following [41, Ch. 11], this line is canonically isomorphic up to multiplication by a positive number with the determinant line of a certain Fredholm operator D_q. The moduli space of pseudo-holomorphic curves is canonically oriented relative to these lines.

In the case $d_1 \leq d_2$, where all the intersections have degree 0, there is a preferred choice of trivialization for $o(q)$. Let $q \in L(d_1) \cap L(d_2)$. Then we have horizontal-vertical splittings

$$T_q L(d_i) = (T_q L(d_i))^h \oplus (T_q L(d_i))^v \tag{95}$$

The intersection point q has degree 0 as a morphism from $L(d_1)$ to $L(d_2)$, and moreover both $(T_q L(d_i))^h$ and $(T_q L(d_i))^v$ tilt clockwise by a small amount as we pass from $L(d_1)$ to $L(d_2)$.

Let H denote the half-plane with a negative puncture. We define the orientation operator $D_q = D_q^h \oplus D_q^v$, acting on the product bundle $\mathbb{C} \times \mathbb{C} \to H$, where the boundary condition in the first factor is the short path $(T_q L(d_1))^h \to (T_q L(d_2))^h$, while that in the second factor is the short path $(T_q L(d_1))^v \to (T_q L(d_2))^v$. By [41, Eq. (11.39)], there is a canonical isomorphism

$$\det(D_q) \cong o(q) \tag{96}$$

which for our purposes we take as the definition of $o(q)$. On the other hand, D_q is the direct sum of the operators D_q^h and D_q^v, which have vanishing kernel and cokernel. Hence

$$\det(D_q) \cong \det(D_q^h) \otimes \det(D_q^v) \cong \mathbb{R} \otimes \mathbb{R} \cong \mathbb{R} \tag{97}$$

where all isomorphisms are canonical. This gives us a preferred choice of isomorphism $o(q) \cong \mathbb{R}$.

Proposition 4.24. *Taking the preferred isomorphisms $o(q) \cong \det(D_q) \cong \mathbb{R}$ for all generators $q \in CF^0(L(d_1), L(d_2))$ for $d_1 \leq d_2$, all of the holomorphic triangles found above have the same sign. Hence, after possibly reversing all off the preferred isomorphisms $o(q) \cong \mathbb{R}$, all of them have a positive sign.*

Proof. Let $u : S \to X(B)$ be a triangle with positive punctures at $q_1 \in CF^0(L(d_1), L(d_2))$ and $q_2 \in CF^0(L(d_2), L(d_3))$ and negative puncture at $q_0 \in CF^0(L(d_1), L(d_3))$. Let D_u denote the linearized operator at u. Gluing onto D_u the chosen orientation operators D_{q_2} and D_{q_1} in that order gives another orientation operator D'_{q_0} for the point q_0, and we want to compare the two orientations for the line $o(q_0)$.

The linearized operator D_u is an operator in the pulled back tangent bundle $u^*TX(B)$. The bundle $TX(B)$ has a splitting into horizontal and vertical subspaces $TX(B) = TX(B)^h \oplus TX(B)^v$ given by the symplectic connection. Over the boundaries of S, the Lagrangian boundary condition also splits. Recall from the discussion on gradings in Section 3.3.5 that we have phase maps on horizontal and vertical subspaces, which measure rotation with respect to the foliations by circles on the base and fiber respectively. These serve to give us preferred trivializations of $u^*TX(B)^h$

and $u^*TX(B)^v$. With respect to these trivializations, the Lagrangian boundary conditions move only by a small amount: in the horizontal space, TL^h has essentially constant phase, while in the vertical space, TL^v moves in such a way that it never crosses the the line corresponding to the circle foliation on the fibers.

The next step is to observe that the boundary conditions for the operator D_u are essentially the same independent of the map u and of the chosen degree zero generators q_0, q_1, q_2, and that hence all of the maps u must contribute with the same sign. In all cases we have an operator in a trivial rank two vector bundle, with split Lagrangian boundary conditions, each factor of which is trivial. Additionally, each pair of Lagrangians at a puncture has the same local form, namely they are related by a small clockwise rotation in both the horizontal and vertical directions. The orientation operators D_{q_i} used to define the orientation lines are also essentially the same for every point q_i. Hence the relative sign between the determinant of D'_{q_0} and $o(q_0)$ must be the same for each map u.

If the signs are all positive, we are done. Otherwise by reversing our choice of generator for every line $o(q)$ we obtain bases for the groups $CF^0(L(d_1), L(d_2))$ in which all curves contribute positively.

□

Remark 10. The preceding proposition is another example of the phenomenon observed by Abouzaid [1, Lemma 3.21]. A similar argument is found in [41, (13c)].

5. A tropical count of triangles

Abouzaid, Gross and Siebert have proposed a definition of a category defined from the tropical geometry of an integral affine manifold, which is meant to describe some part of the Fukaya category of the corresponding symplectic manifold. The starting point for this definition is an integral affine manifold B. The objects are then the non-negative integers, with $\hom^0(d_1, d_2) = \operatorname{span} B(\frac{1}{d_2-d_1}\mathbb{Z})$ when $d_1 < d_2$, and chains on B when $d_1 = d_2$. The composition is defined by counting a certain type of tropical curve that is balanced after addition of *tropical disks*.

The motivation is that the non-negative integers correspond to certain Lagrangian sections $L(n)$ of a special Lagrangian torus fibration over B, such that the intersection points between $L(d_1)$ and $L(d_2)$ lie precisely over the points in $B(\frac{1}{d_2-d_1}\mathbb{Z})$. The tropical curves then correspond to the pseudo-holomorphic polygons counted in the A_∞ operations.

The Lagrangians considered above are essentially an example of this symplectic setup, so it is encouraging that our computation agrees with the expectation of Abouzaid-Gross-Siebert. The tropical triangles counted in their definition correspond closely to the pseudo-holomorphic triangles found in Section 4.

5.1. Tropical polygons.

Let ∇ denote the canonical torsion-free flat connection on B associated to the affine structure. The following definitions are due to Abouzaid [4].

Definition 7. Let B be a two-dimensional singular affine manifold with focus-focus singularities, and let $x \in B$ be a non-singular point. Let Γ be a tree all of whose edges are finite. Label one univalent vertex p_{out}, and denote the other univalent vertices by $p_1, \dots, p_r$. A *tropical disk* in B ending at x is a balanced tropical embedding $v : \Gamma \to B$, satisfying the following conditions.

(1) $v(p_{\mathrm{out}}) = x$,
(2) For $i = 1, \dots, r$, $v(p_i)$ is a singular point of the affine structure, and v maps the (unique) edge incident to p_i into a monodromy-invariant line of the singularity.

In the case at hand, there is only one singularity, so the only such tropical disks are line segments emanating from the singular point in the monodromy-invariant direction. If there were more singularities with intersecting monodromy-invariant lines, we could obtain more complicated tropical disks.

Definition 8. Let $q_0, q_1, \dots, q_k$ be points of B, with $q_j \in B(\frac{1}{d_j}\mathbb{Z})$. Let Γ be a metric ribbon tree with $k+1$ infinite edges. One infinite edge, the *root*, is labeled with q_0 and it is the output. The other k infinite edges, the *leaves*, are labeled with $q_1, \dots, q_k$ in counterclockwise order and these are the inputs. Assign to the region between q_j and q_{j+1} the weight $\sum_{i=1}^{j} d_i$, and give the region between q_0 and q_1 weight 0. Orient the tree upward from the root, so that each edge has a "left" and a "right" side coming from the ribbon structure. To each edge e, assign a weight w'_e given by the weight on the left side of e minus the weight on the right side of e. Define a corrected weight w_e by

$$\begin{aligned} w_e &= 0 \text{ if } w'_e < 0 \text{ and } e \text{ contains a leaf} \\ w_e &= 0 \text{ if } w'_e > 0 \text{ and } e \text{ contains the root} \\ w_e &= w'_e \text{ otherwise} \end{aligned} \tag{98}$$

Then a *tropical polygon* modeled on Γ is a map $u : \Gamma \to B$ such that:

(1) u converges on the root edge to q_0 and on the j-th leaf edge to q_j;
(2) on the edge e, the tangent vector $\dot{u}_e$ to the component u_e satisfies
$$\nabla_{\dot{u}_e} \dot{u}_e = w_e \dot{u}_e \tag{99}$$
and $\dot{u}_e$ converges to 0 on the infinite ends of the root and leaf edges; this differential equation only holds outside a finite set of points where tropical disks are attached;
(3) there exists a finite collection of tropical disks $v_1, \dots, v_N$ such that the union $u \cup \{v_1, \dots, v_N\}$ is balanced; the balancing condition at a vertex x is the vanishing of the sum of the derivative vectors $\dot{u}_e$ of the various components of u incident at x and the integral tangent vectors to v_i at x, oriented toward the vertex.

The heuristic motivation for these definitions is that, if we consider a deformation of complex structures in which the torus fibers collapse, we expect that a family of honest holomorphic polygons will likewise collapse onto a piecewise linear complex in B. We further expect that two types of local limiting behavior can occur. Some parts of the curve may limit to surfaces that fiber over straight lines in B intersecting the torus fibers in paths joining two Lagrangian sections, while other parts may limit to surfaces that fiber over straight lines in B intersecting the torus fibers in circles. The latter are what become the tropical disks, while the former become the rest of the tropical polygon. The balancing condition comes from the necessity that these parts can be connected up to form a topological polygon with boundary on the Lagrangian sections. See also [6, Ch. 8].

To unpack this definition, let us restate it in the simplest case, which is that of tropical triangles (this mainly simplifies the issues regarding weights):

Proposition 5.1. *Let $q_1 \in B(\frac{1}{n}\mathbb{Z})$ and $q_2 \in B(\frac{1}{m}\mathbb{Z})$, with $n > 0$ and $m > 0$. Let $q_0 \in B(\frac{1}{n+m}\mathbb{Z})$. Let Γ be the ribbon tree with one vertex and three infinite edges. Then a tropical triangle modeled on Γ consists of three maps $u_0 : [0,\infty) \to B$, $u_1 : (-\infty, 0] \to B$, $u_2 : (-\infty, 0] \to B$ such that*

1. *$u_0 \equiv q_0$ is a constant map;*
2. *$u_1(0) = u_2(0) = q_0$;*
3. *$u_1(-\infty) = q_1$, $u_2(-\infty) = q_2$;*
4. *We have $\nabla_{\dot{u}_1}\dot{u}_1 = n\dot{u}_1$ and $\nabla_{\dot{u}_2}\dot{u}_2 = m\dot{u}_2$, outside of a finite set of points where tropical disks are attached;*
5. *there exists a finite collection of tropical disks $v_1, \dots, v_N$ such that the balancing condition holds.*

We can also unpack the equation $\nabla_{\dot{u}}\dot{u} = n\dot{u}$. The key outcome of the following Lemmas is the insight that the tangent vector $\dot{u}$ increases by n times the distance the path u travels. Recall that a path γ is called a geodesic for a connection ∇ if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

Lemma 5.2. *Let $\gamma : [0,1] \to B$ be a geodesic for ∇. Define $u : (-\infty, 0] \to B$ by $u(t) = \gamma(\exp(nt))$. Then at the point $x = \gamma(s)$, we have $\dot{u} = ns\dot{\gamma}$, and $\nabla_{\dot{u}}\dot{u} = n\dot{u}$.*

Proof. We set $s = \exp(nt)$. The equation $\dot{u} = ns\dot{\gamma}$ is just the chain rule (note that $\dot{u} = du/dt$ while $\dot{\gamma} = d\gamma/ds$). We have

$$\nabla_{\dot{\gamma}}\dot{u} = \nabla_{\dot{\gamma}}(ns\dot{\gamma}) = n\dot{\gamma} + ns\nabla_{\dot{\gamma}}\dot{\gamma} = n\dot{\gamma} \tag{100}$$

since γ is a geodesic. Multiplying this equation by ns, and using the fact that ∇ is tensorial in its subscript, we obtain $\nabla_{\dot{u}}\dot{u} = n\dot{u}$. □

Lemma 5.3. *Let $u : [a,b] \to B$ solve $\nabla_{\dot{u}}\dot{u} = n\dot{u}$. Then if we patch together affine charts along u to embed a neighborhood of u into $\mathbb{R}^n$, we have*

$$\dot{u}(b) - \dot{u}(a) = n(u(b) - u(a)) \tag{101}$$

where we use the affine structure of $\mathbb{R}^n$ to take differences of points and vectors at different points.

Proof. Define $\gamma : [\exp(na), \exp(nb)] \to B$ by $\gamma(s) = u((\log s)/n)$, so that γ is a geodesic. We can embed a neighborhood of γ into $\mathbb{R}^n$ by patching together affine coordinate charts along γ. Using addition in this embedding, we can write $\gamma(s) = \gamma(\exp(na)) + (s - \exp(na))\dot{\gamma}$. We have $\dot{u}(\log(s)/n) = ns\dot{\gamma}(s)$, and

$$\dot{u}(b) - \dot{u}(a) = n(\exp(nb) - \exp(na))\dot{\gamma} = n(\gamma(\exp(nb)) - \gamma(\exp(na))) = n(u(b) - u(a)) \tag{102}$$

□

One more thing to note regarding these tangent vectors $\dot{u}$ is how they represent homology classes on the torus fibers of the fibration over B.

When considering a tropical curve C^{trop} corresponding to a closed holomorphic curve C, each edge of the tropical curve carries an integral tangent vector, which morally represents the class $[C \cap T_b^2] \in H_1(T_b^2;\mathbb{Z})$ that measures how the holomorphic curve intersects the torus fiber $T_b^2 = \pi^{-1}(b)$. Because the curve is closed, this class is locally constant along each edge of C^{trop}.

When considering a tropical curve C^{trop} representing a holomorphic curve C with boundary on Lagrangian sections $L(i), L(j)$, the intersection of C with T_b^2 would morally be a path on T_b^2 from $L(i)_b$ to $L(j)_b$, where $L(i)_b$ is intersection of T_b^2 and $L(i)$. Let $A(L(i)_b, L(j)_b) \subset H_1(T_b^2, \{L(i)_b, L(j)_b\};\mathbb{Z})$ be the subset consisting of such cycles, which could also be described as the preimage of $L(j)_b - L(i)_b \in H_0(\{L(i)_b, L(j)_b\};\mathbb{Z})$ under the boundary homomorphism. Hence $A(L(i)_b, L(j)_b)$ is a torsor for the kernel of that homomorphism, which is $H_1(T_b^2;\mathbb{Z})$. Using the group structure on T_b^2, we can identify $A(L(i)_b, L(j)_b)$ with the coset

$$[L(j)_b - L(i)_b] + H_1(T_b^2;\mathbb{Z}) \subset H_1(T_b^2;\mathbb{R}) \tag{103}$$

On the other hand, there is an isomorphism $(T_bB)_{\mathbb{R}} \cong H_1(T_b^2;\mathbb{R})$. Hence the class of $[C \cap T_b^2]$ can be regarded as a tangent vector to the base, which is in general real and varies along the tropical curve as $L(i)$ and $L(j)$ move relative to one another. The tangent vector $\dot{u}$ to the tropical curve is this class.

The balancing condition at a vertex b of the tropical polygon amounts to requiring that the three paths $L(i)_b \to L(j)_b$, $L(j)_b \to L(k)_b$ and $L(k)_b \to L(i)_b$ form a contractible loop. At a point where a tropical disk is attached, the path $L(i)_b \to L(j)_b$ changes discontinuously by a loop in the homology class in $H_1(T_b^2;\mathbb{Z})$ corresponding to the integer tangent vector to the tropical disk.

The last thing to describe for tropical polygons is their multiplicities. There is a multiplicity coming from the different ways to attach a disk v to the tropical polygon. If one incoming edge of the polygon has tangent vector $\dot{u}_e$ corresponding to a path $\gamma_1 : L(i)_b \to L(j)_b$ on T_b^2, and the disk has tangent vector w corresponding to a loop γ_2 on T_b^2, there are $|\gamma_1.\gamma_2| = |\det(\dot{u}_e, w)|$ points where the disk can be attached, assuming this determinant is an integer (as it is in the special case below). Otherwise, one must look carefully at exactly where on the torus the paths γ_1 and γ_2 are located.

For a tropical disk v, we also expect the Gross–Siebert theory to associate a multiplicity $m(v)$. Morally speaking, this number should be a virtual count of holomorphic disks corresponding to v, and it should possible to extract this from the "structure" on the affine base in the sense of [22]. In the case at hand, where the base has just a single focus-focus singularity, all of these factors are expect to be one for simple disks, and zero for multiply covered disks [2]. In the rest of this section, we adopt this as an ansatz. Modulo this ansatz, we show in Proposition 5.4 that the tropical curve counts agree with the holomorphic curve counts computed in Section 4. From a certain point of view, this may be regarded as evidence for the ansatz itself, since tropical curve invariants are intentionally designed to correspond to holomorphic curve invariants, and we have computed the latter.

In summary, the multiplicity of a tropical polygon will have both the Gross–Siebert factors $m(v)$ counting how many holomorphic disks are in each class, as well as simpler factors counting how many ways these classes of disks can be attached.

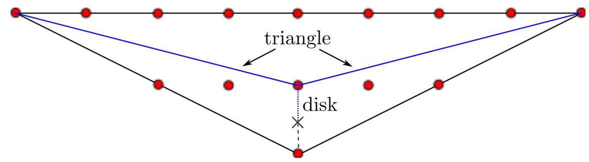

FIGURE 13. A tropical triangle. The dotted line extending up from the singular point is a tropical disk.

Remark 11. After this paper was originally written, an exposition of the closely related idea of *jagged paths* appeared [18, Definition 3.2]. In that paper, the various multiplicities are packaged in a different way, but it turns out the that tropical curve counts defined there also agree, in the case of $(\mathbb{CP}^2, D)$, with the holomorphic curve counts computed in Section 4. The multiplicities we seek are packaged into certain series $f_{\mathfrak{d}} = 1 + \sum_p c_p z^p$ where $\mathfrak{d}$ runs over certain distinguished rays and lines in B. See [18, §3] for more explanation. In our case, there are two such rays emanating from the singularity of the affine structure in the monodromy invariant directions. Our ansatz is equivalent to the fact that the corresponding series reduce, in appropriate choices of coordinates, to $1 + z$ and $1 + w$ respectively. The salient feature is that these series have a linear term with coefficient 1 and no higher degree terms.

5.2. **Tropical triangles for** $(\mathbb{CP}^2, D)$**.** We now write out explicitly the tropical curves contributing to the triangle products in the case of $(\mathbb{CP}^2, D)$. Let us use coordinates (η, ξ) where the point $q_{a,i} \in B(\frac{1}{n}\mathbb{Z})$ has coordinates $(a/n, -i/n)$

There is one family of simple tropical disks that emanate from the singularity on the $\eta = 0$ line in the vertical direction. Their primitive tangent vectors are $\pm(0, 1)$.

Figure 13 shows the the tropical triangle representing the contribution of y^2p to the product of x^2 and z^2. This triangle has multiplicity 2. The singularity of the affine structure is placed so as to emphasize the tropical disk ending at the singularity.

Proposition 5.4. *Let $n > 0$ and $m > 0$, and take $q_{a,i} \in B(\frac{1}{n}\mathbb{Z})$, $q_{b,j} \in B(\frac{1}{m}\mathbb{Z})$, and $q_{a+b,h} \in B(\frac{1}{n+m}\mathbb{Z})$.*

Suppose a and b have different signs, and let $k = \min(|a|, |b|)$, and $h = i + j + s$. If $0 \leq s \leq k$, there is one tropical triangle connecting these three points. It is balanced after the addition of either s or $k - s$ tropical disks, depending on the position of the singularity. The multiplicity of this curve is $\binom{k}{s}$. If s does not lie in this range, there is no triangle.

Suppose that a and b have the same sign. Then unless $h = i + j$ there is no triangle, and when $h = i + j$ there is exactly one, which is represented geometrically by the line segment joining $q_{a,i}$ and $q_{b,j}$.

Proof. When a and b have the same sign the tropical triangle can have no tropical disks attached to it. Therefore the tropical triangle is simply a line segment which passes through the three points, and this only exists if $q_{a+b,h}$ lies on the line between $q_{a,i}$ and $q_{b,j}$.

As for when a and b have different signs, let us consider the case $a \leq 0$, $b \geq 0$, $a + b \geq 0$, and hence $k = -a$. The other cases are related to this by obvious reflections and re-namings.

By Proposition 5.1, a tropical triangle consists essentially of two maps $u_1, u_2 : (-\infty, 0] \to B$, together with some copies of the tropical disk and their multiples.

- The leg u_2 of the tree connecting $q_{b,j}$ to $q_{a+b,h}$ cannot have any tropical disks attached, since both endpoints lie on the same side of the line $\eta = 0$. We can apply Lemma 5.3 to obtain the tangent vector $\dot{u}_2$ at $q_{a+b,h}$ as m times the difference between the endpoints, or

$$\dot{u}_2(q_{a+b,h}) = m(q_{a+b,h} - q_{b,j}) = \left(\frac{ma - nb}{n+m}, \frac{-(mi - nj + ms)}{n+m}\right) \tag{104}$$

- The leg u_1 of the tree connecting $q_{a,i}$ to $q_{a+b,h}$ crosses the line $\eta = 0$ at some point x, where tropical disks can be attached. It may bend there, and continue on to $q_{a+b,h}$, where the balancing condition $\dot{u}_1 + \dot{u}_2 = 0$ must hold. This shows that the portion of u_1 connecting x to $q_{a+b,h}$ must be parallel to u_2. Hence x must be on the line joining $q_{b,j}$ and $q_{a+b,h}$. We obtain the position of x:

$$x : (\eta, \xi) = (0, (-aj + bi + bs)/(ma - nb)) \tag{105}$$

- By Lemma 5.3 at x the tangent vector $\dot{u}_1$ is given by n times the difference of the endpoints x and $q_{a,i}$:

$$\dot{u}_1(x)_L = n(x - q_{a,i}) = (-a, [a(mi - nj) + nbs]/(ma - nb)) \tag{106}$$

  we use the subscript L to denote this is the tangent vector coming from the left.
- If the singularity of the affine structure occurs below the point x, then tropical disks are line segments going up from the singularity to x in the direction $(0, 1)$. Adding the vector $(0, s)$ to $\dot{u}_1(x)_L$, we obtain $\dot{u}_1(x)_R$, the tangent vector from the right,

$$\dot{u}_1(x)_R = \dot{u}_1(x)_L + (0, s) = (-a, a(mi - nj + ms)/(ma - nb)) \tag{107}$$

  which is parallel to $\dot{u}_2(q_{a+b,h})$, as it must be. This shows that we must attach a collection of tropical disks whose total weight is s.
- If the singularity of the affine structure occurs above x, then tropical disks are line segments going down from the singularity to x in the direction $(0, -1)$, but there is also the monodromy to be taken into account. First we must act on $\dot{u}_1(x)_L$ by the monodromy $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ to get

$$M\dot{u}_1(x)_L = (-a, [a(mi - nj) + nbs]/(ma - nb) - a) \tag{108}$$

  Adding the vector $(0, -(k - s))$ to this, with $k = -a$, gives the same result as before for $\dot{u}_1(x)_R$. However, in this case we are attaching a collection of disks with total weight $(k - s)$.
- The leg u_1 propagates in the direction $\dot{u}_1(x)_R$ from x to $q_{a+b,h}$. As it does so, the tangent vector $\dot{u}_1$ increases by $\Delta = n(q_{a+b,h} - x)$, which is parallel to $\dot{u}_1(x)_R$. By comparing affine lengths of the segment of u_1 from $q_{a,i}$ to x and the segment of u_1 from x to $q_{a+b,h}$, we have the proportion

$$\dot{u}_1(x)_R : \Delta = [-a/n] : [(a + b)/(n + m)] \tag{109}$$

and

$$(\dot{u}_1(x)_R+\Delta) : \dot{u}_1(x)_R = [(a+b)/(n+m)-a/n] : [-a/n] = [-(ma-nb)/(n+m)] : [-a] \tag{110}$$

Thus

$$\dot{u}_1(q_{a+b,h}) = \left(-\frac{ma-nb}{n+m}, \frac{mi-nj+ms}{n+m}\right) = -\dot{u}_2(q_{a+b,h}) \tag{111}$$

Verifying the balancing condition at $q_{a+b,h}$.

Now that we know which tropical curves contribute, we must compute their multiplicities. The tropical curve constructed above uses the tropical disk s or $(k-s)$ times. This means that either we attach a single simple disk s times, or we attach some multiple covers of the disk in some fashion as to achieve a total multiplicity of s. By our ansatz, only the simple disks contribute, and each with multiplicity one.

Attaching simple disks does introduce a multiplicity, since there are multiple places to attach this disk. In fact, we have $\det(\dot{u}_1(x)_L, (0,1)) = -a = k$, so there are a total of k places for disks to be attached. Thus we get the multiplicity $\binom{k}{s}$ or $\binom{k}{k-s}$, which are equal and give the desired result.

□

Remark 12. The fact that the multiplicities $\binom{k}{s}$ and $\binom{k}{k-s}$ are equal is an illustration of the general phenomenon that the exact position of the singularity along its invariant line does not matter for tropical curve counts.

6. Parallel monodromy–invariant directions

In this section we describe a class of affine manifolds to which the results obtained for $(\mathbb{CP}^2, D)$ naturally generalize. Consider an integral affine manifold B, compact with boundary. We require that B has the following properties. Recall that a focus-focus singularity has a monodromy invariant direction, corresponding to the tangent vector field that is fixed by the monodromy.

(1) The boundary faces of B are straight with respect to the affine structure.
(2) The singularities of B are focus-focus singularities, and the monodromy–invariant directions of all singularities are parallel.
(3) There is an affine linear function $\eta : B \to \mathbb{R}$ such that any corners of B occur at extreme values of η.

Remark 13. The condition that the monodromy–invariant directions are parallel is equivalent to the existence of a global integral vector field, or a vector invariant under parallel transport along an arbitrary path in B minus the singular points. This vector field spans $\ker d\eta$.

6.1. **Symplectic forms.** Let $x_1, \ldots, x_n$ denote the singularities of B. Let $[a_0, a_{n+1}]$ be the image of B under η, and let $a_i = \eta(x_i)$. We split up B along the monodromy invariant lines of each singularity, and obtain intervals $I_i = [a_i + \epsilon, a_{i+1} - \epsilon]$, with fibrations $X(B_i) \to X(I_i)$. Choosing an affine function ξ on B_i such that (η, ξ) are coordinates, $X(B_i)$ has corresponding complex coordinates (w, z_i), while the coordinate on $X(I_i)$ is w.

Each piece B_i is an affine manifold whose horizontal boundary consists of two straight lines (since B has no corners but at the extreme values of η), and hence we are in the situation of section 3.1. We obtain a symplectic form for which the symplectic connection of $X(B_i) \to X(I_i)$ foliates the horizontal boundary facets of $X(B_i)$.

Corresponding to each focus–focus singularity, we glue in a Lefschetz singularity. The discussion in section 3.2 applies directly. The result is a manifold $X(B)$ with a Lefschetz fibration $w : X(B) \to X(I)$, and such that the horizontal boundary faces of $X(B)$ are foliated by the symplectic connection. Let $w_1, \ldots, w_n$ denote the critical values of $w : X(B) \to X(I)$.

If B does not have vertical boundary facets, but rather corners, we simply cut off the corners to introduce horizontal boundary facets. This modification only applies to the symplectic geometry constructions, and the relevant tropical geometry is still that of the original B.

We recall that $X(I)$ is an annulus $\{a_0 \le \log|w| \le a_{n+1}\}$. Let M_0 denote the fiber over e^{a_0}, and M_1 the fiber over $e^{a_{n+1}}$.

6.2. **Lagrangian submanifolds.** As before, the construction of Lagrangian sections proceeds by taking paths in the base and a Lagrangian in the fiber, and sweeping out a Lagrangian in the total space by symplectic parallel transport. Potentially, we have more freedom than in the mirror to $\mathbb{CP}^2$.

The admissible Lagrangian submanifolds we consider have boundary conditions given by a complex curve in $\partial X(B)$ along each boundary face. This gives two complex curves Σ_0 and Σ_1 for the bottom and top horizontal boundary faces. At the vertical boundary faces, we have corresponding complex curves M_0 and M_1. These are fibers of the fibration $X(B) \to X(I)$, and to be admissible requires the Lagrangian to end on this fiber. If B has a corner rather than a vertical boundary, then the fiber M_i corresponds to a vertical boundary created by cutting off the corner, and the admissibility condition requires the Lagrangians to intersect this fiber in a prescribed real curve.

Thus, a Lagrangian submanifold may be constructed by taking a Lagrangian L_0 in M_0, and a path ℓ in the base joining M_0 to M_1, and taking the parallel transport. If M_0 corresponds to a corner, we have only one choice for L_0, and if M_1 corresponds to a corner, this imposes a constraint on L_0 and ℓ. In order to obtain sections of the torus fibration, we choose L_0 to be a curve which is a section of the fibration by circles of constant ξ on each fiber, and ℓ to be a section of the fibration of the base by circles of constant η. Thus when drawing $X(I)$ as an annulus, ℓ appears as a spiral.

As an example, consider the degree six del Pezzo surface X_6, which may be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up two points that do not lie on the same ruling line. The moment polytope of $\mathbb{P}^1 \times \mathbb{P}^1$ is a rectangle. By doing toric blowups at opposite corners, we obtain X_6 with a toric boundary divisor consisting of six (-1)-curves, and the moment polytope is a hexagon. Alternatively, by doing almost toric blowups [43] on opposite sides of the hexagon, we obtain X_6 with an anticanonical divisor of four components with self-intersections $-1, 0, -1, 0$. The base affine manifold B has four sides and two singularities with parallel monodromy invariant directions.

Passing over to the mirror side, we consider Lagrangians in the symplectic manifold $X(B)$. Then we can choose independently

(1) the number of times the initial Lagrangian L_0 winds around the fiber M_0,

(2) the number of times ℓ winds around the base between the M_0 and the first singularity,
(3) the number of times ℓ winds around the base between the two singularities, and
(4) the number of times ℓ winds around the base between the second singularity and M_1.

This gives rise to a 4-parameter family of Lagrangians. Under mirror symmetry, all of them correspond to line bundles. We expect that $X(B)$ is a mirror to the degree 6 del Pezzo surface X_6 with a 4-component anticanonical divisor. The 4 parameters correspond to $\mathrm{Pic}(X_6) \cong \mathbb{Z}^4$.

Though we can construct many Lagrangians this way, in order to compute Floer cohomology and identify the basis with $B(\frac{1}{d}\mathbb{Z})$, we must choose a family of Lagrangians $\{L(d)\}_{d\in\mathbb{Z}}$ corresponding to the tensor powers of an ample line bundle. First we choose $\ell(0)$ as a reference path in the base, over which lies $L(0)$, a Lagrangian satisfying the boundary conditions. We take $\ell(1)$ to be a certain path in the base: the number of times that $\ell(1)$ must wind between the singularities and the vertical boundaries/corners is determined by B: it is essential that the number of turns $\ell(1)$ makes between two consecutive singularities (or between a boundary and the neighboring singularity) is the affine width of the corresponding portion of B. An equivalent condition is that the $\ell(1)$ winds at unit speed. If B has a vertical boundary face rather than a corner, the intersection of $L(1)$ with the fiber at that boundary must be a curve that, relative to $L(0)$, makes a number of turns equal to the affine length of the corresponding vertical boundary face. Then we choose $\ell(d)$ to be a path in the base whose slope is d times the slope of $\ell(1)$, relative to $\ell(0)$. We also make sure that in the fiber, the slope $L(d)$ is d times the slope of $L(1)$ (relative to $L(0)$ in each fiber).

6.3. **Holomorphic and tropical triangles.** Having chosen a family of Lagrangians $\{L(d)\}_{d\in\mathbb{Z}}$ corresponding to the powers of an ample line bundle, we find that after positively perturbation, $HF^*(L(d_1), L(d_2))$ is concentrated in degree 0 when $d_1 \leq d_2$. The techniques of section 4 allow us to compute the holomorphic triangles contributing to the multiplication

$$HF^*(L(d_2), L(d_3)) \otimes HF^*(L(d_1), L(d_2)) \to HF^*(L(d_1), L(d_3)) \tag{112}$$

Looking at the winding numbers of the Lagrangians in the base once again yields an auxiliary $\mathbb{Z}$-grading. For fixed values of this $\mathbb{Z}$-grading on the input, one can determine the number of times that triangles contributing to the product cover the critical values $w_1, \dots, w_n$; call these numbers $k_1, \dots, k_n$. Then the degeneration process breaks the triangle into $k = \sum_{i=1}^n k_i$ copies of the fibration over a disk with single critical value, as well a trivial fibration over a $(k+3)$-gon. Over the disks the count of sections is 1, while the analysis of sections over the $(k+3)$-gon still goes through because, in the fiber, the Lagrangian boundary condition is still a sequence of curves on the cylinder whose slope changes monotonically. Hence the matrix coefficients of this product are binomial coefficients of the form $\binom{k}{s}$.

In this degeneration argument, the Lefschetz singularities that come from different focus-focus singularities are not distinguished, while in the case of tropical triangles, different singularities of the affine structure contribute differently to the tropical curves. We find that this family of triangles, with total count $\binom{k}{s}$, corresponds to

Variety	Anticanonical divisor	Mirror space	Superpotential
$\mathbb{CP}^2$	$D = C \cup L$	$X^\vee = \{(u,v) \mid uv \neq 1\}$	$W = u + \frac{e^{-\Lambda}v^2}{uv-1}$
$U_L = \mathbb{CP}^2 \setminus L$	$C \setminus (C \cap L)$	$X^\vee$	$W_L = u$
$U_C = \mathbb{CP}^2 \setminus C$	$L \setminus (L \cap C)$	$X^\vee$	$W_C = \frac{e^{-\Lambda}v^2}{uv-1}$
$U_D = \mathbb{CP}^2 \setminus D$	$\emptyset$	$X^\vee$	$W_D = 0$

Table 1. Mirrors to divisor complements.

several tropical triangles $T_{s_1,\ldots,s_n}$ with $s_1 + \cdots + s_n = s$, indexed by ordered partitions of s, with 0 allowed as a part (of which there are $\binom{s+n}{s}$). The triangle $T_{s_1,\ldots,s_n}$ uses the tropical disk emanating from the i-th singularity either s_i or $k_i - s_i$ times, and the multiplicity of $T_{s_1,\ldots,s_n}$ is $\binom{k_1}{s_1}\binom{k_2}{s_2}\cdots\binom{k_n}{s_n}$. The equality of the total counts

$$\binom{k}{s} = \sum_{\{s_1,\ldots,s_n \mid s_i \geq 0, \sum_{i=1}^n s_i = s\}} \prod_{i=1}^n \binom{k_i}{s_i} \tag{113}$$

follows from comparing the coefficients of x^s in the equation

$$(1+x)^k = \prod_{i=1}^n (1+x)^{k_i} \tag{114}$$

7. Mirrors to divisor complements

In this section we examine the relationship between the wrapped Floer cohomology of our Lagrangians $L(d)$ and the cohomology of coherent sheaves on complements of components of the anticanonical divisor in $\mathbb{CP}^2$. Let $D = C \cup L$ denote the anticanonical divisor which is the union of a conic C and a line L. We can consider the divisor complements $U_D = \mathbb{CP}^2 \setminus D$, $U_C = \mathbb{CP}^2 \setminus C$, and $U_L = \mathbb{CP}^2 \setminus L$. The torus fibration $\mathbb{CP}^2 \setminus D$ can be restricted to such a complement, and T-duality gives the same space $X^\vee$ as before, but with a different superpotential, reflecting the counts of holomorphic disks intersecting the remaining components of the anticanonical divisor.

Once again, the cohomology of coherent sheaves $\mathcal{O}(d)$ corresponds to Floer cohomology of the Lagrangian submanifolds $L(d)$.

Removing a divisor D from a compact variety X changes the cohomology of a coherent sheaf $\mathcal{F}$, since, for example, sections of $\mathcal{F}$ with poles along D are regular on the complement $U = X \setminus D$, so that $H^0(U, \mathcal{F})$ is not finitely generated in general.

On the symplectic side, changing the superpotential by dropping a term modifies the boundary condition for our Lagrangian submanifolds $L(d)$. Some parts of $L(d)$ that were required to lie on the fiber of W are no longer so constrained, and it is appropriate to *wrap* these parts of $L(d)$. The algebraic structure associated to $L(d)$ is then *wrapped Floer cohomology* $HW^*(L(d_1), L(d_2))$, which is the limit $\lim_{w\to\infty} HF^*(\phi_{wH}(L(d_1)), L(d_2))$, where H is an appropriate Hamiltonian function (a more precise definition is given below). The limit $HW^*(L(d_1), L(d_2))$ will not be finitely generated in general, since it potentially contains trajectories of H joining $L(d_1)$ to $L(d_2)$ of *any* length. The general theory of wrapped Floer cohomology is developed in [5].

7.1. **Algebraic motivation.** In order to motivate the symplectic constructions of wrapped Floer cohomology, it is useful to understand the algebraic side first. The starting point is the following proposition ([40], statement (1.10)).

Proposition 7.1. *Let X be a smooth quasiprojective variety over $\mathbb{C}$, $Y \subset X$ a hypersurface, and $U = X \setminus Y$ the complement. Write $Y = s^{-1}(0)$, where s is the canonical section of the line bundle $\mathcal{L} = \mathcal{O}_X(Y)$. Let $\mathcal{F}$ be a coherent sheaf on X. Multiplication by s defines an inductive system*

$$H^*(X, \mathcal{F} \otimes \mathcal{L}^{r-1}) \longrightarrow H^*(X, \mathcal{F} \otimes \mathcal{L}^{r}) \longrightarrow H^*(X, \mathcal{F} \otimes \mathcal{L}^{r+1}) \longrightarrow \cdots \tag{115}$$

and the limit is

$$\lim_{r\to\infty} H^*(X, \mathcal{F} \otimes \mathcal{L}^r) \cong H^*(U, \mathcal{F}|U) \tag{116}$$

We spell out the application of this proposition to each of the cases we consider

- U_L: Since $L = \{y = 0\}$ is a line, we identify $\mathcal{L} \cong \mathcal{O}(1)$ and take $s = y$. Thus
$$H^*(U_L, \mathcal{O}(d)) \cong \lim_{r\to\infty} H^*(\mathbb{CP}^2, \mathcal{O}(d + r)) \tag{117}$$
where the limit is formed with respect to multiplication by y. An element of $H^0(U_L, \mathcal{O}(d))$ is a rational function $f(x, y, z)/y^r$, where f is a homogeneous polynomial of degree $d + r$.
- U_C: Since $C = \{xz - y^2 = 0\}$ is a conic, we identify $\mathcal{L} \cong \mathcal{O}(2)$ and take $s = p = xz - y^2$. Thus
$$H^*(U_C, \mathcal{O}(d)) \cong \lim_{r\to\infty} H^*(\mathbb{CP}^2, \mathcal{O}(d + 2r)) \tag{118}$$
where the limit is formed with respect to multiplication by $p = xz - y^2$. An element of $H^0(U_C, \mathcal{O}(d))$ is a rational function $f(x, y, z)/p^r$, where f is a homogeneous polynomial of degree $d + 2r$.
- U_D: Since $D = \{yp = xyz - y^3 = 0\}$ is a cubic, we identify $\mathcal{L} \cong \mathcal{O}(3)$ and take $s = yp$. Thus
$$H^*(U_D, \mathcal{O}(d)) \cong \lim_{r\to\infty} H^*(\mathbb{CP}^2, \mathcal{O}(d + 3r)) \tag{119}$$
where the limit is formed with respect to multiplication by yp. An element of $H^0(U_D, \mathcal{O}(d))$ is a rational function $f(x, y, z)/(yp)^r$, where f is a homogeneous polynomial of degree $d + 3r$.

For the purposes of computation, a useful simplification comes from noting that the line bundles $\mathcal{O}(d)$ may become isomorphic over the complements.

- U_L: Since $U_L \cong \mathbb{C}^2$, all the line bundles $\mathcal{O}(d)$ are isomorphic over it.
- U_C: The complement of a smooth conic in $\mathbb{CP}^2$ has $H^2(U_C; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, generated by $c_1(\mathcal{O}(1))$. The defining section $p : \mathcal{O} \to \mathcal{O}(2)$ is an isomorphism over U_C, and so $\mathrm{Pic}(U_C) \cong \mathbb{Z}/2\mathbb{Z}$ as well.
- U_D: Since $U_D \subset U_L$, all the line bundles $\mathcal{O}(d)$ are isomorphic over it as well.

7.2. **Wrapping.** In this subsection we describe the geometric setup for wrapped Floer cohomology in the mirrors of U_L, U_C, and U_D.

7.2.1. *Completions.* Wrapped Floer cohomology is formulated in terms of noncompact manifolds containing noncompact Lagrangian submanifolds. These can be defined as (partial) completions of compact manifolds with boundary.

The starting point for all three cases is the original Lefschetz fibration $X(B) \to X(I)$ containing the Lagrangians $\{L(d)\}_{d\in\mathbb{Z}}$. We define completions $\hat{X}_L$, $\hat{X}_C$, and $\hat{X}_D$. This process involves replacing a boundary component with an infinite end, either on the base or in the fiber of the Lefschetz fibration.

- L: The manifold $\hat{X}_L$ retains a boundary component at the top horizontal boundary, corresponding to the fiber of the superpotential $W_L = u$. The fibers of $X(B) \to X(I)$ are completed at the other, bottom, end. The base annulus $X(I)$ is completed to a cylinder $\hat{X}(I)$, and the fibration is extended over this cylinder.
- C: The manifold $\hat{X}_C$ retains a boundary component at the bottom horizontal boundary, corresponding to the fiber of the superpotential $W_C = \frac{e^{-\Lambda}v^2}{uv-1}$. The fibers are completed at the other, top, end. The base annulus is completed to a cylinder $\hat{X}(I)$, and the fibration is extended over this cylinder.
- D: The manifold $\hat{X}_D$ has no boundary, and the fibers are completed at both ends. The base annulus is completed to a cylinder $\hat{X}(I)$, and the fibration is extended over this cylinder.

The torus fibration on $X(B)$ extends to these completions, and yields torus fibrations over completed bases $\hat{B}_L$,$\hat{B}_C$, $\hat{B}_D$.

- $\hat{B}_L$ is a half-plane with singular affine structure given by removing the bottom boundary from B and extending in that direction.
- $\hat{B}_C$ is a half-plane with singular affine structure given by removing the top boundary from B and extending in that direction.
- $\hat{B}_D$ is an entire plane with singular affine structure given by removing all boundaries from B and extending in all directions.

The Lagrangian submanifolds $L(d)$ are extended to $\hat{L}(d)$; In all cases, we extend $L(d)$ into whatever ends are attached so as to be invariant under the Liouville flow within the end. However, in the case of L, respectively C, we still have the boundary condition that $\hat{L}(d)$ is required to end on Σ_1 (the complex hypersurface contained in the top boundary), respectively Σ_0 (contained in the bottom boundary).

7.2.2. *Hamiltonians.* The most crucial difference between the three cases comes from the choices of Hamiltonians that are be used to perform the wrapping. The Hamiltonians we consider are the sum of contributions from the base and the fiber.

Let $H_b : \hat{X}(B) \to \mathbb{R}$ be the pullback of a function on the base cylinder which is a function of the radial coordinate $\eta = \log|w|$ only. Writing the symplectic form on the base as $d\rho \wedge d\theta$, where ρ is a function of η, we take H_b to be a convex function of ρ on the compact part $X(I)$, and linear in ρ on the ends. We also require $H_b \geq 0$, with minimum on the central circle $\eta = 0$. Since dH_b vanishes on the fibers, X_{H_b} is horizontal.

The fiber Hamiltonian H_f is chosen differently in each case. The main constraint is that its differential must vanish at any boundary component which may still be present. The construction is most convenient if we assume the completion preserves the S^1-symmetry that rotates the fibers. If μ denotes the moment map for this action, we can take H_f to be a function of μ. Since X_{H_f} is tangent to the fibers, we have

$$\{H_b, H_f\} = \omega(X_{H_b}, X_{H_f}) = 0 \tag{120}$$

which allows us to compute the flow of $H_b + H_f$ term by term.

We use the same base Hamiltonian H_b for all cases. The specific choice of H_f in each case is as follows.

- L: Let $H_{f,L} \geq 0$ be a function with a minimum at the top of the fiber, convex in μ on the compact part, and linear in μ on the bottom end.
- C: Let $H_{f,C} \geq 0$ be a function with a minimum at the bottom of the fiber, convex in μ on the compact part, and linear in μ on the top end.
- D: Let $H_{f,D} \geq 0$ be a function with a minimum in the two-thirds of the way down from the top of the compact part of the fiber, convex in μ on the compact part, and linear in μ on the ends.

In each case, the total Hamiltonian $H = H_b + H_f$ achieves its minimum along a torus in $X(B)$, which is one of the fibers of the torus fibration over B. Since $L(r)$ is a section of the torus fibration, H has a unique minimum on $L(r)$, and hence there is a unique constant chord for $L(r)$, which corresponds to the identity element $e_r \in HW^*(L(r), L(r))$. The cases are:

- L: The minimum is along the torus corresponding to the midpoint of the top edge of B, where the generator y lies. All of the Lagrangians $L(r)$ for $r \in \mathbb{Z}$ intersect this torus at the same point. As an intersection point of $L(0)$ and $L(d)$, the minimum corresponds to $q_{a,i}$ for $(a, i) = (0, 0)$.
- C: The minimum is along the torus corresponding to the midpoint of the bottom edge of B, where the generator p lies. All of the Lagrangians $L(r)$ for $r \in 2\mathbb{Z}$ intersect this torus at the same point. As an intersection point of $L(0)$ and $L(r)$, the minimum corresponds to $q_{a,i}$ for $(a, i) = (0, r/2)$.
- D: The minimum is along the torus two-thirds of the way from the top of the middle fiber of the map $B \to I$, where the generator yp lies. All of the Lagrangians $L(r)$ for $r \in 3\mathbb{Z}$ intersect this torus in the same point. As an intersection point of $L(0)$ and $L(r)$, the minimum corresponds to $q_{a,i}$ for $(a, i) = (0, r/3)$.

Remark 14. The Hamiltonians we obtain as $H_b + H_f$ are not admissible in the usual sense, because they vanish at some boundaries, and, even in the case D, are not linear with respect to a cylindrical end. Closer to our situations are the *Lefschetz admissible* Hamiltonians considered by Mark McLean [34], that are precisely those functions on the total space of a Lefschetz fibration that are the sum of admissible Hamiltonians on the base and fiber separately.

7.2.3. *Generators.* Given Lagrangian submanifolds L_1, L_2 of X, equipped with Hamiltonian H, and $r \in \mathbb{R}$, we get Floer cohomology complexes $CF^*(L_1, L_2; rH)$ generated by time-1 trajectories of X_{rH} starting on L_1 and ending on L_2. As usual the differential counts inhomogeneous pseudo-holomorphic strips. We also have continuation maps

$$CF^*(L_1, L_2; rH) \to CF^*(L_1, L_2; r'H), \quad r < r' \tag{121}$$

given by counting strips where the inhomogeneous term interpolates between $r'X_H$ and rX_H. At the homology level, the continuation maps form an inductive system, and we define the *wrapped Floer cohomology*

$$HW^*(L_1, L_2) = \lim_{r\to\infty} HF^*(L_1, L_2; rH) \tag{122}$$

Our purpose in this section is simply to set up an enumeration of the generators of $CF^*(L(d_1), L(d_2); rH)$ in each of the three cases. These generators can also be regarded as intersection points $\phi_{rH}(L(d_1)) \cap L(d_2)$. In order to make the situation as convenient as possible for our later arguments, we refine our choice of Hamiltonians so as to ensure that $\phi_{rH}(L(d))$ is actually $L(d')$ for some d'; thus we can identify wrapped Floer cohomology generators with intersection points of our original Lagrangians. This is done by adjusting the slopes of our Hamiltonians on the ends.

- We take the base Hamiltonian H_b so that the time–1 flow completes 1 turn on the cylindrical ends of the base.
- For cases L and C, we take the fiber Hamiltonian H_f so that the time–1 flow completes 1/2 turn on the cylindrical end of the fiber.
- For case D, we take the fiber Hamiltonian H_f so that the time–1 flow completes 1/3 turn at the top of the fiber, and 1/6 turn at the bottom of the fiber.

In all cases the total Hamiltonian we use is $H = H_b + H_f$.

The way to understand the flow of H is to first apply H_f, then H_b. The flow of H_f wraps $L(d)$ in the fiber, while the flow of H_b, when it completes a loop in the base, performs the monodromy of the Lefschetz fibration around that loop, which undoes some of the wrapping due to H_f.

We can relate $\phi_H(L(d))$ to $L(d')$ as follows:

- L: we have $\phi_{rH}(L(d)) = L(d - r)$.
- C: we have $\phi_{2rH}(L(d)) = L(d-2r)$. Note that the same *cannot* be said with r in place of $2r$; in that case the two Lagrangians intersect the bottom boundary (where no wrapping occurs) in different points.
- D: we have $\phi_{3rH}(L(d)) = L(d - 3r)$. Again the same cannot be said with r in place of $3r$.

In order to identify generators with intersection points, we perturb the boundary intersection points in a positive sense just as before. In the cases with boundary, where the Hamiltonian is supposed to have a minimum at the boundary, it is useful to perform this perturbation in an extra collar attached to the boundary, so that the Lagrangians still intersect at the minimum of H if they did prior to the perturbation. Once this is done, we can identify

$$CF^*(L(d_1), L(d_2); rH) \cong CF^*(\phi_{rH}(L(d_1)), L(d_2)) \cong CF^*(L(d_1 - r), L(d_2)) \tag{123}$$

where $r \in \mathbb{Z}$ in case L, $r \in 2\mathbb{Z}$ in case C, and $r \in 3\mathbb{Z}$ in case D. This identification is compatible with the gradings, which shows that for $r > d_1 - d_2$, the Floer complex $CF^*(L(d_1), L(d_2); rH)$ is concentrated in degree zero, and so has vanishing differential. Recall that the generators of the last group are identified with $B(\frac{1}{d_2-d_1+r}\mathbb{Z})$.

Once we are in the range $d_2 - d_1 + r > 0$, we find that as r increases, new generators are created, none are destroyed, and the generators that already exist are "compressed" toward the minimum of H. This gives rise to naive inclusion maps $i : CF^*(L(d_1), L(d_2); rH) \to CF^*(L(d_1), L(d_2); r'H)$ for $r < r'$, where

$r > d_1 - d_2$. In terms of fractional integral points, this i corresponds to the map $B(\frac{1}{d_2-d_1+r}\mathbb{Z}) \to B(\frac{1}{d_2-d_1+r'}\mathbb{Z})$ which is dilation by the appropriate factor centered at the point corresponding to the minimum of H.

We can index the points of $\hat{B}(\frac{1}{d}\mathbb{Z})$ with two indices. The index $a \in \mathbb{Z}$ corresponds to the column lying at $\eta = a/d$, while the index i that indexes points within a column, and which lies in $\{0, \dots, \left\lfloor \frac{d-|a|}{2} \right\rfloor\}$ in the compact case, is now unbounded in the positive direction in case L, in the negative direction in case C, and in both directions in case D. We use the notation $q_{a,i}$ for these points.

7.3. **Continuation maps and products.** We saw above that for $r > d_1 - d_2$, the generators of $CF^*(L(d_1), L(d_2); rH)$ all have degree 0, so there are no differentials, and we can identify these complexes with their homologies. In order to obtain the wrapped Floer cohomology, we must determine the continuation maps.

Let it be understood that we require $r \in \mathbb{Z}$ in case L, $r \in 2\mathbb{Z}$ in case C, and $r \in 3\mathbb{Z}$ in case D.

To get started, we consider the wrapped Floer cohomology of $L(d_1)$ with itself. Each complex $CF^*(L(d_1), L(d_1); rH)$ has a distinguished element e_r, sitting at the minimum of H. The complexes are filtered by action, and the continuation maps are non-increasing with respect to this filtration. Under the $r \to r'$ continuation map, $e_r \mapsto e_{r'}$; e_r and $e_{r'}$ are the unique generators of minimal action in their respective complexes, and in fact their actions are equal, so the only strip is the constant map to the minimum.

At this point we bring in the product structure.

Proposition 7.2. *Let L_i one of the Lagrangians $L(d_i)$ for $i = 1, 2, 3$. Under the identification*

$$CF^*(L_i, L_j; rH) \cong CF^*(\phi_{rH}(L_i), L_j) \tag{124}$$

The product

$$HF^*(L_2, L_3; rH) \otimes HF^*(L_1, L_2; sH) \to HF^*(L_1, L_3; (r+s)H) \tag{125}$$

which counts inhomogeneous pseudo-holomorphic triangles corresponds to the product

$$HF^*(\phi_{rH}(L_2), L_3) \otimes HF^*(\phi_{(r+s)H}(L_1), \phi_{rH}(L_2)) \to HF^*(\phi_{(r+s)H}(L_1), L_3) \tag{126}$$

counting pseudo-holomorphic triangles with no inhomogeneous term.

Proof. First we recall the equation for the product in the wrapped setting. The disk with three boundary punctures S is equipped with strip-like ends: at the inputs they are parametrized by $(s,t) \in (0,\infty) \times [0,1]$, while at the output we have $(s,t) \in (-\infty, 0) \times [0,1]$. S is also equipped with a closed one form $\beta \in \Omega^1(S)$, whose restriction to ∂S vanishes, such that on the strip-like ends of S, β has the form $r\,dt$, $s\,dt$ and $(r+s)\,dt$ for the two inputs and the output respectively. The curves we count are solutions of

$$(du - X_H \otimes \beta)^{0,1} = 0 \tag{127}$$

Because β is closed and S has no first homology, we may find a primitive $\tau \in C^\infty(S)$ such that $d\tau = \beta$, and we assume $\tau = 0$ on the boundary corresponding to L_3. Denoting by ϕ_H^t the flow of X_H for time t, we can define a new map $v = \phi_H^{-\tau} \circ u$, meaning that for $p \in S$, we take $u(p)$ and flow it for time $-\tau(p)$ by X_H. This has the effect of collapsing each Hamiltonian chord to a single point. Then the map v is pseudo-holomorphic without inhomogeneous term, but with a domain-dependent almost complex structure obtained by conjugating the original J by ϕ_H^τ at each point.

We must relate the products

$$HF^*(\phi_{rH}(L_2), L_3) \otimes HF^*(\phi_{(r+s)H}(L_1), \phi_{rH}(L_2)) \to HF^*(\phi_{(r+s)H}(L_1), L_3) \tag{128}$$

defined with respect to this domain-dependent almost complex structure $\{J_p\}_{p\in S}$ to the operation that we studied in Section 4, where the complex structure J was fixed. We fix attention to one strip-like end, say the one corresponding to L_2 and L_3. On this end the complex structure depends only on $t \in [0,1]$. There is a continuation map

$$HF^*(\phi_{rH}(L_2), L_3; \{J_t\}) \to HF^*(\phi_{rH}(L_2), L_3; J) \tag{129}$$

counting strips with a domain-dependent almost complex structure that interpolates between $\{J_t\}$ and J. We claim that, in our situation, all such strips are constant. This means that the bases of intersection points on the two sides of the continuation map correspond. These continuation maps intertwine the product with respect to $\{J_p\}$ and the one defined using J, implying that these products agree when expressed in the basis of intersection points on either side of the correspondence.

It remains to prove the claim that all the continuation strips are constant. Let u denotes such a strip. Because the Hamiltonian flow preserves the structure of the Lefschetz fibration, each J_p for $p \in S$ is a complex structure with the property that the tangent space to a fiber is complex, and the projection to the base is holomorphic, once we equip the base with a suitable complex structure j_p. As u is pseudo-holomorphic with respect to $\{J_p\}$, this implies that $\pi \circ u$ has the property that its differential at each point has rank either 0 or 2. Thus the image of $\pi \circ u$ is a set whose boundary is contained in the images of the Lagrangians $\phi_{rH}(L_2)$ and L_3. Because there are no topological strips in the base joining different intersection points of the base paths, we conclude that $\pi \circ u$ must be constant, and the image of u is contained in a fiber. But in the fiber there are also no topological strips joining different intersection points. Hence u itself is constant. □

When the differentials on the Floer complexes vanish, the identification of the products holds at the chain level as well. Tracing the isomorphisms through, we find that the product with e_r induces the naive inclusion map on generators

$$\mu^2(e_r, \cdot) = i : HF^*(L(d_1), L(d_2); sH) \to HF^*(L(d_1), L(d_2); (r+s)H) \tag{130}$$

Due to the compatibility of the product with the continuation maps, and the fact that $e_r \mapsto e_{r'}$ under continuation, we find that the naive inclusion maps commute with the continuation maps:

$$\begin{array}{ccc} HF^*(L(d_1), L(d_2); sH) & \xrightarrow{i} & HF^*(L(d_1), L(d_2); (s+r)H) \\ \big\downarrow = & & \big\downarrow \text{cont.} \\ HF^*(L(d_1), L(d_2); sH) & \xrightarrow{i} & HF^*(L(d_1), L(d_2); (s+r')H) \end{array} \tag{131}$$

Thus the continuation map agrees with the naive inclusion map, at least on those generators which are in the image of $HF^*(L(d_1), L(d_2); sH)$. It follows that the continuation maps agree with the naive inclusion maps, at least for r large enough depending on a particular generator. Hence

$$HW^*(L(d_1), L(d_2)) = \lim_{r\to\infty} HF^*(L(d_1), L(d_2); rH), \tag{132}$$

where the limit is formed with respect to the continuation maps, or with respect to the naive inclusion maps, or (what is equal) the multiplications by the various elements e_r.

Spelling this out a bit more gives a precise correspondence with section 7.1. Consider the isomorphism

$$HF^*(L(d_1), L(d_1); rH) \cong HF^*(L(d_1 - r), L(d_1)) \cong H^*(\mathbb{CP}^2, \mathcal{O}(r)) \tag{133}$$

- L: For $r \in \mathbb{Z}$, this isomorphism identifies e_r with y^r.
- C: For $r \in 2\mathbb{Z}$, this isomorphism identifies e_r with $p^{r/2}$.
- D: For $r \in 3\mathbb{Z}$, this isomorphism identifies e_r with $(yp)^{r/3}$.

Thus the directed systems computing wrapped Floer cohomology are identified with those computing the cohomology of line bundles on the divisor complements.

We can identify the basis of $HW^*(L(d_1), L(d_2))$ with $\hat{B}(\frac{1}{d_2-d_1}\mathbb{Z})$, where $\hat{B} = \hat{B}_L, \hat{B}_C, \hat{B}_D$ is the completion of the affine manifold. The sets $B(\frac{1}{d_2-d_1+r}\mathbb{Z})$ embed in $\hat{B}(\frac{1}{d_2-d_1}\mathbb{Z})$ and this latter is their limit as $r \to \infty$; the map is dilation by $\frac{d_2-d_1+r}{d_2-d_1}$ centered at the minimum of H.

Using the products we computed in section 4, we can identify this basis $\hat{B}(\frac{1}{d}\mathbb{Z})$ for $HW^*(L(0), L(d))$ with a basis of $H^*(U; \mathcal{O}(d))$, and complete the proof of Theorem 1.2

- L: The point $q_{a,i}$ of $\hat{B}_L(\frac{1}{d}\mathbb{Z})$ corresponds to the function $x^{-a}p^iy^{d+a-2i}$ for $a \le 0$, and $z^ap^iy^{d-a-2i}$ for $a \ge 0$. In this case $i \ge 0$ can be arbitrarily large, so the exponent of y is allowed to be negative.
- C: The point $q_{a,i}$ of $\hat{B}_C(\frac{1}{d}\mathbb{Z})$ corresponds to the function $x^{-a}p^iy^{d+a-2i}$ for $a \le 0$, and $z^ap^iy^{d-a-2i}$ for $a \ge 0$. In this case $i \le \left\lfloor \frac{d-|a|}{2} \right\rfloor$ can be negative, so the exponent of p is allowed to be negative, while the exponent of y is non-negative.
- D: The point $q_{a,i}$ of $\hat{B}_D(\frac{1}{d}\mathbb{Z})$ corresponds to the function $x^{-a}p^iy^{d+a-2i}$ for $a \le 0$, and $z^ap^iy^{d-a-2i}$ for $a \ge 0$. In this case $i \in \mathbb{Z}$, so the exponents of y and p are allowed to be negative.

References

1. Mohammed Abouzaid, *Homogeneous coordinate rings and mirror symmetry for toric varieties*, Geom. Topol. **10** (2006), 1097–1157 (electronic). MR 2240909 (2007h:14052)
2. Mohammed Abouzaid, 2009, Private communication.
3. Mohammed Abouzaid, *Morse homology, tropical geometry, and homological mirror symmetry for toric varieties*, Selecta Math. (N.S.) **15** (2009), no. 2, 189–270. MR 2529936
4. ———, *Toward a tropical Fukaya category*, Talk given in the MSRI workshop "Tropical Structures in Geometry and Physics", December 2009, video available on MSRI website.
5. Mohammed Abouzaid and Paul Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol. **14** (2010), no. 2, 627–718. MR 2602848

6. Paul S. Aspinwall, Tom Bridgeland, Alastair Craw, Michael R. Douglas, Mark Gross, Anton Kapustin, Gregory W. Moore, Graeme Segal, Balázs Szendrői, and P. M. H. Wilson, *Dirichlet branes and mirror symmetry*, Clay Mathematics Monographs, vol. 4, American Mathematical Society, Providence, RI, 2009. MR 2567952 (2011e:53148)
7. Denis Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. GGT **1** (2007), 51–91. MR 2386535 (2009f:53141)
8. Denis Auroux, Ludmil Katzarkov, and Dmitri Orlov, *Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves*, Invent. Math. **166** (2006), no. 3, 537–582. MR 2257391 (2007g:14045)
9. ______, *Mirror symmetry for weighted projective planes and their noncommutative deformations*, Ann. of Math. (2) **167** (2008), no. 3, 867–943. MR 2415388 (2009f:53142)
10. Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), no. 1, 21–74. MR 1115626 (93b:32029)
11. Bohan Fang, Chiu-Chu Melissa Liu, David Treumann, and Eric Zaslow, *T-duality and homological mirror symmetry for toric varieties*, Adv. Math. **229** (2012), no. 3, 1875–1911. MR 2871160 (2012m:14075)
12. Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic). MR 1887642 (2003f:16050)
13. Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, *Lagrangian intersection floer theory: Anomaly and obstruction, chapter 10*, (2007), preprint.
14. ______, *Lagrangian Floer theory on compact toric manifolds. I*, Duke Math. J. **151** (2010), no. 1, 23–174. MR 2573826 (2011d:53220)
15. Alexander B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices (1996), no. 13, 613–663. MR 1408320 (97e:14015)
16. Mark Gross, *The Strominger-Yau-Zaslow conjecture: from torus fibrations to degenerations*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 149–192. MR 2483935 (2010b:14080)
17. Mark Gross, Paul Hacking, and Sean Keel, *Mirror symmetry for log Calabi–Yau surfaces I*, preprint: arXiv:1106.4977.
18. Mark Gross and Bernd Siebert, *Theta functions and mirror symmetry*, Written for the JDG2011 conference, arXiv:1204.1991.
19. ______, *Affine manifolds, log structures, and mirror symmetry*, Turkish J. Math. **27** (2003), no. 1, 33–60. MR 1975331 (2004g:14041)
20. ______, *Mirror symmetry via logarithmic degeneration data. I*, J. Differential Geom. **72** (2006), no. 2, 169–338. MR 2213573 (2007b:14087)
21. ______, *Mirror symmetry via logarithmic degeneration data, II*, J. Algebraic Geom. **19** (2010), no. 4, 679–780. MR 2669728 (2011m:14066)
22. ______, *From real affine geometry to complex geometry*, Ann. of Math. (2) **174** (2011), no. 3, 1301–1428. MR 2846484
23. Nigel J. Hitchin, *The moduli space of special Lagrangian submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 3-4, 503–515 (1998), Dedicated to Ennio De Giorgi. MR 1655530 (2000c:32075)
24. Kentaro Hori, Amer Iqbal, and Cumrum Vafa, *D-branes and mirror symmetry*, preprint: arXiv:hep-th/0005247.
25. Kentaro Hori and Cumrun Vafa, *Mirror symmetry*, preprint: arXiv:hep-th/0002222.
26. Ilia Itenberg, Grigory Mikhalkin, and Eugenii Shustin, *Tropical algebraic geometry*, second ed., Oberwolfach Seminars, vol. 35, Birkhäuser Verlag, Basel, 2009. MR 2508011 (2010d:14086)
27. Anton Kapustin, Ludmil Katzarkov, Dmitri Orlov, and Mirroslav Yotov, *Homological mirror symmetry for manifolds of general type*, Cent. Eur. J. Math. **7** (2009), no. 4, 571–605. MR 2563433 (2010j:53184)
28. Maxim Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139. MR 1403918 (97f:32040)

29. ———, *Lectures at ENS, Paris*, (Spring 1998), notes taken by J. Bellaiche, J.-F. Dat, I. Marin, G. Racinet and H. Randriambololona.
30. Maxim Kontsevich and Yan Soibelman, *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 203–263. MR 1882331 (2003c:32025)
31. ———, *Affine structures and non-Archimedean analytic spaces*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 321–385. MR 2181810 (2006j:14054)
32. N. C. Leung, S.-T. Yau, and E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., vol. 23, Amer. Math. Soc., Providence, RI, 2001, pp. 209–225. MR 1876070 (2002k:53102)
33. Bong H. Lian, Kefeng Liu, and Shing-Tung Yau, *Mirror principle. I*, Asian J. Math. **1** (1997), no. 4, 729–763. MR 1621573 (99e:14062)
34. Mark McLean, *Lefschetz fibrations and symplectic homology*, Geom. Topol. **13** (2009), no. 4, 1877–1944. MR 2497314 (2011d:53224)
35. Robert C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), no. 4, 705–747. MR 1664890 (99j:53083)
36. Alexander Polishchuk and Eric Zaslow, *Categorical mirror symmetry: the elliptic curve*, Adv. Theor. Math. Phys. **2** (1998), no. 2, 443–470. MR 1633036 (99j:14034)
37. Paul Seidel, *More about vanishing cycles and mutation*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 429–465. MR 1882336 (2003c:53125)
38. ———, *Vanishing cycles and mutation*, European Congress of Mathematics, Vol. II (Barcelona, 2000), Progr. Math., vol. 202, Birkhäuser, Basel, 2001, pp. 65–85. MR 1905352 (2003i:53128)
39. ———, *A long exact sequence for symplectic Floer cohomology*, Topology **42** (2003), no. 5, 1003–1063. MR 1978046 (2004d:53105)
40. ———, *A_∞-subalgebras and natural transformations*, Homology, Homotopy Appl. **10** (2008), no. 2, 83–114. MR 2426130 (2010k:53154)
41. ———, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR 2441780 (2009f:53143)
42. Andrew Strominger, Shing-Tung Yau, and Eric Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), no. 1-2, 243–259. MR 1429831 (97j:32022)
43. Margaret Symington, *Four dimensions from two in symplectic topology*, Topology and geometry of manifolds (Athens, GA, 2001), Proc. Sympos. Pure Math., vol. 71, Amer. Math. Soc., Providence, RI, 2003, pp. 153–208. MR 2024634 (2005b:53142)

Department of Mathematics, MIT, Cambridge, MA 02139, USA

Current address: Department of Mathematics, The University of Texas at Austin, Austin, TX 78712, USA

E-mail address: `jpascaleff@math.utexas.edu`