HIGHER LOCAL SYSTEMS, n-AFFINENESS AND KOSZUL DUALITY

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Abstract

In this paper we study local systems of (∞, n)-categories on spaces. We prove that categorical local systems are captured by (higher) monodromy data. We show that if X is $(n+1)$-connected, local systems of (∞, n)-categories over X can be described as $\mathbb{E}_n$-modules over the iterated loop space $\Omega_*^{n+1}X$. Our main applications are to n-affineness and Koszul duality. We prove that n-truncated Betti stacks are n-affine; and that $\pi_{n+1}(X)$ is an obstruction to n-affineness. Our main result is a general statement of $\mathbb{E}_n$-Koszul duality for pairs of $\mathbb{E}_n$-algebras of the form

$$\mathrm{C}_\bullet(\Omega_*^n X; \Bbbk) \longleftrightarrow \mathrm{C}^\bullet(X; \Bbbk).$$

This takes the shape of an equivalence of (∞, n)-categories

$$n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(X; \Bbbk))) \simeq n\mathbf{LocSysCat}^{n-1}(X; \Bbbk)$$

where $\mathrm{cSpec}(\mathrm{C}^\bullet(X; \Bbbk))$ is the *cospectrum* of the algebra of singular cochains, and $n\mathbf{ShvCat}^{n-1}$ is the (∞, n)-category of quasi-coherent sheaves of $(\infty, n-1)$-categories. Our result is new already in the classical case $n = 1$, although it can be seen to recover well known formulations of $\mathbb{E}_1$-Koszul duality as a Morita equivalence of module categories (up to appropriate completions of the t-structures).

Contents

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Introduction

In this paper we study local systems of higher categories over spaces, as well as $\mathbb{E}_n$-Koszul duality and the problem of n-affineness of Betti stacks. Our aim is to generalize and extend to higher categories the following well-known classical story. Let X be a connected space and let $\Bbbk$ be a field of characteristic 0. Local systems of $\Bbbk$-vector spaces over X are determined by monodromy data, in the sense that the abelian category of such local systems is equivalent to the category of representations of $\pi_1(X)$. Understanding the higher cohomology of local systems requires more information that is not captured by $\pi_1(X)$, and in fact depends on full homotopy type of X. More precisely, the stable category of complexes of sheaves of vector spaces on X whose cohomology sheaves are local systems is equivalent to the stable category of modules over $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$, the algebra of chains on the based loop space of X.

The $\mathbb{E}_1$-Koszul dual of $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ is $\mathrm{C}^\bullet(X;\Bbbk)$, the algebra of cochains on X; under certain finiteness hypotheses, the reciprocal duality also holds, and moreover there is a tight relationship between the categories of modules over these two algebras. Hence, under these hypotheses, local systems over X also admit a description in terms of $\mathrm{C}^\bullet(X;\Bbbk)$. Passing to the n-categorical level, local systems of vector spaces are replaced by local systems of $\Bbbk$-linear (∞,n)-categories, the loop space is replaced by the $(n+1)$-fold iterated loop space $\Omega_*^{n+1}X$, and Koszul duality of $\mathbb{E}_1$-algebras is replaced by Koszul duality of the $\mathbb{E}_{n+1}$-algebras $\mathrm{C}_\bullet(\Omega_*^{n+1}X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$. This paper seeks to sort out how the classical relationships generalize to this setting.

Much of our interest in these questions stems from that fact that categorical local systems and more generally schobers, i.e. categorical local systems with singularities, play an increasing role in symplectic geometry and mirror symmetry. They also feature prominently in recent approaches to 3d mirror symmetry [GHM23]. This is a mysterious duality which is only beginning to be explored, and that has deep connections with many areas of mathematics and particularly geometric representation theory, where is sometimes referred to as symplectic duality [BLPW16]. We are particularly indebted to ideas of Teleman on 3d homological mirror symmetry, and some of our results rigorously formalise insights that first appeared in his [Tel14]. The thesis of Toly Preygel [Pre11] also sketches some ideas that we have formalised.

In the next section of this introduction we explain in greater detail the main ideas underpinning our work, focusing on our results $\mathbb{E}_n$-Koszul duality and n-affineness. For clarity, we will mostly explain the first non-trivial case, namely $\mathbb{E}_2$-Koszul duality. Next, in Section I.2, we shall give an analytic description of the structure of the paper and state our main results. In the final section of the introduction we shall explain more broadly the motivations of our work coming from symplectic geometry.

I.1. **Koszul duality and n-affineness.** In the first part of the paper, we obtain analogues of the fundamental equivalence

$$\mathrm{LocSys}(X;\Bbbk) \simeq \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \tag{I.1.1}$$

in the setting of local systems of *presentable* (∞, n)-categories. Presentable categories have long been familiar to practitioners of ∞-categories, but their (∞, n)-categorical analogues have only been recently introduced by Stefanich in [Ste20]. In its strongest form, our result states that there is an equivalence between two $(\infty, n+1)$-categories: on the one hand, the $(\infty, n+1)$-category of local systems of presentable (∞, n)-categories over X; and on the other hand, a category of iterated modules over the $\mathbb{E}_{n+1}$-algebra of chains over the $(n+1)$-fold loop space $\mathrm{C}_\bullet(\Omega_*^{n+1}X;\Bbbk)$. This fits well with the familiar picture according to which higher local systems should have monodromy along higher dimensional spheres.

Our main goal is to study higher Koszul duality, and the closely related question of *n-affineness* of Betti stacks. To explain the context of our work, we start recalling in some greater detail classical Koszul duality.

Let $\Bbbk$ be an algebraically closed field of characteristic 0. Classical Koszul duality is a duality between certain augmented *associative* $\Bbbk$-algebras, the most well-known example of which is the duality between symmetric and exterior algebras. Topology and the theory of local systems are the source of one of the most important classes of Koszul dual algebras. Let us explain how this works. Let X be a pointed and simply connected finite CW complex. The natural map $X \to \{*\}$ equips the algebra $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ with an augmentation

$$\mathrm{C}_\bullet(\Omega_* X;\Bbbk) \longrightarrow \mathrm{C}_\bullet(\Omega_* \{*\}, \Bbbk) \simeq \Bbbk.$$

The dg algebra of singular cochains $\mathrm{C}^\bullet(X;\Bbbk)$ is augmented via the pointing $\{*\} \to X$

$$\mathrm{C}^\bullet(X;\Bbbk) \longrightarrow \mathrm{C}^\bullet(\{*\}, \Bbbk) \simeq \Bbbk.$$

Then the algebras $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$ are *Koszul dual*. Classically this means that we have the following two closely related statements.

(1) The algebra of endomorphisms of the augmentation of $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ is equivalent to $\mathrm{C}^\bullet(X;\Bbbk)$, and vice versa. In symbols:

$$\mathrm{C}_\bullet(\Omega_* X;\Bbbk) \simeq \underline{\mathrm{Map}}_{\mathrm{C}^\bullet(X;\Bbbk)}(\Bbbk, \Bbbk) \quad \text{and} \quad \mathrm{C}^\bullet(X;\Bbbk) \simeq \underline{\mathrm{Map}}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}(\Bbbk, \Bbbk).$$

(2) The functor between $\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}$ and $\mathrm{LMod}_{\mathrm{C}^\bullet(X;\Bbbk)}$ given by

$$\underline{\mathrm{Map}}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}(\Bbbk, -) : \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \longrightarrow \mathrm{LMod}_{\mathrm{C}^\bullet(X;\Bbbk)} \tag{I.1.2}$$

is *almost*, but not quite, a Morita equivalence.

Under the equivalence $\mathrm{LocSys}(X;\Bbbk) \simeq \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}$ the augmentation module is sent to the constant local system $\underline{\Bbbk}_X$, and the functor $\underline{\mathrm{Map}}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}(\Bbbk, -)$ corresponds to the enhanced global sections

$$\Gamma(X, -) : \mathrm{LocSys}(X;\Bbbk) \longrightarrow \mathrm{Mod}_{\mathrm{C}^\bullet(X;\Bbbk)}. \tag{I.1.3}$$

As it turns out, the enhanced global section functor (I.1.3), and thus functor (I.1.2), are almost never equivalences.[1] Using the terminology of algebraic geometry, we can express this by saying that finite CW complexes, or more precisely their *Betti stacks*, are (almost) never *affine*. Here we understand affineness precisely as the property that global sections define an equivalence between the stable category of quasi-coherent sheaves, and modules over the global sections of the structure sheaf. Now, the Betti stack of a space X, denoted X_{B}, is the constant stack with values X (see Section 4.1 in the main text for more details). The category $\mathrm{QCoh}(X_{\mathrm{B}})$ is naturally equivalent to $\mathrm{LocSys}(X;\Bbbk)$ and under this identification $\mathcal{O}_{X_{\mathrm{B}}}$ goes to the constant local system $\underline{\Bbbk}_X$.

The failure of Koszul duality to give rise to an actual Morita equivalence is one the main subtleties of the theory. There are several ways to obviate this, and turn (2) into a rigorous mathematical statement. It is possible to show that functor (I.1.2) does restrict to an equivalence between categories of appropriately bounded modules: more precisely, there is an equivalence

$$\mathrm{LMod}^{-}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{LMod}^{+}_{\mathrm{C}^\bullet(X;\Bbbk)} \tag{I.1.4}$$

between *bounded above* $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$-modules, and *bounded below* $\mathrm{C}^\bullet(X;\Bbbk)$-modules (compare with [BGS96, Theorem 12.6]). Alternatively, we can modify the notion of module we work with. Namely, the functor (I.1.2) induces an equivalence

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{IndCoh}_{\mathrm{C}^\bullet(X,\Bbbk)} \tag{I.1.5}$$

where the right-hand side is the category of *ind-coherent* modules over $\mathrm{C}^\bullet(X;\Bbbk)$, which we define formally in Section 5 of the main text: suffice it to say for the moment that, in this setting, this is the category generated by the augmentation module. It is this latter formulation of $\mathbb{E}_1$-Koszul duality which will be particularly relevant for our approach to $\mathbb{E}_n$-Koszul duality.

Now let X be a pointed and n-connected finite CW complex. Much as before, we can associate to X two augmented algebras: except now these will be $\mathbb{E}_n$- rather than $\mathbb{E}_1$-algebras. On the one hand, the n-th iterated loop space

$$\Omega^n_* X \coloneqq \Omega_* \ldots \Omega_* X$$

[1] When, in the course of this Introduction, we say informally that a certain statement is "*almost* never true" (or some such formula to this effect) we mean one of two things: either that one can prove that there are no non-trivial examples (such as the point or discrete spaces); or, as in the present instance, that we believe, but do know how to prove, that there are no non-trivial examples; and further that there are obstructions preventing the statement to apply to many commonly occurring spaces.

is a $\mathbb{E}_n$-space; thus, $\mathrm{C}_\bullet(\Omega^n_* X;\Bbbk)$ carries a $\mathbb{E}_n$-product. On the other hand, the algebra of $\Bbbk$-valued cochains $\mathrm{C}^\bullet(X;\Bbbk)$ on X is naturally a $\mathbb{E}_\infty$-algebra, so we can regard it in particular as an $\mathbb{E}_n$-algebra. The key claim is that these two algebras are $\mathbb{E}_n$-Koszul dual to each other:

$$\mathrm{C}_\bullet\big(\Omega^n_* X, \Bbbk\big) \longleftrightarrow \mathrm{C}^\bullet(X;\Bbbk).$$

Applying [Lur11b, Theorem 4.4.5] one can *almost* deduce an $\mathbb{E}_n$-analogue of statement (1). Indeed, using [Lur17, Example 5.3.1.5 and Lemma 5.3.1.11], one can prove that the Koszul dual of an augmented $\mathbb{E}_n$-algebra $A \to \Bbbk$ is the morphism object

$$A^! := \underline{\mathrm{Map}}_{\mathrm{Mod}_A^{\mathbb{E}_n}}(A, \Bbbk).$$

However, no analogue of statement (2) has been established in the literature. In fact, as far as we are aware of, even how to properly formulate (2) in the $\mathbb{E}_n$-setting was not known. No doubt one of the reasons for this gap in the literature is due to the subtle nature of the equivalence: as we discussed this is not a straightforward equivalence between categories of modules; its formulation requires sophisticated ingredients which are not easily adapted to the $\mathbb{E}_n$-setting. In this paper we prove an $\mathbb{E}_n$-analogue of statement (2), and this will yield in particular an $\mathbb{E}_n$-analogue of statement (1). Conceptually, our main innovation consists in reinterpreting statement (2), and in particular equivalence (I.1.5), from a novel perspective which makes categorification possible.

To clarify our results, we shall focus on the case $n = 2$. We will give a more complete summary of our main results, for all n, in (I.2) of this Introduction. Consider the following diagram of $(\infty, 2)$-categories.

$$\begin{tikzcd}
\mathbf{2LMod}_{\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega^2_* X;\Bbbk)}}\big(\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}\big) \arrow[r, leftrightarrow, dotted, "A"] \arrow[d, leftrightarrow, dotted, "B"'] & \mathbf{2LMod}_{\mathrm{LMod}_{\mathrm{C}^\bullet(X;\Bbbk)}}\big(\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}\big) \arrow[d, leftrightarrow, dotted, "C"] \\
\mathbf{2LocSysCat}(X;\Bbbk) \arrow[r, leftrightarrow, dotted, "D"'] & \mathbf{2LMod}_{\mathrm{LocSys}(X;\Bbbk)}\big(\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}\big)
\end{tikzcd} \tag{I.1.6}$$

Here $\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}$ denotes the $(\infty, 2)$-category of presentable categories. All categories appearing in the diagram, except $\mathbf{2LocSysCat}(X;\Bbbk)$, are defined as $(\infty, 2)$-category of modules for appropriate $\mathbb{E}_1$-algebra objects (i.e. monoidal categories) in $\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}$. These four categories all play a role in a categorification of Koszul duality.

Not all arrows in the diagram stand for equivalences. Let us briefly comment on each of them separately.

- The category $\mathbf{2LocSysCat}(X;\Bbbk)$ is the $(\infty, 2)$-category of local systems of k-linear presentable categories over X. Arrow B categorifies the presentation of local systems as monodromy data (I.1.1). We show that B is an equivalence in Section 1 of the main text, where we extend more generally (I.1.1) to local systems of (∞, n)-categories for all n.

- Using the terminology of [Gai15], arrow D is an equivalence when the Betti stack X_{B} associated to X is 1-*affine*. This is a natural categorification of the notion of affineness (see the discussion after (I.1.3) above) which is due to Gaitsgory. In Section 4 we shall prove that 1-truncated Betti stacks are 1-affine; and that the non-vanishing of $\pi_2 \otimes \Bbbk$ is an obstruction to 1-affineness (although not the only one). In fact, recent work of Stefanich allows us to make sense of the notion of n-affineness for all n, and we will also establish analogues of these results in the context of n-affineness.

  We stress that the fact that Betti stacks typically fail to be n-*affine* is the main source of difficulties in higher Koszul duality theory, just as in the classical story. As we discussed, it is precisely because Betti stacks are virtually never *affine* that $\mathbb{E}_1$-Koszul duality fails to be a Morita equivalence. The question of affineness of Betti stacks is therefore closely related to Koszul duality, and this is why we devote Section 4 to an in-depth investigation of it.
- Arrow C is almost never an equivalence. This boils down to the failure of $\mathbb{E}_1$-Koszul duality to induce a Morita equivalence. For the same reason, arrow A is almost never an equivalence. Note that if A were an equivalence, then the two $\mathbb{E}_2$-algebras $C_\bullet(\Omega^2_* X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$ would actually be $\mathbb{E}_2$-Morita equivalent, in the sense that their categories of iterated modules would be equivalent.

  Based on classical $\mathbb{E}_1$-Koszul duality we should not expect such a straightforward statement to hold, and indeed it is typically false. However, in Section 5 we explain how to modify the 2-category $2\mathbf{LMod}_{\mathrm{LMod}_{\mathrm{C}^\bullet(X;\Bbbk)}}\left(2\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$ in such a way that A becomes an equivalence. We regard the resulting equivalence as the analogue of (I.1.5) in the setting of $\mathbb{E}_2$-Koszul duality. Also, we show how to obtain analogous results in the context of $\mathbb{E}_n$-Koszul duality for all n.

As this is one of the main contributions of this article, let us explain it in some more detail. It turns out that instead of viewing $\mathrm{C}^\bullet(X;\Bbbk)$ merely as a $\mathbb{E}_\infty$-algebra, we can do algebraic geometry with it. The algebra of cochains $\mathrm{C}^\bullet(X;\Bbbk)$ can be endowed with a structure of a commutative dg-algebra, but it does not fall within the range of ordinary derived geometry because it fails to be connective: its homology vanishes in positive degrees, and is concentrated in negative degrees; the contrary of what we require of a derived affine scheme. Toën ([Toë06]) and Lurie ([Lur11a]), have explained that we can view such an algebra as the algebra of functions on a *coaffine stack*, which is called its *cospectrum*. The cospectrum of $\mathrm{C}^\bullet(X;\Bbbk)$ is denoted $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$.

Quasi-coherent sheaves on $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ can be viewed as a *renormalization* of the category of $\mathrm{C}^\bullet(X;\Bbbk)$-modules. Under our assumptions on X, they coincide with ind-coherent modules

$$\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \mathrm{IndCoh}_{\mathrm{C}^\bullet(X,\Bbbk)}\,. \tag{I.1.7}$$

This yields a new formulation of the *almost* Morita equivalence which is at the heart of $\mathbb{E}_1$-Koszul duality. Namely, combining (I.1.5) and (I.1.7) we obtain an equivalence

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))). \tag{I.1.8}$$

In this way, the notion of ind-coherent $\mathrm{C}^\bullet(X;\Bbbk)$-module required to turn Koszul duality into an actual Morita equivalence is encoded in the geometry of $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. The great advantage over other formulations of Koszul duality is that equivalence (I.1.8) is well adapted to categorification.

Our main result in Section 5 is that, if X is a 2-connected finite CW complex,[2] there is an equivalence of $(\infty,2)$-categories between iterated modules over $\mathrm{C}_\bullet(\Omega^2_* X;\Bbbk)$ and *quasi-coherent sheaves of categories* over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. This is the analogue of equivalence (I.1.5) in the $\mathbb{E}_2$-setting: as in the classical story, this means in particular that if X is a 2-connected finite CW complex the theory of categorified local systems over X only depends on the algebra of cochains $\mathrm{C}^\bullet(X;\Bbbk)$.

This equivalence fits as the top arrow in the following commutative diagram of equivalences, which should be viewed as a better behaved second take on diagram (I.1.6).

$$\begin{tikzcd}
\mathbf{2LMod}_{\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega^2_* X;\Bbbk)}}\left(\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}\right) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, leftrightarrow, "\wr"'] & \mathbf{2ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \arrow[d, leftrightarrow, "\wr"] \\
\mathbf{2LocSysCat}(X;\Bbbk) \arrow[r, leftrightarrow, "\simeq"] & \mathbf{2ShvCat}(X_{\mathrm{B}})
\end{tikzcd} \tag{I.1.9}$$

Let us explain our notations: here $\mathbf{2ShvCat}(-)$ denotes the symmetric monoidal $(\infty,2)$-category of *quasi-coherent sheaves of categories*, which was first introduced by Gaitsgory. It is a categorification of quasi-coherent sheaves in the precise sense that it is a delooping of $\mathrm{QCoh}(-)$: i.e. $\mathrm{QCoh}(-)$ can be recovered as the endomorphisms of the unit object in $\mathbf{2ShvCat}(-)$. In Section 5 we also prove analogous results for n-connected finite CW complex in the context of $\mathbb{E}_n$-Koszul duality.

I.2. **Main results.**

We shall give next a more analytical description of the contents of the paper, and state our main results. In Section 1 we survey briefly all preliminary material which will be required in the remainder of the paper. In particular we give a thorough overview of the basic definitions and results in the theory of local systems. Let $\mathscr{C}$ be an $(\infty,1)$-category, and let X be a space. We define the category of $\mathscr{C}$-valued local systems on X as the category of functors

$$\mathrm{LocSys}(X;\mathscr{C}) := \mathrm{Fun}(X,\mathscr{C})$$

[2]Our results hold in fact in greater generality, see Section 5 for the precise assumptions we need.

between X (viewed as an ∞-groupoid) and $\mathscr{C}$. It is a fundamental fact proved by Lurie, and then by Beardsley and Péroux [BP19] in a formulation which is more directly relevant for us, that if $\mathscr{C}$ is presentable then $\mathscr{C}$-valued local systems can be encoded as monodromy data. Namely, let X be a connected space, and let $\mathscr{C}$ be a *presentable* $(\infty,1)$-category. Then there exists an equivalence of $(\infty,1)$-categories

$$\mathrm{LocSys}(X;\mathscr{C}) \simeq \mathrm{LMod}_{\Omega_* X}(\mathscr{C}). \tag{I.2.1}$$

Equivalence (I.2.1) is fundamental, but its generality is too narrow to be applicable to local systems of categories. For instance, we would like to take $\mathscr{C}$ to be $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ itself, as this would allow us to describe local systems whose sections are presentable categories; however $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is not presentable, but only cocomplete, and therefore does not fall under the scope of the previous statement. In Section 2 we address this issue, by proving that cocompleteness is sufficient to obtain a monodromy description of local systems.

In Section 3, we generalise equivalence (I.2.1) to local systems of *presentable* (∞,n)-categories. Presentable (∞,n)-categories have been recently introduced by Stefanich [Ste20]. For the benefit of the reader we include a summary of the theory in the main text. For the sake of this Introduction however we will limit ourselves to say that the category $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ of presentable (∞,n)-categories is symmetric monoidal and that it is a n-fold delooping of $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ (i.e. we can recover $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ by taking iterated endomorphisms of the unit). It enjoys many of the formal properties of $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$, and as such it provides a favourable environment for (∞,n)-category theory. We denote the incarnation as a $(\infty,n+1)$-category of $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ as $(n+1)\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$. The next is one of our main results. We comment on the statement below.

Theorem A (Theorem 3.2.24)**.** *Let $n \geqslant 1$ be an integer, let X be a pointed n-connected space (i.e., $\pi_k(X) \cong 0$ for every $k \leqslant n$). Then there exist equivalences of $(\infty,n+1)$-categories*

$$(n+1)\mathbf{LocSysCat}^n(X) \simeq (n+1)\mathbf{LMod}_{n\mathbf{LMod}_{\Omega_*^{n+1}X}(\mathcal{S})}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}.$$

Let us make some comments on the statement, as some of the notations will only be introduced in the main text. The category on the left hand side is the $(\infty,n+1)$-category of local systems of presentable (∞,n)-categories over X; the category on the right hand side is the $(\infty,n+1)$-category of presentable (∞,n)-categories with an action of the presentable (∞,n)-category of iterated left modules over the grouplike topological $\mathbb{E}_{n+1}$-monoid $\Omega_*^{n+1}X$. As we mentioned earlier the connectedness assumptions on X can be dropped, we clarify this point in Paragraph 3.2.30. We also stress that in the main text we always work relative to a monoidal presentable category $\mathscr{A}$: for simplicity we stated above our result only in the absolute case, when $\mathscr{A}$ is the category of topological spaces $\mathcal{S}$. Of particular interest for applications is also the stable setting when $\mathscr{A}$ is e.g. the category of spectra or $\Bbbk$-modules for a field $\Bbbk$. In this latter case, the corresponding $(\infty,n+1)$-category on the right hand side in Theorem A can be interpreted as an $(\infty,n+1)$-category of "presentable $\mathrm{C}_\bullet(\Omega_*^{n+1}X;\Bbbk)$-linear n-categories".

It is implicit in the statement of Theorem A that we can make sense of the action of a topological monoid on a category. This is built in in ∞-category theory: by definition all cocomplete ∞-categories (e.g. $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$) are tensored over the ∞-category of spaces $\mathcal{S}$. This provides a natural notion of the action of a monoid in $\mathcal{S}$ on an object in a cocomplete category, and so in particular on a presentable category (i.e. an object in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$, which is itself cocomplete). Working in the dg setting makes the nature of topological actions on categories less evident, and in fact Teleman proposes various alternative definitions in [Tel14], before settling on one. We show that Teleman's preferred model coincides with the natural concept of topological action in ∞-category theory, and turn Teleman's ansatz into a theorem.[3]

Theorem B (Theorem 2.17)**.** *Topological actions of a connected group G on a differential graded category $\mathscr{C}$ which is linear over some base commutative ring $\Bbbk$ are completely captured, up to contractible choices, by the induced $\mathbb{E}_2$-algebra morphisms*

$$\mathrm{C}_\bullet(\Omega_* G;\Bbbk) \longrightarrow \mathrm{HH}^\bullet(\mathscr{C})$$

where the source is simply the algebra of chains of $\Omega_* G$ with coefficients in $\Bbbk$, endowed with the Pontrjagin product, and the target is the Hochschild cohomology of the differential graded category $\mathscr{C}$.

In Section 4 we study the question of n-affineness for Betti stacks. We obtain a complete characterization of n-affine Betti stacks, which has however the drawback of not being explicit: it reduces the question of n-affineness of a Betti stack X_{B}, which is n-categorical in nature, to a purely 1-categorical condition on the Betti stack of the iterated loop space $\Omega^n_* X$. This condition is however difficult to check in practice, see Theorem C below. To obviate this shortcoming we extract from Theorem C one necessary condition, and one sufficient condition, which are both easily verifiable.

Theorem C (Theorem 4.2.9)**.** *Let X be a space with a choice of a base point. Then its Betti stack X_{B} is n-affine if and only if the global section functor*

$$\Gamma(\Omega^n_* X,-)\colon \mathrm{LocSys}(\Omega^n_* X;\Bbbk) \longrightarrow \mathrm{Mod}_{\Bbbk}$$

is monadic.

Theorem D (Theorem 4.2.8 and Corollary 4.2.24)**.** *Let X be a space, and let $\Bbbk$ be a field of characteristic* 0.

[3] We remark that the statement below appears as [Tel14, Theorem 2.5]. We stress however that in [Tel14] this claim appears without proof: in fact one could argue that rather than a theorem, it is a reformulation of Teleman's definition of a topological action on dg-categories. Within the framework of ∞-categories, however, it becomes a a non-tautological statement about actions of topological monoids, and we will prove it rigorously. In this respect, we believe that our contribution consists in providing a formalisation of Teleman's insight within the theory of ∞-categories.

- *If X is n-truncated, then its Betti stack X_{B} is n-affine.*
- *Suppose that $\pi_{n+1}(X)$ does not vanish for some choice of a base point in X. Then the Betti stack X_{B} is not n-affine over $\Bbbk$.*

In Section 5 we turn our attention to $\mathbb{E}_n$-Koszul duality. In addition to our results proper, we believe that one of our main contributions in this section is of a conceptual nature. We propose that the *cospectrum* of the coconnective cdga of cochains on X, $\mathrm{C}^\bullet(X;\Bbbk)$, should play a key role in the study of Koszul duality for this class of algebras. We test this idea first in the classical case, where we show that if X is simply connected (and sufficiently finite) there is an equivalence of categories

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))). \tag{I.2.2}$$

As we explained, the category $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ is not equivalent to the category of $\mathrm{C}^\bullet(X;\Bbbk)$-modules (for which the equivalence above does not hold), though it is closely related: as noted in [Lur11a], $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ is the left completion of the natural t-structure on $\mathrm{C}^\bullet(X;\Bbbk)$-modules. Equivalence (I.2.2) shows that if we replace $\mathrm{C}^\bullet(X;\Bbbk)$-modules with quasi-coherent sheaves on $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ we can formulate Koszul duality as an actual equivalence of categories. Our main result in Section 5 is a categorification of (I.2.2).

Theorem E (Theorem 5.22)**.** *Let $n \geqslant 1$ be an integer, let $\Bbbk$ be a field of characteristic* 0*, and let X be a pointed $(n+1)$-connected space satisfying appropriate finiteness conditions. Then there is a natural equivalence of $(\infty, n+1)$-categories*

$$(n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq (n+1)\mathbf{LocSysCat}^n(X;\Bbbk). \tag{I.2.3}$$

Combining this with Theorem A we obtain an equivalence of $(\infty, n+1)$-categories

$$(n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq (n+1)\mathbf{Mod}_{n\mathbf{Mod}^{n-1}_{\mathrm{C}_\bullet(\Omega^{n+1}_* X;\Bbbk)}}\Big((n+1)\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\Big)$$

which is an n-fold categorification of equivalence (I.2.2).

In the statement of Theorem E, the $(n+1)$-category $n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ is the $(n+1)$-category of quasi-coherent sheaves of (presentably $\Bbbk$-linear) n-categories over the cospectrum of the $\Bbbk$-valued cochains $\mathrm{C}^\bullet(X;\Bbbk)$ of X. This is an n-categorification of the usual category of quasi-coherent sheaves: when $n=2$, this was defined in [Gai15], while for arbitrary n it has been recently introduced in [Ste21].

Before proceeding, let us comment further on equivalence (I.2.3), which we consider to be the deepest result of this paper. The appearance of $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ in the statement might seem only a technical artefact of our approach; on the contrary, we believe that our work clarifies the true nature of Koszul duality in the topological setting. There is a canonical map of stacks

$$\mathrm{aff}_X : X_{\mathrm{B}} \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$$

called the *affinization map*. Equivalence (I.2.3) is given precisely by the pull back along aff_X. The real content of Koszul duality in this setting is therefore that, under appropriate connectivity assumptions on X, the theory of (higher) local systems does not distinguish between X_{B} and $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. We believe this to be a more transparent statement already in the classical case $n=1$ where, as we discussed, the standard formulation of the duality between $\mathrm{C}^\bullet(X;\Bbbk)$ and $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ requires otherwise artificial size restrictions, or t-structure renormalizations.

I.3. **Local systems of categories and symplectic geometry.** Like Teleman, our interest in the questions studied in this article stems in large part from the fact that symplectic geometry furnishes examples of local systems of categories, by applying the theory of Fukaya categories to Hamiltonian fibrations. As this is one of the main motivations underlying our work we find it worthwhile to explain this story in some detail. Let (S,s_0) be a connected based space. A *Hamiltonian fibration* over S is a smooth fibration of manifolds $\pi : X \to S$ where each fiber $X_s = \pi^{-1}(s)$ is equipped with a symplectic form ω_s, and the fibration is equipped with a reduction of the structure group to $\mathrm{Ham}(X_{s_0},\omega_{s_0})$. By this definition, to a Hamiltonian fibration there is an associated classifying map

$$S \longrightarrow \mathbf{B}\mathrm{Ham}(X_{s_0},\omega_{s_0}).$$

A Hamiltonian fibration has an underlying symplectic fibration classified by a map $S \to \mathbf{B}\mathrm{Symp}(X_{s_0},\omega_{s_0})$. When S is simply-connected, the reduction of the structure group from Symp to Ham is equivalent to a choice of closed two-form $\tau \in \Omega^2_{\mathrm{cl}}(X)$ such that $\tau|X_s = \omega_s$ for every $s \in S$ [MS98, Theorem 6.36]. Such a two-form τ defines a Ehresmann connection on $\pi : X \to S$ by taking the τ-orthogonals to the fibers, and the looped classifying map

$$\Omega_* S \longrightarrow \mathrm{Ham}(X_{s_0},\omega_{s_0})$$

admits an interpretation as the holonomy of such a connection (at least if π is proper or if the connection has appropriately tame behavior at infinity).

When (X_{s_0},ω_{s_0}) is a monotone symplectic manifold, Savelyev [Sav23] has constructed a map of ∞-categories

$$\mathbf{B}\mathrm{Ham}(X_{s_0},\omega_{s_0}) \longrightarrow \widehat{\mathrm{Cat}}_{(\infty,1)}$$

(the source is an ∞-groupoid) whose value at the base point is $\mathrm{Fuk}(X_{s_0},\omega_{s_0})$. Thus one obtains a map $S \to \widehat{\mathrm{Cat}}_{(\infty,1)}$ by composing the classifying map of the Hamiltonian fibration with Savelyev's map. In the terminology of the present paper, this is nothing but a local system of $(\infty,1)$-categories over S.

From another perspective, Oh and Tanaka [OT22] have constructed a topological action of $\mathrm{Ham}(X_s,\omega_s)$ on $\mathrm{Fuk}(X_s,\omega_s)$ when (X_s,ω_s) is a Liouville sector. We shall show that such a topological action is equivalent to a local system of $(\infty,1)$-categories over $\mathbf{B}\mathrm{Ham}(X_s,\omega_s)$

(Corollary 1.2.8, a special case of Theorem A), and hence one once again obtains a local system of $(\infty,1)$-categories over S.

Proposition I.3.1. *Let $\pi : X \to S$ be a Hamiltonian fibration, such that the fibers (X_s, ω_s) are either compact monotone or are Liouville sectors. Then there is an associated local system of $(\infty,1)$-categories over S whose fiber over $s \in S$ is the Fukaya category of (X_s, ω_s).*

A related case is where (X,ω) carries a Hamiltonian action of a Lie group G. The remarks above then apply with $S = \mathbf{B}G$. Following Teleman's insight we will show (see Theorem B of this Introduction for a precise statement) that we obtain a map

$$\mathrm{C}_\bullet(\Omega^2_* S; k) \longrightarrow \mathrm{HH}^*(\mathrm{Fuk}(X_{s_0}, \omega_{s_0})) \cong \mathrm{QH}(X_{s_0}, \omega_{s_0}).$$

When $S = \mathbf{B}\mathrm{Ham}(X_{s_0}, \omega_{s_0})$, this recovers the celebrated Seidel homomorphism.

Notations and conventions.

- We will use throughout the language of $(\infty,1)$-categories and higher homotopical algebra, as developed in [Lur09; Lur17], from which we borrow most of the notations and conventions.
- Since our work heavily relies on intrinsically derived and homotopical concepts, we shall simply write "limits", "colimits", "tensor product", suppressing adjectives such as "homotopy" or "derived" in our notations. Similarly, we shall simply write "categories" instead of "$(\infty,1)$-categories", and "n-categories" instead of "(∞,n)-categories".
- We will work with *local systems* and *sheaves* of categories, and it will be important pay attention to size issues. We fix a sequence of nested universes $\mathcal{U} \in \mathcal{V} \in \mathcal{W} \in \dots$. We shall say that a category $\mathscr{C}$ is *small* if it is $\mathcal{U}$-small, that $\mathscr{C}$ is *large* if it is $\mathcal{V}$-small without being $\mathcal{U}$-small, that $\mathscr{C}$ is *very large* if it is $\mathcal{W}$-small without being $\mathcal{V}$-small, and that $\mathscr{C}$ is *huge* if it is not even $\mathcal{W}$-small. When dealing with categories of (possibly decorated) categories, we shall adopt the following notations in order to distinguish the size: large categories of categories will be denoted with a normal font; very large categories of categories will be denoted with $\widehat{(-)}$; huge categories of categories will be denoted with $\widehat{(-)}$ and capital letters.
  For example, $\mathrm{Cat}_{(\infty,1)}$ is the large category of small categories, while $\widehat{\mathrm{Cat}}_{(\infty,1)}$ is the very large category of large categories, and $\widehat{\mathrm{CAT}}_{(\infty,1)}$ is the huge category of very large categories.
- We shall denote the large category of small spaces by $\mathcal{S}$. In particular, by *space* we always mean *small space*.
- The large category $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ of large presentable categories and the very large category of $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ of large cocomplete categories are both symmetric monoidal categories: $\mathbb{E}_k$-algebras inside $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ and $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ are (respectively) presentable and cocomplete

categories endowed with an $\mathbb{E}_k$-monoidal structure that commutes with colimits separately in each variable. In order to compactify our notations, in the rest of our paper we shall refer to an $\mathbb{E}_k$-algebra in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ as a *presentably $\mathbb{E}_k$-monoidal category*, and to an $\mathbb{E}_k$-algebra in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ as a *cocompletely $\mathbb{E}_k$-monoidal category*; in the case $k = \infty$ we shall simply write *symmetric monoidal* in place of $\mathbb{E}_\infty$-monoidal. The notation for $\mathbb{E}_k$-algebras in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ can sound ambiguous, since an $\mathbb{E}_k$-monoidal structure on a cocomplete category can fail to be compatible with colimits: for an easy counterexample, just consider the category of pointed spaces endowed with the Cartesian symmetric monoidal structure. However, we shall never be interested in such kind of monoidal structures in this work.

- In a similar fashion, for any $k \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}$ and a cocompletely (resp. presentably) $\mathbb{E}_k$-monoidal ∞-category $\mathscr{A}$, we shall say that a category $\mathscr{C}$ is *cocompletely* (resp. *presentably*) left tensored over $\mathscr{A}$ if it is a left $\mathscr{A}$-module in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ (resp. in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$). This formula amounts to the datum of a cocomplete (or presentable) category $\mathscr{C}$ which is left tensored over $\mathscr{A}$ in such a way that the tensor action functor commutes with colimits separately in each argument.
- Starting from Section 3.2, we shall deal with higher (i.e., n-)categories, and in particular with $(n+1)$-categories of (possibly decorated) n-categories. We shall denote such n-categories with a bold font. In order to avoid confusion concerning the “categorical height” we are working at, we shall also adopt the following highly non-standard notation as well: if we want to refer to the (very large) higher category of large m-categories seen as a n-category, we shall write $n\widehat{\mathbf{Cat}}_{(\infty,m)}$. In the particular case $n = 1$, we shall drop both the bold font and the 1 before our notations, and simply write $\widehat{\mathrm{Cat}}_{(\infty,m)}$. For example, $3\widehat{\mathbf{Cat}}_{(\infty,2)}$ is the very large 3-category of all large 2-categories, while $2\widehat{\mathbf{Cat}}_{(\infty,2)}$ is its underlying 2-category, and $\widehat{\mathrm{Cat}}_{(\infty,2)}$ is its underlying 1-category. (See also Notation 3.0.1.)
- Most of the times we will consider categories which are enriched over some preferred category (e.g., modules in spectra which are enriched over themselves, or presentably enriched categories which are enriched over themselves, and so forth). At the same time, we will need to consider the underlying spaces of maps between objects in such categories. For this reason, when $\mathscr{C}$ is enriched over a category $\mathscr{A}$, we will denote as $\mathrm{Map}_{\mathscr{C}}(-,-)$ the space of maps in $\mathscr{C}$, and as $\underline{\mathrm{Map}}_{\mathscr{C}}(-,-)$ the morphism object of $\mathscr{A}$ providing the enrichment, so as to to highlight whether we are seeing a morphism object as a space or as something more structured. If $\mathscr{C}$ is a higher category of categories (e.g., $\mathscr{C} = \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ or $\mathscr{C} = \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$) we will also use $\underline{\mathrm{Fun}}(-,-)$, possibly with decorations, to mean the category of structure-preserving functors which serves as the category of morphisms in $\mathscr{C}$.

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1. Categorical local systems and categorical loop space representations

1.1. **Preliminaries on local systems in the homotopy setting.**

In this section we collect the first definitions and notations concerning categorical local systems over spaces, i.e., local systems over spaces with coefficients in some category of categories. Our preferred coefficients shall be the 2-category of presentable categories, possibly enriched over a presentably symmetric monoidal category $\mathscr{A}$.

Given a (strict) topological space, we have a natural way to define what a local system with coefficients in some category is.

1.1.1. Let X be a topological space, let $\mathrm{Op}(X)$ be the poset of its open subsets equipped with the Grothendieck topology τ generated by jointly surjective maps, and let $\mathscr{C}$ be any category. We can either consider the topos

$$\mathrm{Shv}(X;\mathscr{C}) := \mathrm{Shv}_\tau(\mathrm{Op}(X);\mathscr{C})$$

of $\mathscr{C}$-valued sheaves over X, or its hypercompletion

$$\mathrm{Shv}^{\mathrm{hyp}}(X;\mathscr{C}) := \widehat{\mathrm{Shv}}(X;\mathscr{C}).$$

The latter is a localization of the former, i.e., it is a full subcategory closed under limits which admits a hypersheafification left adjoint

$$(-)^{\mathrm{hyp}}\colon \mathrm{Shv}(X;\mathscr{C}) \longrightarrow \mathrm{Shv}^{\mathrm{hyp}}(X;\mathscr{C}).$$

Definition 1.1.2. Let X be a topological space and let $\mathscr{C}$ be any category.

(1) We say that a $\mathscr{C}$-valued sheaf $\mathscr{F}$ over X is *constant* if it lies in the essential image of the pullback functor

$$\Gamma^*\colon \mathscr{C} \simeq \mathrm{Shv}(\{*\};\mathscr{C}) \longrightarrow \mathrm{Shv}(X;\mathscr{C}).$$

(2) We say that a sheaf $\mathscr{F}$ is *locally constant* if there exists a small collection of objects $\{U_\alpha \hookrightarrow X\}_\alpha$ which is jointly surjective over X such that $\mathscr{F}|_{U_\alpha}$ is constant in $\mathrm{Shv}(U_\alpha;\mathscr{C})$.

(3) We say that a hypersheaf $\mathscr{F}$ over X is *hyperconstant* if it belongs to the essential image of the functor

$$\Gamma^{\mathrm{hyp},*}\colon \mathscr{C} \xrightarrow{\Gamma^*} \mathrm{Shv}(X;\mathscr{C}) \xrightarrow{(-)^{\mathrm{hyp}}} \mathrm{Shv}^{\mathrm{hyp}}(X;\mathscr{C}).$$

(4) We say that a hypersheaf $\mathscr{F}$ over X is *hyperlocally hyperconstant* if there exists a small collection of objects $\{U_\alpha \hookrightarrow X\}_\alpha$ which is jointly surjective over X such that $\mathscr{F}|_{U_\alpha}$ is hyperconstant in $\mathrm{Shv}^{\mathrm{hyp}}(U_\alpha;\mathscr{C})$.

Locally constant sheaves and locally hyperconstant hypersheaves form two full subcategories of $\mathrm{Shv}(X;\mathscr{C})$ and $\mathrm{Shv}^{\mathrm{hyp}}(X;\mathscr{C})$. Call them $\mathrm{LC}(X;\mathscr{C})$ and $\mathrm{LC}^{\mathrm{hyp}}(X;\mathscr{C})$, respectively.

Warning 1.1.3 ([HPT23, Warning 1.19])**.** It is not in general true that the natural inclusion

$$\mathrm{LC}(X;\mathscr{C}) \cap \mathrm{Shv}^{\mathrm{hyp}}(X;\mathscr{C}) \subseteq \mathrm{LC}^{\mathrm{hyp}}(X;\mathscr{C})$$

is an equivalence of categories, unless X is *locally of constant shape* in the sense of [Lur17, Definition A.4.15]. In this case, all locally constant sheaves are locally hyperconstant ([Lur17, Corollary A.1.7]), hence the two expressions obviously match.

[HPT23, Corollary 3.7 and Observation 3.8] show that the correct notion of locally constant $\mathscr{C}$-valued sheaf over a topological space X is given by locally hyperconstant hypersheaves. If $\mathscr{C}$ is a presentable category, then

$$\mathrm{LC}^{\mathrm{hyp}}(X;\mathscr{C}) \simeq \mathcal{S}_{/X} \otimes \mathscr{C} \simeq \mathrm{Fun}(X,\mathscr{C}) \tag{1.1.4}$$

where $\otimes$ denotes the Lurie tensor product of presentable categories and where we have implicitly identified X and its fundamental groupoid $\Pi_\infty(X)$.

We are mostly interested in considering topological spaces X as objects in the category $\mathcal{S}$ of homotopy types, rather than in the point-set theoretic sense. Equivalence (1.1.4) gives us a way to think about local systems on a strict topological spaces in terms of data that only depend on its underlying homotopy type, i.e. $\mathscr{C}$-valued functors out of X (at least if $\mathscr{C}$ is presentable). This motivates the following definition.

Notation 1.1.5. For X a space and for $\mathscr{C}$ *any* category, we set

$$\mathrm{LocSys}(X;\mathscr{C}) := \mathrm{Fun}(X,\mathscr{C}).$$

If $\mathscr{C} := \mathcal{S}$ is the category of spaces itself, we shall simplify notations and set

$$\mathrm{LocSys}(X) := \mathrm{LocSys}(X;\mathcal{S}).$$

In the same way, if $\Bbbk$ is any $\mathbb{E}_1$-ring spectrum and $\mathscr{C} := \mathrm{Mod}_\Bbbk$ is the category of $\Bbbk$-modules in spectra, then we set

$$\mathrm{LocSys}(X;\Bbbk) := \mathrm{LocSys}(X;\mathrm{Mod}_\Bbbk).$$

In the rest of the paper, we shall often abuse notations and identify a topological space X with its underlying homotopy type $\Pi_\infty(X)$, and simply refer to it as a *space*.

For later use, we recall the following fundamental monodromy equivalence statement.

Lemma 1.1.6 (Monodromy equivalence, [BP19, Lemma 3.9])**.** *Let X be a connected space, and let $\mathscr{C}$ be a presentable category. Then there exists an equivalence of categories*

$$\mathrm{LocSys}(X;\mathscr{C}) \simeq \mathrm{LMod}_{\Omega_* X}(\mathscr{C}).$$

1.2. **The monodromy equivalence for cocomplete categories of coefficients.** Our goal in this Section is generalizing Lemma 1.1.6 to cocomplete categories which are not necessarily presentable. In order to explain why this is important for our project, we have to introduce first some objects which will play a key role in the sequel.

Let $\mathscr{A}$ be a presentably symmetric monoidal category. Let $\mathrm{Mod}_{\mathscr{A}}\left(\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$ the category of presentable categories which are presentably tensored over $\mathscr{A}$. By [Hei23, Theorem 1.2], this is the same as the category of presentable categories enriched over $\mathscr{A}$ in the sense of [GH15]. In symbols, there is an equivalence

$$\mathrm{Mod}_{\mathscr{A}}\left(\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \simeq \mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}. \tag{1.2.1}$$

In the sequel we shall always use the notation $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ to refer to this category. In the particular case in which $\mathscr{A}$ is the category of $\Bbbk$-modules in spectra for some $\mathbb{E}_\infty$-ring spectrum $\Bbbk$ we shall simply write $\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. This includes the case in which $\Bbbk = \mathbb{S}$ is the sphere spectrum, hence $\mathrm{Lin}_{\mathbb{S}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \simeq \mathrm{Pr}^{\mathrm{L,st}}_{(\infty,1)}$ denotes the category of stable presentable categories.

Remark 1.2.2. Note that if we are considering left $\mathscr{A}$-modules in the larger category of categories $\widehat{\mathrm{Cat}}_{(\infty,1)}$, left $\mathscr{A}$-modules and $\mathscr{A}$-enriched categories are *not* equivalent anymore. Indeed [Hei23, Theorem 1.1] guarantees only an equivalence between *closed* left $\mathscr{A}$-modules in $\widehat{\mathrm{Cat}}_{(\infty,1)}$ and a *non-full* subcategory of all $\mathscr{A}$-enriched categories.

Notation 1.2.3. We set

$$\mathrm{LocSysCat}(X) \coloneqq \mathrm{LocSys}\left(X;\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \quad \text{and} \quad \mathrm{LocSysCat}(X;\mathscr{A}) \coloneqq \mathrm{LocSys}\left(X;\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right).$$

Note that, although both $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ and $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ are complete and cocomplete categories, they are *not* presentable. Thus we cannot apply Lemma 1.1.6 directly to $\mathrm{LocSysCat}(X)$ and $\mathrm{LocSysCat}(X;\mathscr{A})$. The main result of this Section, Proposition 1.2.7, generalizes Lemma 1.1.6 to cocomplete categories and thus will allow us to circumvent this difficulty.

The following recent result of Stefanich is the key ingredient in the generalization of Lemma 1.1.6 to cocomplete categories.

Proposition 1.2.4 ([Ste20, Propositions 5.1.4 and 5.1.14])**.** *Let $\mathscr{A}$ be any presentably symmetric monoidal category, let $\mathrm{Lin}_{\mathscr{A}}\,\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ be the category of $\mathscr{A}$-linear cocomplete categories, and let κ_0 be the smallest large cardinal of the theory. Then $\mathrm{Lin}_{\mathscr{A}}\,\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ is κ_0-compactly generated by the category $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ of presentably $\mathscr{A}$-linear categories.*

Using Proposition 1.2.4, we shall write any cocomplete category $\mathscr{C}$ as a (large) filtered colimit of presentable categories, and then use Lemma 1.1.6 to deduce a more general version of the monodromy equivalence for categorical local systems. First, we need to establish that the operation of taking modules for a topological $\mathbb{E}_k$-monoid commutes with colimits of large cocomplete categories.

Lemma 1.2.5. *Let $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ be the very large category of large cocomplete categories with cocontinuous functors between them. For any $\mathbb{E}_k$-monoid in spaces A, the functor*

$$\mathrm{LMod}_A\colon \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)} \longrightarrow \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$$

is part of an ambidextrous adjunction

$$\mathrm{LMod}_A\colon \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)} \rightleftarrows \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)} : \mathrm{RMod}_A.$$

Proof. First, let us remark that both functors LMod_A and RMod_A are actually well defined, since $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ is a symmetric monoidal category under Lurie's tensor product ([Lur17, Corollary 4.8.1.4]) with unit provided by the category of spaces $\mathcal{S}$. In particular,

$$\mathrm{Mod}_{\mathcal{S}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \simeq \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)},$$

hence any cocontinuous functor between cocomplete categories is $\mathcal{S}$-linear, in the sense of [Lur17, Definition 4.6.2.7]. Moreover, for any cocomplete category $\mathscr{C}$ the categorical Eilenberg-Watts Theorems (see [Lur17, Theorems 4.8.4.1 and 4.8.4.6]) yield equivalences:

(1) $\underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LMod}_A(\mathcal{S}), \mathscr{C}) \simeq \mathrm{RMod}_A(\mathscr{C})$ and $\underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{RMod}_A(\mathcal{S}), \mathscr{C}) \simeq \mathrm{LMod}_A(\mathscr{C})$.
(2) $\mathscr{C} \otimes \mathrm{RMod}_A(\mathcal{S}) \simeq \mathrm{RMod}_A(\mathscr{C})$ and $\mathrm{LMod}_A(\mathcal{S}) \otimes \mathscr{C} \simeq \mathrm{LMod}_A(\mathscr{C})$.

In the first statement, $\underline{\mathrm{Fun}}^{\mathrm{L}}$ denotes the category of cocontinuous functors, which plays the role of a internal mapping object for the *closed* symmetric monoidal structure of $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ ([Lur17, Remark 4.8.1.6]), while in the second statement $\otimes$ denotes Lurie's tensor product of cocomplete of categories. We make the following remarks:

(1) Let $\mathscr{A}$ be a cocomplete monoidal category, let A be a $\mathbb{E}_k$-monoid inside $\mathscr{A}$, and let $\mathscr{C}$ be a cocomplete category which is left tensored over $\mathscr{A}$. Then the cocomplete category $\mathrm{RMod}_A(\mathscr{C})$ is only a left $\mathscr{A}$-module and $\mathrm{LMod}_A(\mathscr{C})$ is only a right $\mathscr{A}$-module. However, in our case left and right modules are equivalent because $\mathcal{S}$ is symmetric monoidal; in particular, we can harmlessly swap the factors in the formulas $\mathscr{C} \otimes \mathrm{RMod}_A(\mathcal{S})$ and $\mathrm{LMod}_A(\mathcal{S}) \otimes \mathscr{C}$.
(2) In principle, the equivalences in the statement of the categorical Eilenberg-Watts Theorems might not be natural in $\mathscr{C}$. It turns out, however, that these equivalences are in fact natural. We will prove this in Lemma 1.2.6 below.

So, given two cocomplete categories $\mathscr{C}$ and $\mathscr{D}$ we have a chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}}(\mathrm{LMod}_A(\mathscr{C}), \mathscr{D}) &\simeq \mathrm{Map}_{\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}}(\mathscr{C} \otimes \mathrm{LMod}_A(\mathcal{S}), \mathscr{D}) \\
&\simeq \mathrm{Map}_{\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}}\left(\mathscr{C}, \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LMod}_A(\mathcal{S}), \mathscr{D})\right) \\
&\simeq \mathrm{Map}_{\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}}(\mathscr{C}, \mathrm{RMod}_A(\mathscr{D})),
\end{aligned}$$

which are moreover natural in both $\mathscr{C}$ and $\mathscr{D}$, thanks to Lemma 1.2.6. It follows that $\mathrm{LMod}_A(-)$ is a left adjoint to $\mathrm{RMod}_A(-)$, hence preserves small colimits. $\square$

Lemma 1.2.6. *The equivalences of cocomplete categories*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LMod}_A(\mathcal{S}), \mathscr{C}) \simeq \mathrm{RMod}_A(\mathscr{C})$$

and

$$\mathscr{C} \otimes \mathrm{RMod}_A(\mathcal{S}) \simeq \mathrm{RMod}_A(\mathscr{C})$$

are natural in $\mathscr{C}$.

Proof. First, suppose that $\mathscr{C} = \mathcal{S}$ is the category of spaces. In this case, [Lur17, Remark 4.8.4.8] guarantees that the first equivalence exhibits $\mathrm{LMod}_A(\mathcal{S})$ as a *left dual* to $\mathrm{RMod}_A(\mathcal{S})$, hence as a *weak left dual* in the sense of [Lur17, Remark 5.2.5.6]. The choice of a weak left dual in a closed symmetric monoidal category can be made functorially ([Lur17, Remark 5.2.5.10]). Moreover, in any closed symmetric monoidal category, fixing any left dualizable object X and an arbitrary object Y, the tensor product $Y \otimes X^{\vee}$ serves as an exponential of Y by X ([Lur17, Lemma 4.6.1.5]); again, exponentials can be chosen functorially in Y ([Lur17, Remark 4.6.1.3]). Combining this argument with [Lur17, Lemma 4.6.1.6], we deduce the existence of a string of equivalences

$$\begin{aligned}\underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LMod}_A(\mathcal{S}), \mathscr{C}) &\simeq \mathscr{C} \otimes \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LMod}_A(\mathcal{S}), \mathcal{S}) \\ &\simeq \mathscr{C} \otimes \mathrm{RMod}_A(\mathcal{S}),\end{aligned}$$

which are all natural in $\mathscr{C}$. So, proving the naturality in $\mathscr{C}$ of the equivalence

$$\mathscr{C} \otimes \mathrm{RMod}_A(\mathcal{S}) \simeq \mathrm{RMod}_A(\mathscr{C})$$

will yield that also the equivalence

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LMod}_A(\mathcal{S}), \mathscr{C}) \simeq \mathrm{RMod}_A(\mathscr{C})$$

is natural in $\mathscr{C}$. But under the (natural) equivalence $\mathscr{C} \otimes \mathrm{RMod}_A(\mathcal{S}) \simeq \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LMod}_A(\mathcal{S}), \mathscr{C})$, the functor

$$\mathrm{RMod}_A(\mathscr{C}) \longrightarrow \mathscr{C} \otimes \mathrm{RMod}_A(\mathcal{S}) \simeq \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LMod}_A(\mathcal{S}), \mathscr{C})$$

is readily seen to be the functor obtained via adjunction from the functor

$$\mathrm{LMod}_A(\mathcal{S}) \otimes \mathrm{RMod}_A(\mathscr{C}) \longrightarrow \mathcal{S} \otimes \mathscr{C} \simeq \mathscr{C}$$

given by the tensor product of the obvious forgetful functors. Since forgetting the action of a monoid A in $\mathcal{S}$ is functorial with respect to colimit-preserving functors (which are obviously $\mathcal{S}$-linear), we can conclude in virtue of the naturality of the correspondence between adjoint morphisms.

$\square$

Proposition 1.2.7 (Monodromy equivalence, second take)**.** *Let X be a connected space, and let $\mathscr{C}$ be a cocomplete (not necessarily presentable) category. Then there exists an equivalence of categories*

$$\mathrm{LocSys}(X;\mathscr{C}) \simeq \mathrm{LMod}_{\Omega_* X}(\mathscr{C}).$$

Proof. When $\mathscr{A} := \mathcal{S}$, Proposition 1.2.4 guarantees that $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ is κ_0-compactly generated under large colimits by presentable categories, where κ_0 is the smallest large cardinal for our theory. So, for any cocomplete category $\mathscr{C}$ we can choose a presentation as a large colimit of presentable categories $\mathscr{C}_i$. Using Lemma 1.2.5 and the fact that small spaces are compact with respect to large colimits in virtue of [Lur09, Proposition 5.4.1.2], we obtain that

$$\mathrm{LocSys}(X;\mathscr{C}) \simeq \operatorname*{colim}_i \mathrm{LocSys}\,(X;\mathscr{C}_i) \simeq \operatorname*{colim}_i \mathrm{LMod}_{\Omega_* X}(\mathscr{C}_i) \simeq \mathrm{LMod}_{\Omega_* X}(\mathscr{C})$$

□

Corollary 1.2.8. *Let X be a connected space. Then there is an equivalence of categories*

$$\mathrm{LocSysCat}(X) \simeq \mathrm{LMod}_{\Omega_* X}\left(\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right).$$

More generally, if $\mathscr{A}$ is a presentably symmetric monoidal category, there is an equivalence of categories

$$\mathrm{LocSysCat}(X;\mathscr{A}) \simeq \mathrm{LMod}_{\Omega_* X}\left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right).$$

Proof. Note that the second part of the statement specializes to the first when we take $\mathscr{A} := \mathcal{S}$ to be the category of spaces. As $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is cocomplete, the statement follows immediately from Proposition 1.2.7. □

1.2.9. In fact, it is possible to prove Corollary 1.2.8 directly without appealing to Stefanich's Proposition 1.2.4. For semplicity we shall focus on the case where the category of coefficients in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. The point is that we can explicitly write $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ as a colimit of presentable categories. For completeness, let us sketch the argument.

The first ingredient is given by [Lur17, Lemmas 5.3.2.9 and 5.3.2.11], which we summarize here for the convenience of the reader. First, if we fix a regular cardinal κ, the large category $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ of κ-compactly generated presentable categories, together with left exact functors which preserve κ-compact objects, *is* presentable and admits small colimits (which agree with small colimits in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$). Moreover, the tensor product of two small presentable and κ-compactly generated categories $\mathscr{C}$ and $\mathscr{D}$ is again κ-compactly generated: indeed, it is generated by κ-filtered colimits of objects of the form $C \otimes D$, where C and D are κ-compact generators of $\mathscr{C}$ and $\mathscr{D}$ respectively. The restriction of the tensor product $\otimes\colon \mathrm{Pr}^{\mathrm{L}}_{\kappa} \times \mathrm{Pr}^{\mathrm{L}}_{\kappa} \to \mathrm{Pr}^{\mathrm{L}}_{\kappa}$ commutes again with small colimits, hence [Lur17, Remark 4.2.1.34] guarantees that $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$ is enriched over itself.

Note that $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ may be regarded as the filtered colimit inside $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ of the categories $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$'s, where the colimit ranges all over regular cardinals κ which are $\mathscr{V}$-small. A priori, this is only a colimit in $\widehat{\mathrm{Cat}}_{(\infty,1)}$; however, the discussion above guarantees that all the functors making up this diagram are cocontinuous functors between cocomplete (actually presentable) categories, hence the diagram lies in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$. Since the inclusion

$$\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)} \subseteq \widehat{\mathrm{Cat}}_{(\infty,1)}$$

preserves filtered colimits ([Lur09, Proposition 5.5.7.11]), we can regard $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ as the colimit of $\mathrm{Pr}^{\mathrm{L}}_{\kappa}$'s in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$, as we claimed. Then arguing exactly as in the proof of Proposition 1.2.7 we conclude that there is an equivalence

$$\mathrm{LocSysCat}(X) \simeq \mathrm{LMod}_{\Omega_* X}\left(\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right).$$

1.3. **Naturality of the correspondence.** The statements of Lemma 1.1.6, Proposition 1.2.7 and Corollary 1.2.8 are not obviously *natural* in X, for X a connected and pointed space. In fact even making sense of naturality in this context requires some care. Indeed, the functor

$$\mathrm{LocSys}(-;\mathscr{C}) \coloneqq \mathrm{Fun}(-,\mathscr{C})\colon \mathcal{S} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

does not depend on a choice of a pointing. On the other hand, the functor $\mathrm{LMod}_{\Omega_*(-)}(\mathscr{C})$ makes sense only for pointed spaces. In particular, the domain of these two functors is, *a priori*, quite different. We will explain how to get around these issues, and upgrade the monodromy equivalence of Lemma 1.1.6 to a natural equivalence of functors. The main goal of this section is to prove the following.

Proposition 1.3.1. *Let $\mathcal{S}_*^{\geqslant 1}$ be the category of connected pointed spaces. Let $\mathscr{C}$ be a cocomplete category. Then there is a natural equivalence of functors*

$$\mathrm{LMod}_{\Omega_*(-)}(\mathscr{C}) \simeq \mathrm{LocSys}(-;\mathscr{C})|_{\mathcal{S}_*^{\geqslant 1}}\colon \mathcal{S}_*^{\geqslant 1} \longrightarrow \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$$

More generally, if $\mathscr{A}$ is a presentably symmetric monoidal category and $\mathscr{C}$ is a cocomplete category which is cocompletely tensored over $\mathscr{A}$ (i.e., it is an $\mathscr{A}$-module in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$), then there is a natural equivalence of functors

$$\mathrm{LMod}_{\Omega_*(-)}(\mathscr{C}) \simeq \mathrm{LocSys}(-;\mathscr{C})|_{\mathcal{S}_*^{\geqslant 1}}\colon \mathcal{S}_*^{\geqslant 1} \longrightarrow \mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}.$$

Proposition 1.3.1 will play an important role in the later sections of the article, since it will be a stepping stone in proving that the Day convolution monoidal structure on the category of functors over a connected topological monoid G naturally corresponds to the relative tensor product monoidal structure over the $\mathbb{E}_2$-monoid $\Omega_* G$ (Proposition 2.10).

1.3.2. The second part of Proposition 1.3.1 specializes to the first when we set $\mathscr{A} \coloneqq \mathcal{S}$. For ease of exposition, we will limit ourselves to prove Proposition 1.3.1 in this latter case; the general case is proved in the same way.

The proof of Lemma 1.1.6 given in [BP19, Lemma 3.9] depends on a sequence of equivalences of categories. We will show that each of them is natural in X in a series of Lemmas (Lemma 1.3.7, Lemma 1.3.8, Lemma 1.3.9 and Lemma 1.3.10). This will show that

Proposition 1.3.1 holds when $\mathscr{C}$ is presentable. We will then conclude that the statement holds for an arbitrary cocomplete category $\mathscr{C}$ using Proposition 1.2.4.

Let $\mathscr{C}$ be a presentable category. Consider the category $\mathrm{LocSys}(X;\mathscr{C})$ of $\mathscr{C}$-valued local systems over X. The proof of Lemma 1.1.6 depends on the following chain of equivalences:

$$\begin{aligned}\mathrm{LocSys}(X;\mathscr{C}) &:= \mathrm{Fun}(X,\mathscr{C})\\ &\overset{(a)}{\simeq} \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{Fun}(X^{\mathrm{op}},\mathcal{S}),\mathscr{C})\\ &\overset{(b)}{=} \underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LocSys}(X^{\mathrm{op}}),\mathscr{C})\\ &\overset{(c)}{\simeq} \underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathcal{S}_{/X},\mathscr{C}\big)\\ &\overset{(d)}{\simeq} \underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathrm{LMod}_{\Omega_*X}(\mathcal{S}),\mathscr{C}\big)\\ &\overset{(e)}{\simeq} \mathrm{RMod}_{\Omega_*X}(\mathscr{C})\\ &\overset{(f)}{\simeq} \mathrm{LMod}_{\Omega_*X}(\mathscr{C}).\end{aligned} \tag{1.3.3}$$

Next, let us explain why each of these equivalences holds.

- Equivalence (a) follows from the Yoneda Lemma.
- Equality (b) is definitional.
- Equivalence (c) follows from the straightening/unstraightening construction of [Lur09], which gives precisely
$$\mathrm{LocSys}(X^{\mathrm{op}}) := \mathrm{Fun}(X^{\mathrm{op}},\mathcal{S}) \simeq \mathcal{S}_{/X}. \tag{1.3.4}$$
- Equivalence (d) follows from the fact that when X is pointed and connected, [Lur17, Remark 5.2.6.28] yields an equivalence
$$\mathcal{S}_{/X} \simeq \mathrm{LMod}_{\Omega_*X}(\mathcal{S}). \tag{1.3.5}$$
- Equivalence (e) follows from the categorical Eilenberg-Watts theorem ([Lur17, Theorem 4.8.4.1]).
- Equivalence (f) depends on the fact that Ω_*X is a *grouplike* topological monoid ([Lur17, Definition 5.2.6.2]). In particular, it can be regarded as a group object in spaces ([Lur17, Remark 5.2.6.5]), and the antipode map $\iota\colon \Omega_*X \xrightarrow{\simeq} \Omega_*X$ yields an equivalence between Ω_*X and Ω_*X^{rev}, where Ω_*X^{rev} is the same underlying space as Ω_*X endowed with the reverse monoid structure ([Lur17, Remark 4.1.1.7]). Hence, left and right modules over Ω_*X are the same because of [Lur17, Remark 4.6.3.2].

Let us add some comments on this last point. In [BP19], $\mathrm{LMod}_{\Omega_*X}(\mathcal{S})$ and $\mathrm{RMod}_{\Omega_*X}(\mathcal{S})$ are in fact used interchangeably. The assignment

$$X \mapsto X^{\mathrm{rev}}$$

is a functor induced by the canonical involution of the operad Assoc $\simeq \mathbb{E}_1$ i.e., the equivalence between left and right modules over $\Omega_* X$ is *natural* in X. Thus, as in [BP19], we shall often abuse notations and blur the difference between $\mathrm{LMod}_{\Omega_* X}(\mathcal{S})$ and $\mathrm{RMod}_{\Omega_* X}(\mathcal{S})$.

In order to establish Proposition 1.3.1, we will analyze in turn each of the equivalences in the chain (1.3.3), and show that that they are natural in X. By our previous discussion, the naturality of (b) and (f) is clear, so we will focus on the remaining four equivalences.

We start with equivalence (1.3.3).(a). The naturality of the Yoneda Lemma is well-known in ordinary category theory. In the ∞-categorical setting it was recently established in [Mos23].

Theorem 1.3.6. *[Mos23, Theorem 3.6] Let $\mathcal{A}$ be a presentably monoidal category, and let $\mathrm{Lin}_{\mathcal{A}}\,\mathrm{Cat}_{(\infty,1)}$ (resp. $\mathrm{Lin}_{\mathcal{A}}\,\widehat{\mathrm{Cat}}_{(\infty,1)}$) be the category of small (resp. large) $\mathcal{A}$-enriched categories. The inclusion*

$$\mathrm{Lin}_{\mathcal{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \hookrightarrow \mathrm{Lin}_{\mathcal{A}}\,\widehat{\mathrm{Cat}}_{(\infty,1)}$$

admits a left adjoint relative to the inclusion $\mathrm{Lin}_{\mathcal{A}}\,\mathrm{Cat}_{(\infty,1)} \subseteq \mathrm{Lin}_{\mathcal{A}}\,\widehat{\mathrm{Cat}}_{(\infty,1)}$, given by taking the category of presheaves with values in $\mathcal{A}$. The partial unit is provided by the $\mathcal{A}$-enriched Yoneda embedding $よ_{\mathcal{A}}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$.

Theorem 1.3.6 immediately implies the following.

Lemma 1.3.7 (Equivalence (1.3.3).(a) is natural). *There exists a natural equivalence of functors*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LocSys}((-)^{\mathrm{op}});\mathcal{C}) \simeq \mathrm{LocSys}((-)^{\mathrm{op}};\mathcal{C})\colon \mathcal{S} \subseteq \mathrm{Cat}_{(\infty,1)} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\,.$$

Lemma 1.3.8 (Equivalence (1.3.3).(c) is natural). *There is a natural equivalence of functors*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{LocSys}((-)^{\mathrm{op}}),\mathcal{C}) \simeq \underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathcal{S}_{/(-)},\mathcal{C}\big)\colon \mathcal{S} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\,.$$

Proof. This is an immediate consequence of the straightening process underlying the Grothendieck construction of [Lur09]; in particular, the naturality is a consequence of [Lur09, Proposition 2.2.1.1]. □

We now consider equivalence (1.3.3).(d). The functoriality of the equivalence

$$\mathcal{S}_{/X} \simeq \mathrm{LMod}_{\Omega_* X}(\mathcal{S})$$

is actually already proved in [Lur17, Remark 5.2.6.28]. Hence, after precomposition with the forgetful functor

$$\mathcal{S}_*^{\geqslant 1} \longrightarrow \mathcal{S}^{\geqslant 1} \subseteq \mathcal{S}$$

we obtain the naturality statement we need.

Lemma 1.3.9 (Equivalence (1.3.3).(d) is natural). *There is a natural equivalence of functors*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathcal{S}_{/(-)},\mathcal{C}\big) \simeq \underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathrm{LMod}_{\Omega_*(-)}(\mathcal{S}),\mathcal{C}\big)\colon \mathcal{S}_*^{\geqslant 1} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\,.$$

Finally, we are left to prove the following.

Lemma 1.3.10 (Equivalence (1.3.3).(e) is natural)**.** *There is a natural equivalence of functors*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathrm{LMod}_{\Omega_*(-)}(\mathcal{S}),\mathcal{C}\big)\simeq \mathrm{RMod}_{\Omega_*(-)}(\mathcal{C})\colon \mathcal{S}_*^{\geqslant 1}\longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\,.$$

Proof. We first assume that $\mathcal{C}$ is the category of spaces $\mathcal{S}$. In this case, there is a well defined functor

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(-,\mathcal{S})\circ \mathrm{LMod}_{\Omega_*(-)}(\mathcal{S})\colon \mathcal{S}\longrightarrow \left(\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)^{\mathrm{op}}\simeq \mathrm{Pr}^{\mathrm{R}}_{(\infty,1)}\longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

which point-wise agrees with

$$\mathrm{RMod}_{\Omega_*(-)}(\mathcal{S})\colon \mathcal{S}\longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\,.$$

So, let σ be an n-simplex in the category of spaces $\mathcal{S}$: the image of such n-simplex under $\underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathrm{LMod}_{\Omega_*(-)}(\mathcal{S}),\mathcal{S}\big)$ produces an n-dimensional commutative diagram $\widehat{\sigma}$ of categories of left modules inside $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. Each 1-simplex

$$\{f:X\to Y\}\subseteq\sigma$$

becomes, as a 1-simplex of $\widehat{\sigma}$, a cocontinuous functor between categories of right modules

$$f_*\colon \mathrm{RMod}_{\Omega_*X}(\mathcal{S})\longrightarrow \mathrm{RMod}_{\Omega_*Y}(\mathcal{S}).$$

Using again the categorical Eilenberg-Watts Theorem, any functor as above is canonically equivalent to taking the tensor product with some (Ω_*X,Ω_*Y)-bimodule over Ω_*X: an immediate inspection shows that such bimodule is the pullback along the forgetful functor

$$f^*\colon \mathrm{LMod}_{\Omega_*Y}(\mathcal{S})\longrightarrow \mathrm{LMod}_{\Omega_*X}(\mathcal{S})$$

of Ω_*Y, seen as a left Ω_*Y-module. In particular, the functor f_* is canonically equivalent to

$$-\otimes_{\Omega_*X}\Omega_*Y:\mathrm{RMod}_{\Omega_*X}(\mathcal{S})\longrightarrow \mathrm{RMod}_{\Omega_*Y}(\mathcal{S}).$$

Using the fact that the relative tensor product is associative up to *canonical* homotopy ([Lur17, Section 4.4.3]), it follows that the n-simplex $\widehat{\sigma}$ agrees naturally with the homotopy coherence witnessing the associativity of the relative tensor product; hence, we have the desired equivalence of functors

$$\underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathrm{LMod}_{\Omega_*(-)}(\mathcal{S}),\mathcal{S}\big)\simeq \mathrm{RMod}_{\Omega_*(-)}(\mathcal{S}).$$

The case of a general presentable category $\mathcal{C}$ is implied by the case $\mathcal{C}=\mathcal{S}$ since both functors $\underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathrm{LMod}_{\Omega_*(-)}(\mathcal{S}),\mathcal{C}\big)$ and $\mathrm{RMod}_{\Omega_*(-)}(\mathcal{C})$ naturally agree with the composition of the functor

$$\underline{\mathrm{Fun}}^{\mathrm{L}}\big(\mathrm{LMod}_{\Omega_*(-)}(\mathcal{S}),\mathcal{S}\big)\simeq \mathrm{RMod}_{\Omega_*(-)}(\mathcal{S})\colon \mathcal{S}\longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

with the functor

$$-\otimes\mathcal{C}\colon \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

where $\otimes$ denotes Lurie's tensor product of cocomplete categories. $\square$

Proof of Proposition 1.3.1. Lemmas 1.3.7 to 1.3.10 together imply Proposition 1.3.1 when $\mathscr{C}$ is presentable. Arguing as in Proposition 1.2.7, we deduce that the statement holds also when $\mathscr{C}$ is a general (not necessarily presentable) cocomplete category. The $\mathscr{A}$-linear case is proved in the same way. □

2. Categorical local systems and Teleman's topological actions

Proposition 1.2.7 gives a description of of local systems on a space X with coefficients in a cocomplete category in terms of monodromy data. In this Section we refine this statement. We will show that when X is simply connected, categorical local systems can be described in terms of *higher monodromy*: i.e. in terms of appropriate actions of the iterated loop space $\Omega^2_* X$. This will allow us to revisit, from the perspective of ∞-categories, an interesting proposal of Teleman on topological group actions on categories.

In [Tel14], Teleman argues that the datum of a G-action on $\Bbbk$-linear differential graded category $\mathscr{C}$ should be equivalent to a morphism of $\mathbb{E}_2$-algebras

$$C_\bullet(\Omega_* G; \Bbbk) \longrightarrow \mathrm{HH}^\bullet(\mathscr{C}).$$

Here the source is the $\mathbb{E}_2$-algebra of chains on the based loop space $\Omega_* G$, endowed with its Pontrjagin product (we are assuming that G is connected); while the target is the Hochschild cohomology of $\mathscr{C}$. While motivating the plausibility of this statement via a couple of examples, Teleman does not actually propose a proof of it. However, in the context of ∞-category theory there is a natural way to interpret the action of a group object G on any cocomplete category $\mathscr{C}$. Indeed by [Lur09, Section 4.4.4] every cocomplete category $\mathscr{C}$ is tensored over the category of spaces $\mathscr{S}$. For any topological group G one can consider the category $\mathrm{LMod}_G(\mathscr{C})$ of left G-modules in $\mathscr{C}$. We will prove that when G is connected the datum of such a G-action on $\mathscr{C}$, where $\mathscr{C}$ is a presentable category which is $\Bbbk$-linear over a ring spectrum $\Bbbk$, is indeed encoded equivalently as a map of $\mathbb{E}_2$-ring spectra

$$\Sigma^\infty_+ \Omega_* G \wedge \Bbbk \longrightarrow \mathrm{HH}^\bullet(\mathscr{C}). \tag{2.1}$$

When $\Bbbk$ is a ordinary commutative ring, this recovers precisely Teleman's statement.

The main result of this Section is Corollary 2.12. It is the key ingredient in the proof of (2.1). Corollary 2.12 refines Corollary 1.2.8 by describing local systems of categories in terms of higher monodromy data. Namely, we show that if X is *simply connected*, local systems of presentable $\Bbbk$-linear categories on X can be described as iterated modules over the $\mathbb{E}_2$-algebra $\Sigma^\infty_+(\Omega_*\Omega_* X) \wedge \Bbbk$.

The first part of this Section will be dedicated to the proof of Corollary 2.12. This will require several preliminary steps, starting with a *linearization* statement which we prove next (see Lemma 2.2 below). Then in the second part of the Section we will turn our attention to the comparison with Teleman's notion of topological action.

Lemma 2.2. *Let $\mathscr{A}$ be a cocompletely symmetric monoidal category. Let $\mathbb{1}_{\mathscr{A}}$ denote the unit for the monoidal structure on $\mathscr{A}$, and let G be an $\mathbb{E}_{k+1}$-monoid in spaces. Then there is an equivalence of $\mathbb{E}_k$-monoidal categories*

$$\mathrm{LMod}_G(\mathscr{A}) \simeq \mathrm{LMod}_{G\otimes \mathbb{1}_{\mathscr{A}}}(\mathscr{A}).$$

Proof. Since the monoidal unit for the symmetric monoidal structure on $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ is the category of spaces $\mathcal{S}$, we have a chain of equivalences

$$\mathrm{CAlg}\Big(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\Big) \simeq \mathrm{CAlg}\Big(\mathrm{Mod}_{\mathcal{S}}\Big(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\Big)\Big) \simeq \mathrm{CAlg}\Big(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\Big)_{\mathcal{S}/}$$

where the second equivalence is provided by [Lur17, Corollary 3.4.1.7]. In particular, there exists an essentially unique symmetric monoidal and colimit-preserving functor

$$-\otimes \mathbb{1}_{\mathscr{A}} \colon \mathcal{S} \longrightarrow \mathscr{A}$$

which is uniquely determined by the assignment $\{*\} \mapsto \mathbb{1}_{\mathscr{A}}$. It follows that, if G is a topological $\mathbb{E}_{k+1}$-monoid, then $G\otimes \mathbb{1}_{\mathscr{A}}$ is an $\mathbb{E}_{k+1}$-algebra object in $\mathscr{A}$. This means that the (essentially unique) action of $\mathcal{S}$ on $\mathscr{A}$ is encoded in the functor $-\otimes \mathbb{1}_{\mathscr{A}}$. Hence for any topological $\mathbb{E}_{k+1}$-monoid G we have the desired equivalence of categories of left modules.

Let us explain next why this equivalence is $\mathbb{E}_k$-monoidal. This follows from the fact that the $\mathbb{E}_k$-monoidal structures on $\mathrm{LMod}_G(\mathscr{A})$ and $\mathrm{LMod}_{G\otimes \mathbb{1}_{\mathscr{A}}}(\mathscr{A})$ are both induced by the $\mathbb{E}_{k+1}$-monoid structures of G and $G\otimes \mathbb{1}_{\mathscr{A}}$, respectively, via the symmetric monoidal functor

$$\mathrm{LMod} \colon \mathrm{Alg}(\mathscr{A}) \longrightarrow \Big(\mathrm{Lin}_{\mathscr{A}}\, \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big)_{\mathscr{A}/}.$$

Since the $\mathbb{E}_{k+1}$-monoid structure of $G\otimes \mathbb{1}_{\mathscr{A}}$ is in turn induced by the one on G via the symmetric monoidal functor $-\otimes \mathbb{1}_{\mathscr{A}}$, it follows that the two $\mathbb{E}_k$-monoidal structures on $\mathrm{LMod}_G(\mathscr{A})$ and $\mathrm{LMod}_{G\otimes \mathbb{1}_{\mathscr{A}}}(\mathscr{A})$ agree as we claimed, and this concludes the proof. $\square$

Remark 2.3. An important setting for Lemma 2.2 is when

$$\mathscr{A} = \mathrm{Mod}_{\Bbbk} := \mathrm{Mod}_{\Bbbk}(\mathcal{S}\mathrm{p})$$

for some $\mathbb{E}_\infty$-ring spectrum $\Bbbk$. Then $\mathbb{1}_{\mathscr{A}} = \Bbbk$ and $\Omega_* G \otimes \Bbbk$ computes the $\Bbbk$-valued chains of $\Omega_* G$ with coefficients in $\Bbbk$. Indeed, $\mathrm{Mod}_{\Bbbk}$ is presentable and stable ([Lur17, Corollaries 4.2.3.7 and 7.1.1.5]), hence the essentially unique cocontinuous functor $\mathcal{S} \to \mathrm{Mod}_{\Bbbk}$ factors through the essentially unique cocontinuous functor

$$-\wedge \Bbbk \colon \mathcal{S}\mathrm{p} \longrightarrow \mathrm{Mod}_{\Bbbk}.$$

In fact, the category of spectra $\mathcal{S}\mathrm{p}$ is initial among stable presentably symmetric monoidal categories ([Lur17, Proposition 4.8.2.18]). In particular, for any space X we have that

$$X \otimes \mathbb{1}_{\mathscr{A}} = X \otimes \Bbbk := \Sigma^\infty_+ X \wedge \Bbbk$$

where $\Sigma^\infty_+ \colon \mathcal{S} \to \mathcal{S}\mathrm{p}$ is the suspension spectrum functor. But $\Sigma^\infty_+ X \wedge E$ is precisely the spectrum computing the homology of X with coefficients in the generalized homology theory E. Because of this, in the following, for any space X and for any $\mathbb{E}_\infty$-ring $\Bbbk$ we shall write $\mathrm{C}_\bullet(X;\Bbbk)$ for the $\Bbbk$-module $X \otimes \Bbbk$. Its $\Bbbk$-linear dual, which computes the $\Bbbk$-linear *cochains* of X, shall similarly be denoted as $\mathrm{C}^\bullet(X;\Bbbk)$. When $\Bbbk$ is discrete, these objects agree with the usual $\Bbbk$-valued chain and cochain complexes of classical algebraic topology.

Corollary 2.4. *Let $\mathcal{A}$ be a presentably symmetric monoidal category, and let X be a connected space. Then there is an equivalence of categories*

$$\mathrm{LocSysCat}(X;\mathcal{A}) \simeq \mathrm{Lin}_{\Omega_* X \otimes \mathcal{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

Proof. This follows from Lemma 2.2 since $\mathrm{Lin}_{\mathcal{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is cocompletely symmetric monoidal. □

Our next goal is to better understand $\Omega_* X \otimes \mathcal{A}$. We find it convenient to study the general problem of describing $G \otimes \mathcal{A}$ where G is an $\mathbb{E}_k$-monoid in spaces. Our results in this direction are Proposition 2.8 and Proposition 2.10 below. This will be key to establish the main result of this Section, Corollary 2.12. We start by stating a couple of Lemmas.

Lemma 2.5. *Let $\mathcal{C}$ be a symmetric monoidal category which is additionally both tensored and cotensored over a symmetric monoidal category $\mathcal{A}$ (in the sense of [Lur17, 4.2.1.28]). Then the cotensor functor $\mathbb{1}_{\mathcal{C}}^{(-)} \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$ is lax monoidal.*

Proof. For A and C objects in $\mathcal{A}$ and $\mathcal{C}$ respectively, let us denote by $\{A\} \otimes C$ the object obtained from A and C via the tensor action of $\mathcal{A}$ over $\mathcal{C}$. Then, the contravariant bifunctor

$$\mathrm{Map}_{\mathcal{C}}(\{-\} \otimes -, \mathbb{1}_{\mathcal{C}}) \colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}$$

is classified by a pairing $\mathcal{M} \to \mathcal{A} \times \mathcal{C}$, which is left representable (in the sense of [Lur17, Definition 5.2.1.8]). Indeed, by the very definition of *cotensored category*, for every object A in $\mathcal{A}$ the functor

$$\mathrm{Map}_{\mathcal{C}}(\{A\} \otimes -, \mathbb{1}_{\mathcal{C}}) \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}$$

is represented *precisely* by $\mathbb{1}_{\mathcal{C}}^A$. So, the duality map $\mathbb{1}_{\mathcal{C}}^{(-)} \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}$ is lax monoidal by [Lur17, Remark 5.2.2.25]. □

The following useful observations have already been established in existing literature (see for example [CCRY23; GHM23; HM23]). We still provide proofs for the convenience of the reader.

Lemma 2.6 ([GHM23, Theorem 3.2]). *Let $\mathcal{A}$ be a presentably symmetric monoidal category, let X be a space, and let $F \colon X \to \mathrm{Lin}_{\mathcal{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ be a diagram of shape X. Then, there is a natural equivalence $\lim F \simeq \operatorname{colim} F$ in $\mathrm{Lin}_{\mathcal{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$.*

Sketch of proof. Limits in $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ are computed as in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ ([Lur17, Corollary 4.2.3.3]); and these in turn are computed as in $\widehat{\mathrm{Cat}}_{(\infty,1)}$. On the other hand, colimits in $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ are computed as in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$, because it is a cocompletely symmetric monoidal category, hence we can use [Lur17, Corollary 4.2.3.5]. However, colimits in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ do not agree with colimits $\widehat{\mathrm{Cat}}_{(\infty,1)}$: rather, they agree with *limits* in $\widehat{\mathrm{Cat}}_{(\infty,1)}$ *after passing to the diagram of right adjoints*. Since any equivalence can be promoted to an *adjoint equivalence* ([RV22, Proposition 2.1.12]), and since X is a groupoid, it follows that the "adjoint diagram" F^{op} is equivalent to F itself, so the limit and the colimit coincide. □

Remark 2.7. During the final stages of preparation of this paper, Lemma 2.6 was further generalized in [Ben24]. The author shows that in fact the statement already holds in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$. Even if this probably allows to harmlessly generalize our arguments and results to the cocomplete setting, we do not investigate this direction in the present work.

Proposition 2.8 ([CCRY23, Corollary 4.12])**.** *Let $\mathscr{A}$ be a presentably symmetric monoidal category and let G be an $\mathbb{E}_k$-monoid in spaces. Then we have an equivalence of $\mathscr{A}$-enriched presentably $\mathbb{E}_k$-monoidal categories*

$$G \otimes \mathscr{A} \simeq \mathrm{LocSys}(G;\mathscr{A}).$$

Proof. Recall that $G \otimes \mathscr{A}$ is the image of G under the unique symmetric monoidal and colimit-preserving functor $\mathcal{S} \to \mathscr{A}$ determined by the assignment

$$\{*\} \mapsto \mathbb{1}_{\mathscr{A}}.$$

We will show that $\mathrm{LocSys}(-;\mathscr{A})$ is also a symmetric monoidal and colimit-preserving functor mapping $\{*\}$ to $\mathbb{1}_{\mathscr{A}}$. This implies that there is a canonical equivalence of functors

$$-\otimes \mathscr{A} \simeq \mathrm{LocSys}(-;\mathscr{A})$$

and so in particular proves the claim.

Note that the functor $\mathrm{LocSys}(-;\mathscr{A})$ can be interpreted as the natural copowering functor

$$\mathscr{A}^{(-)} \colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)},$$

This immediately shows that it satisfies the condition that $\{*\} \mapsto \mathbb{1}_{\mathscr{A}}$. Next, since the category $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is a cocomplete closed symmetric monoidal category which is both (cocompletely) tensored and cotensored over spaces, we deduce that $\mathrm{LocSys}(-;\mathscr{A})$ is a lax monoidal functor thanks to Lemma 2.5. It remains to show that the functor $\mathrm{LocSys}(-;\mathscr{A})$ preserves colimits, and that for any spaces X and Y the natural map

$$\alpha_{XY} \colon \mathrm{LocSys}(X;\mathscr{A}) \otimes_{\mathscr{A}} \mathrm{LocSys}(Y;\mathscr{A}) \longrightarrow \mathrm{LocSys}(X \times Y;\mathscr{A})$$

is an equivalence. Using Lemma 2.6, we can now prove the following statements.

(1) The functor $\mathrm{LocSys}(-;\mathscr{A})$ is cocontinuous. Indeed, let Y be the colimit of a diagram $I \to \mathcal{S}$ with values in the category of spaces. Recall that every space X is a colimit of

a diagram of shape X itself with constant valute at the point $\{*\}$. Obviously, we have a natural equivalence of $\mathcal{A}$-linear presentable categories

$$Y \otimes \mathcal{A} := \operatorname*{colim}_{Y} \mathcal{A} \simeq \operatorname*{colim}_{i \in I} \operatorname*{colim}_{X_i} \mathcal{A}.$$

We have also natural equivalences of $\mathcal{A}$-linear presentable categories

$$\begin{aligned} \mathrm{LocSys}(Y;\mathcal{A}) &\simeq \mathrm{LocSys}\Big(\operatorname*{colim}_{Y} \{*\};\mathcal{A}\Big) \\ &\simeq \lim_{Y} \mathrm{LocSys}(\{*\};\mathcal{A}) \\ &\overset{2.6}{\simeq} \operatorname*{colim}_{Y} \mathrm{LocSys}(\{*\};\mathcal{A}) \simeq \operatorname*{colim}_{Y} \mathcal{A}. \end{aligned}$$

Analogously,

$$\mathrm{LocSys}(X_i;\mathcal{A}) \simeq \operatorname*{colim}_{X_i} \mathcal{A},$$

so we can conclude that $\mathrm{LocSys}(Y;\mathcal{A}) \simeq \mathrm{colim}_i \mathrm{LocSys}(X_i;\mathcal{A})$.

(2) The functor $\mathrm{LocSys}(-;\mathcal{A})$ is strongly monoidal. Consider two spaces X and Y, and let us again present each as a colimit of a constant diagram whose value is the point. Combining the compatibility of the monoidal structure of $\mathrm{Lin}_{\mathcal{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ with colimits and Lemma 2.6, we immediately see that the map α_{XY} above boils down to the natural equivalence

$$\operatorname*{colim}_{X} \operatorname*{colim}_{Y} \mathcal{A} \xrightarrow{\simeq} \operatorname*{colim}_{X \times Y} \mathcal{A}$$

provided by the *Fubini theorem for homotopy colimits* (see for example [CS02, Theorem 24.9]).

□

Remark 2.9. It is implicit in the statement of the Proposition 2.8 that $\mathrm{LocSys}(G;\mathcal{A})$ carries a natural $\mathbb{E}_k$-monoidal structure. This is clarified by the proof of Proposition 2.8. Indeed, we show that the functor $\mathrm{LocSys}(-;\mathcal{A})$ is strongly monoidal. Hence, in particular, it preserves $\mathbb{E}_k$-monoids. We remark that the resulting $\mathbb{E}_k$-monoidal structure on $\mathrm{LocSys}(G;\mathcal{A})$ agrees with (the reverse of) the $\mathbb{E}_k$-monoidal Day convolution product of [Lur17, Remark 2.2.6.8].

Proposition 2.10. *Let $\mathcal{A}$ be a presentably symmetric monoidal category. For any $k \geqslant 1$, we have a commutative diagram of functors*

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{E}_k}\left(\mathcal{S}^{\geqslant 1}\right) \arrow[rr] \arrow[d, "\mathrm{oblv}_{\mathbb{E}_k}"'] & & \mathrm{Alg}_{\mathbb{E}_k}\left(\mathrm{Lin}_{\mathcal{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \arrow[d, "\mathrm{oblv}_{\mathbb{E}_k}"] \\ \mathcal{S}^{\geqslant 1}_{*} \arrow[rr, "\mathrm{LMod}_{\Omega_*(-)}(\mathcal{A})"] \arrow[d, "\mathrm{oblv}_*"'] & & \left(\mathrm{Lin}_{\mathcal{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)_{\mathcal{A}/} \arrow[d, "\mathrm{oblv}_*"] \\ \mathcal{S}^{\geqslant 1} \arrow[r, hook] & \mathcal{S} \arrow[r, "\mathrm{LocSys}(-;\mathcal{A})"] & \mathrm{Lin}_{\mathcal{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}. \end{tikzcd}$$

In particular, for G a connected $\mathbb{E}_k$-monoid in spaces, there is an equivalence of $\mathscr{A}$-enriched presentably $\mathbb{E}_k$-monoidal categories

$$\mathrm{LocSys}(G;\mathscr{A}) \simeq \mathrm{LMod}_{\Omega_* G}(\mathscr{A})$$

which is natural in G and agrees with the equivalence of Lemma 1.1.6.

Proof. Proposition 2.8 and Remark 2.9 together imply that whenever X is a pointed and connected space, we can reinterpret the natural equivalence $\mathrm{LocSys}(X;\mathscr{A}) \simeq \mathrm{LMod}_{\Omega_* X}(\mathscr{A})$ of Proposition 1.3.1 as follows. Notice that the natural functor

$$\mathrm{LocSys}(-;\mathscr{A})|_{\mathcal{S}^{\geqslant 1}} \colon \mathcal{S}^{\geqslant 1} \subseteq \mathcal{S} \longrightarrow \mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

is again strongly monoidal. Here, $\mathcal{S}^{\geqslant 1}$ is seen as a Cartesian symmetric monoidal category thanks to the fact that finite products of connected spaces are again connected ([Lur09, Corollary 6.5.1.13]); in particular, the inclusion $\mathcal{S}^{\geqslant 1} \subseteq \mathcal{S}$ is a strongly monoidal functor. Therefore, $\mathrm{LocSys}(-;\mathscr{A})|_{\mathcal{S}^{\geqslant 1}}$ induces a functor at the level of $\mathbb{E}_k$-algebras for every $k \geqslant 0$, that we again denote by

$$\mathrm{LocSys}(-;\mathscr{A}) \colon \mathrm{Alg}_{\mathbb{E}_k}\left(\mathcal{S}^{\geqslant 1}\right) \longrightarrow \mathrm{Alg}_{\mathbb{E}_k}\left(\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right).$$

For $k=0$, this is merely a functor from $\mathcal{S}^{\geqslant 1}_*$ to $\left(\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)_{\mathscr{A}/}$, since $\mathbb{E}_0$-algebras in a monoidal category are simply objects pointed by the monoidal unit, without further requirements ([Lur17, Proposition 2.1.3.9]). In particular, Proposition 1.3.1 simply states that such functor agrees with

$$\mathrm{LMod}_{\Omega_*(-)}(\mathscr{A}) \colon \mathcal{S}^{\geqslant 1}_* \simeq \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_1}(\mathcal{S}) \subseteq \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{S}) \longrightarrow \left(\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)_{\mathscr{A}/}, \tag{2.11}$$

where the first equivalence is given by May's delooping theorem ([Lur17, Theorem 5.2.6.10]). Endowing the category of $\mathbb{E}_k$-algebras in connected spaces with its natural Cartesian monoidal structure (i.e., with the monoidal structure provided by the underlying tensor product inside $\mathcal{S}^{\geqslant 1}_*$) and endowing the category $\left(\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)_{\mathscr{A}/}$ with its natural symmetric monoidal structure given by the relative tensor product of presentable categories over $\mathscr{A}$, it follows that the functor (2.11) is again strongly monoidal. Indeed, it is a composition of strongly monoidal functors: this follows from the fact that, given any presentably symmetric monoidal category $\mathscr{C}$ which is presentably tensored over a presentably symmetric monoidal category $\mathscr{A}$, for any associative algebra A in $\mathscr{A}$ the assignation $A \mapsto \mathrm{LMod}_A(\mathscr{C})$ is strongly monoidal ([Lur17, Theorem 4.8.5.16]). Since

$$\mathrm{Alg}_{\mathbb{E}_k}\left(\mathrm{Alg}_{\mathbb{E}_0}(\mathscr{C})\right) \simeq \mathrm{Alg}_{\mathbb{E}_k}(\mathscr{C})$$

in virtue of Dunn's Additivity Theorem ([Lur17, Theorem 5.1.2.2]), we deduce that indeed the diagram of functors pictured above exists and is commutative, deducing our claim. □

Corollary 2.12. *Let $\mathscr{A}$ be a presentably symmetric monoidal category, and let X be a simply connected space. Then there are equivalences of categories*

$$\mathrm{LocSysCat}(X;\mathscr{A}) \simeq \mathrm{LMod}_{\Omega_*X}\left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \simeq \mathrm{Lin}_{\mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\,.$$

Proof. By Lemma 2.2 and Proposition 2.8, we have an equivalence of categories

$$\mathrm{LMod}_{\Omega_*X\otimes\mathscr{A}}\left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \simeq \mathrm{Lin}_{\mathrm{LocSys}(\Omega_*X;\mathscr{A})}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\,.$$

Since X is simply connected, Ω_*X is connected. It follows from Proposition 2.10 that we have an equivalence of $\mathbb{E}_1$-monoidal categories

$$\mathrm{LocSys}(\Omega_*X;\mathscr{A}) \simeq \mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})$$

which implies an equivalence between their categories of left modules in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. Our statement follows by combining these results with the equivalence provided by Corollary 2.4. ☐

2.13. In the last paragraph of this section, we explain the connection between Corollary 2.12 and Teleman's notion of topological action from [Tel14]. We start by recalling the definition of the Hochschild cohomology of a presentable category $\mathscr{C}$ enriched over some presentably symmetric monoidal category $\mathscr{A}$. We follow the construction presented in [Iwa20], which is obtained by combining various results from [Lur17, Sections 4.8.5 and 5.3.2].

Let $\mathscr{A}$ be a presentably symmetric monoidal category. By [Lur17, Theorem 4.8.5.5] we have a fully faithful functor

$$\mathrm{LMod}_{(-)}\colon \mathrm{Alg}(\mathscr{A}) \longrightarrow \left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)_{\mathscr{A}/}$$

sending an associative algebra A in $\mathscr{A}$ to the category of its left modules $\mathrm{LMod}_A(\mathscr{A})$, with the pointing $\mathscr{A} \to \mathrm{LMod}_A(\mathscr{A})$ given by the essentially unique colimit-preserving functor sending $\mathbb{1}_{\mathscr{A}}$ to A.

Remark 2.14. In [Iwa20] the author considers instead the functor

$$\mathrm{RMod}_{(-)}\colon \mathrm{Alg}(\mathscr{A}) \longrightarrow \left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)_{\mathscr{A}/}.$$

This discrepancy however does not impact the present discussion, as $\mathrm{LMod}_{(-)}$ can be obtained from $\mathrm{RMod}_{(-)}$ by precomposing with the involution of $\mathrm{Alg}(\mathscr{A})$ sending an associative algebra to its opposite algebra ([Lur17, Remark 4.1.1.7]).

As we recalled in the proof of Proposition 2.10, the functor $\mathrm{LMod}_{(-)}$ is symmetric monoidal, so we can promote it to a functor between categories of $\mathbb{E}_1$-algebras:

$$\mathrm{LMod}_{(-)}\colon \mathrm{Alg}(\mathrm{Alg}(\mathscr{A})) \simeq \mathrm{Alg}_{\mathbb{E}_2}(\mathscr{A}) \longrightarrow \mathrm{Alg}\left(\left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)_{\mathscr{A}/}\right) \simeq \mathrm{Alg}\left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right),$$

where we used again Dunn's Additivity together with the fact that objects pointed by the unit in any monoidal category $\mathscr{C}$ are the same as $\mathbb{E}_0$-algebras in $\mathscr{C}$. By the general machinery of [Lur17, Proposition 2.2.1.1], it follows that the functor $\mathrm{LMod}_{(-)}$ admits a right adjoint

$$\Phi\colon \mathrm{Alg}\big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\big) \longrightarrow \mathrm{Alg}_{\mathbb{E}_2}(\mathscr{A})$$

that sends a presentably monoidal and $\mathscr{A}$-enriched category $\mathscr{C}$ to the $\mathbb{E}_2$-algebra of endomorphisms $\mathrm{End}_{\mathscr{C}}(\mathbb{1}_{\mathscr{C}})$ in $\mathscr{A}$ ([Lur17, Remark 4.8.5.12]).

Definition 2.15. Let $\mathscr{A}$ be a presentably symmetric monoidal category, and let $\mathscr{C}$ be a presentable category enriched over $\mathscr{A}$. Let $\underline{\mathrm{End}}(\mathscr{C})$ be the endomorphism category of $\mathscr{C}$ in $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ in the sense of [Lur17, Section 4.7.1] – i.e., it is the presentably $\mathscr{A}$-linear category $\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathscr{A}}(\mathscr{C},\mathscr{C})$, seen as a monoidal category via the composition of functors. Then the *Hochschild cohomology* of $\mathscr{C}$ is the $\mathbb{E}_2$-algebra in $\mathscr{A}$

$$\mathrm{HH}^{\bullet}(\mathscr{C}) := \Phi\big(\underline{\mathrm{End}}(\mathscr{C})\big).$$

Proposition 2.16. *Let $\mathscr{A}$ be a presentably symmetric monoidal category, let $\mathscr{C}$ be a presentable category enriched over $\mathscr{A}$, and let G be a connected topological group. Let $\mathrm{LMod}_{\Omega_*G\otimes\mathbb{1}_{\mathscr{A}}}(\mathscr{A})\text{-}\mathrm{ModStr}(\mathscr{C})$ denote the space of all possible left $\mathrm{LMod}_{\Omega_*G\otimes\mathbb{1}_{\mathscr{A}}}(\mathscr{A})$-module structures on $\mathscr{C}$, and let $G\text{-}\mathrm{ModStr}(\mathscr{C})$ denote the space of all possible left G-module structures on $\mathscr{C}$. Then, there are equivalences of spaces*

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathscr{A})}(\Omega_*G\otimes\mathbb{1}_{\mathscr{A}},\mathrm{HH}^{\bullet}(\mathscr{C})) \simeq \mathrm{LMod}_{\Omega_*G\otimes\mathbb{1}_{\mathscr{A}}}(\mathscr{A})\text{-}\mathrm{ModStr}(\mathscr{C}) \simeq G\text{-}\mathrm{ModStr}(\mathscr{C}).$$

Proof. Let us start from the equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathscr{A})}(\Omega_*G\otimes\mathbb{1}_{\mathscr{A}},\mathrm{HH}^{\bullet}(\mathscr{C})) \simeq \mathrm{LMod}_{\Omega_*G\otimes\mathbb{1}_{\mathscr{A}}}(\mathscr{A})\text{-}\mathrm{ModStr}(\mathscr{C}).$$

The adjunction between $\mathrm{LMod}_{(-)}$ and Φ yields an equivalence of spaces

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathscr{A})}(\Omega_*G\otimes\mathbb{1}_{\mathscr{A}},\mathrm{HH}^{\bullet}(\mathscr{C})) \simeq \mathrm{Map}_{\mathrm{Alg}\left(\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)}\big(\mathrm{LMod}_{\Omega_*G\otimes\mathbb{1}_{\mathscr{A}}}(\mathscr{A}),\underline{\mathrm{End}}(\mathscr{C})\big).$$

But the right hand side is equivalent to the space of left $\mathrm{LMod}_{\Omega_*G\otimes\mathbb{1}_{\mathscr{A}}}(\mathscr{A})$-module structures on $\mathscr{C}$, in virtue of [Lur17, Corollary 4.7.1.41]. Then, the equivalence

$$\mathrm{LMod}_{\Omega_*G\otimes\mathbb{1}_{\mathscr{A}}}(\mathscr{A})\text{-}\mathrm{ModStr}(\mathscr{C}) \simeq G\text{-}\mathrm{ModStr}(\mathscr{C})$$

follows immediately from the second equivalence in Corollary 2.12 applied to the simply connected space $X := \mathbf{B}G$. □

Proposition 2.16 states that, for any presentably symmetric monoidal category $\mathscr{A}$ and any presentable category $\mathscr{C}$ enriched over $\mathscr{A}$, giving $\mathscr{C}$ an action of the topological monoid G on $\mathscr{C}$ is equivalent to giving $\mathscr{C}$ a $\mathrm{LMod}_{\Omega_*G\otimes\mathbb{1}_{\mathscr{A}}}(\mathscr{A})$-module structure; in turn, this is equivalent to providing an $\mathbb{E}_2$-algebra map

$$\Omega_*G\otimes\mathbb{1}_{\mathscr{A}} \longrightarrow \mathrm{HH}^{\bullet}(\mathscr{C}).$$

This can be interpreted as a generalization, rephrased in purely ∞-categorical terms, of the following result due to Teleman.

Theorem 2.17 ([Tel14, Theorem 2.5]). *Topological actions of a connected group G on a differential graded category $\mathscr{C}$ which is linear over some base commutative ring $\Bbbk$ are completely captured, up to contractible choices, by the induced $\mathbb{E}_2$-algebra morphisms*

$$\mathrm{C}_\bullet(\Omega_* G; \Bbbk) \longrightarrow \mathrm{HH}^\bullet(\mathscr{C})$$

where the source is simply the algebra of chains of $\Omega_* G$ with coefficients in $\Bbbk$, endowed with the Pontrjagin product, and the target is the Hochschild cohomology of the differential graded category $\mathscr{C}$.

Indeed, when $\mathscr{A} := \mathrm{Mod}_\Bbbk$ is the presentable category of $\Bbbk$-modules over a classical commutative ring $\Bbbk$, we have already seen that the $\mathbb{E}_2$-algebra $\Omega_* G \otimes \mathbb{1}_{\mathscr{A}}$ boils down to the $\mathbb{E}_2$-algebra of $\Bbbk$-chains on $\Omega_* G$ (Remark 2.3). Since differential graded $\Bbbk$-linear categories are the same as compactly generated $\Bbbk$-linear presentable categories up to Morita equivalence ([Coh13, Corollary 5.7]), Proposition 2.16 implies Theorem 2.17.

We include for completion also a neat characterization of invertible objects inside the symmetric monoidal category $\mathrm{LocSysCat}(X; \Bbbk)$, when X is assumed to be simply connected and $\Bbbk$ to be an algebraically closed field. We start with the following easy remark.

Proposition 2.18. *Let X be a connected space, and let $\eta \colon \{*\} \to X$ be any choice of a base point. Let $\Bbbk$ be any commutative ring spectrum. Then an object $\mathscr{F}$ inside $\mathrm{LocSysCat}(X; \Bbbk)$ is invertible if and only if its stalk at the base point $\mathscr{F}_\eta$ is invertible in $\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$.*

Proof. Recall that an object in a monoidal category $\mathscr{C}^\otimes$ is invertible if it is fully dualizable and both the evaluation and the coevaluation map are equivalences. In virtue of [Gai15, Lemma 1.4.6], the fully dualizable objects inside the symmetric monoidal category

$$\mathrm{LocSysCat}(X; \Bbbk) \simeq \lim_{x \to X} \mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

are precisely the objects which are fully dualizable when projecting to each copy of $\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. Under the above equivalence, the projection corresponds to taking the stalk at a point $x \to X$. Since X is connected, a local systems of categories $\mathscr{F}$ over X is fully dualizable if and only if the stalk $\mathscr{F}_\eta$ is fully dualizable as a presentably $\Bbbk$-linear category. Moreover, since η^* is symmetric monoidal, we know that for a fully dualizable local system of categories $\mathscr{F}$ the dual in $\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ of the stalk $\mathscr{F}_\eta$ is the stalk at η of the dual $\mathscr{F}^\vee$ in $\mathrm{LocSysCat}(X; \Bbbk)$.

Thus, we are left to prove that the evaluation and the coevaluation morphisms that testify the dualizability of $\mathscr{F}$ are equivalences if and only if the functors induced at the stalk η are equivalences. Since the pullback along η is functorial, the "only if" direction is obvious. On the other hand, since X is connected, the functor η^* is conservative (it corresponds to forgetting the $\Omega_* X$-action, under the equivalence of Corollary 1.2.8), hence we deduce also the "if" direction. $\square$

2.19. When X is connected and $\Bbbk$ is a commutative ring spectrum, Proposition 2.18 implies that an object in the subgroupoid

$$\left(\mathrm{LocSysCat}(X;\Bbbk)^{\mathrm{inv}}\right)^{\simeq} \subseteq \mathrm{LocSysCat}(X;\Bbbk)^{\simeq}$$

spanned by all invertible local systems of $\Bbbk$-linear categories on X consists of the datum of an invertible presentably $\Bbbk$-linear category $\mathscr{C}$ together with a Ω_*X-action on $\mathscr{C}$. In virtue of [AG14, Theorem 3.15 and Proposition 7.3], we deduce that the connected components of $\left(\mathrm{LocSysCat}(X;\Bbbk)^{\mathrm{inv}}\right)^{\simeq}$ are equivalently described as classes in the Brauer group of $\Bbbk$

$$\mathrm{Br}(\Bbbk) \simeq \pi_0\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L,inv}}_{(\infty,1)}$$

together with all possible Ω_*X-actions over each of them.

2.20. When X is moreover *simply* connected we can apply the machinery of Corollary 2.12 and of Theorem 2.17 to deduce that an invertible local system of $\Bbbk$-linear categories on X consists of the datum of an equivalence class in the Brauer group $[\mathscr{C}] \simeq [\mathrm{Mod}_A] \in \mathrm{Br}(\Bbbk)$, where A is an Azumaya algebra over $\Bbbk$, with a morphism of $\mathbb{E}_2$-algebras $\mathrm{C}_\bullet(\Omega^2_*X;\Bbbk) \to \mathrm{HH}^\bullet(\mathscr{C})$. In particular, suppose that the Brauer group $\mathrm{Br}(\Bbbk)$ is trivial. This happens for every algebraically closed field ([Toë12, Proposition 1.9]) and for every commutative ring spectrum whose π_0 is either $\mathbb{Z}$ or the ring of Witt vectors $\mathbb{W}_p$ over $\mathbb{F}_p$ ([AG14, Theorem 7.16]); in particular, this holds also for the sphere spectrum. Then the invertible objects of $\mathrm{LocSysCat}(X;\Bbbk)$ consists of all possible Ω_*X-action on the category of modules over the essentially unique Azumaya algebra over $\Bbbk$ up to Morita equivalence – that is, $\Bbbk$ itself. Together with Proposition 2.16, we obtain the following.

Proposition 2.21. *Let X be a simply connected space and let $\Bbbk$ be an algebraically closed field. Then we have an isomorphism of abstract groups*

$$\pi_0\left(\mathrm{LocSysCat}(X;\Bbbk)^{\mathrm{inv}}\right)^{\simeq} \cong \mathrm{Hom}_{\mathrm{Grp}}\left(\pi_2(X),\, \Bbbk^\times\right)$$

between the group of equivalence classes of invertible local systems of $\Bbbk$-linear categories on X, and the group of multiplicative characters of $\pi_2(X)$.

Proof. In virtue of the discussion in Paragraph 2.20, we only need to characterize the set of connected components of the space $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Mod}_{\Bbbk})}\left(\mathrm{C}_\bullet(\Omega^2_*X;\Bbbk),\, \mathrm{HH}^\bullet(\Bbbk)\right)$. Notice that the Hochschild cohomology of $\mathrm{Mod}_{\Bbbk}$ computes the ordinary Hochschild cohomology of $\Bbbk$, which is

$$\mathrm{HH}^\bullet(\Bbbk) := \underline{\mathrm{Map}}_{\Bbbk\otimes\Bbbk}(\Bbbk,\Bbbk) \simeq \Bbbk.$$

So we are left to study the mapping space as $\mathbb{E}_2$-algebras from $\mathrm{C}_\bullet(\Omega^2_*X;\Bbbk)$ and $\Bbbk$. We claim that maps of $\mathbb{E}_2$-algebras from a connective $\mathbb{E}_2$-algebra A to a discrete algebra R over a field $\Bbbk$ always factor through maps of $\mathbb{E}_2$-algebras from π_0A. Indeed, the adjunction

$$\tau_{\leqslant 0}\colon \mathrm{Mod}_{\Bbbk} \rightleftarrows (\mathrm{Mod}_{\Bbbk})_{\leqslant 0} : \iota_{\leqslant 0}$$

restricts to an adjunction

$$\tau_{\leqslant 0}^{\heartsuit}\colon \mathrm{Mod}_{\Bbbk}\bigcap(\mathrm{Mod}_{\Bbbk})_{\geqslant 0}=(\mathrm{Mod}_{\Bbbk})_{\geqslant 0}\rightleftharpoons(\mathrm{Mod}_{\Bbbk})_{\leqslant 0}\bigcap(\mathrm{Mod}_{\Bbbk})_{\geqslant 0}=\mathrm{Mod}_{\Bbbk}^{\heartsuit}:\iota_{\leqslant 0}^{\heartsuit}.$$

The right adjoint is strongly monoidal, because over a field every object is flat; the left adjoint is strongly monoidal as well, because of Künneth formula. So we can safely apply [Lur17, Corollary 7.3.2.12 and Remark 7.3.2.13] to deduce the existence of an adjunction

$$\tau_{\leqslant 0}^{\heartsuit}\colon \mathrm{Alg}_{\mathbb{O}}\left((\mathrm{Mod}_{\Bbbk})_{\leqslant 0}\right)\rightleftharpoons\mathrm{Alg}_{\mathbb{O}}\left(\mathrm{Mod}_{\Bbbk}^{\heartsuit}\right):\iota_{\leqslant 0}^{\heartsuit},$$

for any operad $\mathbb{O}$. In the case $\mathbb{O}=\mathbb{E}_2$ we obtain

$$\begin{aligned}\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Mod}_{\Bbbk})}\left(\mathrm{C}_{\bullet}(\Omega^2_*X;\Bbbk),\,\mathrm{HH}^{\bullet}(\Bbbk)\right)&\simeq\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}((\mathrm{Mod}_{\Bbbk})_{\geqslant 0})}\left(\mathrm{C}_{\bullet}(\Omega^2_*X;\Bbbk),\,\Bbbk\right)\\&\simeq\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}\left(\mathrm{Mod}_{\Bbbk}^{\heartsuit}\right)}\left(\tau_{\leqslant 0}\mathrm{C}_{\bullet}(\Omega^2_*X;\Bbbk),\,\Bbbk\right).\end{aligned}$$

Notice that $\mathrm{Mod}_{\Bbbk}^{\heartsuit}$ is the ordinary discrete category of $\Bbbk$-modules, so $\mathbb{E}_2$-algebras are the same as commutative ($\mathbb{E}_\infty$-)algebras ([Lur17, Corollary 5.1.1.7]). On the other hand, $\tau_{\leqslant 0}\mathrm{C}_{\bullet}(\Omega^2_*X;\Bbbk)$ selects the 0-th homology of $\mathrm{C}_{\bullet}(\Omega^2_*X;\Bbbk)$, which is

$$\mathrm{H}_0(\Omega^2_*X;\Bbbk)\cong\Bbbk[\pi_0\Omega^2_*X]\cong\Bbbk[\pi_2(X)].$$

So we are simply looking at the space of maps from $\Bbbk[\pi_2(X)]$ to $\Bbbk$ seen as discrete commutative algebras, which is a discrete set. In particular, using the group ring-group of units adjunction between the category of discrete algebras $\mathrm{Alg}_{\Bbbk}^{\mathrm{disc}}$ and the category of discrete groups Grp, which restricts to the subcategories of commutative algebras and abelian groups, we have

$$\mathrm{Hom}_{\mathrm{CAlg}_{\Bbbk}^{\mathrm{disc}}}(\Bbbk[\pi_2(X)],\,\Bbbk)\cong\mathrm{Hom}_{\mathrm{Grp}}\left(\pi_2(X),\,\Bbbk^{\times}\right).$$

□

3. Local systems of higher categories

In the previous sections we presented several descriptions of local systems of (enriched) categories in terms of monodromy data, culminating in Proposition 1.2.7 and Corollary 2.12. Those results, however, were all formulated as equivalences of 1-categories, whereas local systems of categories naturally form a 2-category. In Section 3.1 we explain how to lift our results to equivalences of appropriate 2-categories. This is the content of Theorem 3.1.4. Our strategy can be summarized as follows: we shall prove that the equivalences of Proposition 1.2.7 and Corollary 2.12 can be regarded as equivalences between $\widehat{\mathrm{Cat}}_{(\infty,1)}$-*tensored categories*. Indeed, n-categories can be modeled by Θ_n-spaces ([Rez10a; Rez10b]), which in turn are enriched over Θ_{n-1}-spaces by [BR13; BR20]. One of the key technical inputs in the proof will be provided by the theory of enriched ∞-categories developed by Gepner–Haugseng [GH15] and others.

Next, in Section 3.2, we will study local systems of *presentable* n-categories. Our main results in this direction is Theorem 3.2.24 which gives a description of local systems of higher presentable categories in terms of *higher monodromy data*, i.e. actions of iterated based loop spaces. While all the relevant tools and concepts which are needed to state and prove Theorem 3.1.4 are well known among category theorists, this is not the case for Theorem 3.2.24. In particular, we shall carefully revisit the theory of *presentable n-categories* and *higher n-categories of modules* introduced in [Ste20]. Finally, we will also explain how to lift Teleman's picture of topological actions in terms of Hochschild cohomology to the n-categorical setting.

We remark that Theorem 3.2.24 subsumes Theorem 3.1.4, and its proof is logically independent from it. However the proof of Theorem 3.2.24 relies on the same key steps as Theorem 3.1.4, which we believe are more easily grasped in the more familiar setting of ordinary presentable categories. Thus the proof of Theorem 3.1.4, which is presented in Section 3.1, should be viewed as a practice run of our general argument, which will then be fully expounded in Section 3.2. We start by fixing notations.

Notation 3.0.1. In the rest of this paper we shall often work with higher categories. Given an n-category $\mathscr{C}$ there are two basic operations we can perform. If $m < n$, we can consider the underlying m-category of $\mathscr{C}$, by discarding all non-invertible k-simplices such that $k > m$; viceversa, if $m > n$, we can promote $\mathscr{C}$ to a m-category such that all its k-simplices, for $k > n$, are invertible (for example, we will sometimes consider a space X as an n-category). In order to avoid confusion we shall adopt the following *non standard* notations.

(1) For any $n > 1$, an n-category $\mathscr{C}$ admitting non-invertible n-simplices will be denoted as $n\boldsymbol{\mathscr{C}}$ in order to highlight its "categorical height".
(2) For $1 < m \leqslant n$, the m-category obtained by $n\boldsymbol{\mathscr{C}}$ discarding non-invertible k-simplices for all $m < k \leqslant n$ will be denoted as $m\boldsymbol{\mathscr{C}}$. For instance, we shall denote the underlying n-category of the $(n+1)$-category $(n+1)\widehat{\mathbf{Cat}}_{(\infty,n)}$ as $n\widehat{\mathbf{Cat}}_{(\infty,n)}$. When $m = 1$, we shall drop the 1 and simply write $\mathscr{C}$: in particular, $\widehat{\mathrm{Cat}}_{(\infty,n)}$ is the 1-category of n-categories.
(3) If $m \leqslant n$, then any m-category seen as an n-category will still be denoted in the same way, e.g. $m\boldsymbol{\mathscr{C}}$. For example, any space X seen as a trivial n-category will still be denoted as X (instead of $\iota_n \cdots \iota_1 X$, which is a convention sometimes adopted in the literature).

For a precise technical formulation of the above constructions, we refer the reader to Remark 3.2.2.

3.1. **2-categorical equivalences.** We start by explaining a technical construction due to Gepner–Haugseng, which allows us to change enrichment along lax monoidal functors. Next, using this, we will show that the categories appearing in the statements of Proposition 1.2.7 and Corollary 2.12 admit natural 2-categorical enhancements. This will be explained in Construction 3.1.5, Construction 3.1.7 and Construction 3.1.11 below. The main result is Theorem 3.1.4, that shows that the equivalences proved in Proposition 1.2.7 and Corollary 2.12 can be promoted to equivalences of 2-categories.

Proposition 3.1.1 (Change of base enrichment, [GH15, Corollary 5.7.6]). *Let $\mathscr{V}$ and $\mathscr{W}$ be two monoidal categories, and let $F\colon \mathscr{V} \to \mathscr{W}$ be a lax monoidal functor. Then there is a canonical functor*

$$\mathrm{Lin}_{\mathscr{V}}\,\widehat{\mathrm{Cat}}_{(\infty,1)} \longrightarrow \mathrm{Lin}_{\mathscr{W}}\,\widehat{\mathrm{Cat}}_{(\infty,1)}$$

from the category of $\mathscr{V}$-enriched categories to the category of $\mathscr{W}$-enriched categories.

Recall that $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is naturally symmetric monoidal. Indeed, limits and colimits inside $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ are computed as in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ (see the proof of Lemma 2.6). Then, by [Lur17, Theorem 4.5.2.1], the category $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ carries a natural symmetric monoidal structure given by the relative tensor product relative over $\mathscr{A}$; moreover, such monoidal structure commutes with colimits separately in each variable

Corollary 3.1.2. *Let $\mathscr{A}$ be a presentably symmetric monoidal category. Any category $\mathscr{C}$ enriched over $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is enriched over $\widehat{\mathrm{Cat}}_{(\infty,1)}$ hence is a 2-category.*

Proof. Note first that the natural inclusion functor $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \subseteq \widehat{\mathrm{Cat}}_{(\infty,1)}$ is lax monoidal, because it is a composition of the strongly monoidal inclusion $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \subseteq \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ ([Lur17, Proposition 4.8.1.15]) with the lax monoidal inclusion $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)} \subseteq \widehat{\mathrm{Cat}}_{(\infty,1)}$ ([Lur17, Corollary 4.1.8.4]). Moreover, given any presentably symmetric monoidal category $\mathscr{A}$, the natural cocontinuous and symmetric monoidal functor $\mathscr{S} \to \mathscr{A}$ yields a strongly monoidal functor

$$\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \simeq \mathrm{Lin}_{\mathscr{S}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

which is left adjoint to the natural forgetful functor $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \to \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. Therefore, such forgetful functor is lax monoidal ([HHLN20, Theorem B]), and we have a chain of forgetful lax monoidal functors

$$\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \hookrightarrow \widehat{\mathrm{Cat}}_{(\infty,1)}.$$

The statement then follows from Proposition 3.1.1. □

Remark 3.1.3. Let single out an important special case of Corollary 3.1.2. As $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is symmetric monoidal it is naturally enriched over itself, by [Lur17, Proposition 4.2.1.33.(2)]. Hence, in virtue of Corollary 3.1.2, it is enriched over $\widehat{\mathrm{Cat}}_{(\infty,1)}$.This provides the 2-categorical enhancement of $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ and $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$, respectively. Following the conventions introduced in Notation 3.0.1, we denote these enhancements as $\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}$ and $\mathbf{2Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$.

Theorem 3.1.4. *Let X be a connected space, and let $\mathscr{A}$ be a presentably symmetric monoidal category. Then, there is an equivalences of 2-categories*

$$\mathbf{2LocSysCat}(X;\mathscr{A}) \simeq \mathbf{2LMod}_{\Omega_* X}\left(\mathbf{2Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$$

Additionally, if X is simply connected, there are equivalences of 2-categories

$$\mathbf{2LocSysCat}(X;\mathscr{A}) \simeq \mathbf{2LMod}_{\Omega_* X}\Big(\mathbf{2Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}\Big) \simeq \mathbf{2LMod}_{\mathrm{LMod}_{\Omega^2_* X}(\mathscr{A})}\Big(\mathbf{2Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}\Big).$$

Let us briefly explain our strategy to prove Theorem 3.1.4. The three categories appearing in the statement of Corollary 2.12 are naturally cocompletely tensored over $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$, and the tensor action is closed. This yields an enrichment over $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ and hence, by Corollary 3.1.2, a structure of 2-categories. We will prove that the equivalences of Corollary 2.12 intertwine the tensor action of $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$: this imples that the three categories have compatible $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$-enrichments and readily implies Theorem 3.1.4.

Construction 3.1.5 (The 2-category of local systems). Let $\mathscr{C}$ be a cocompletely closed symmetric monoidal category and let $\mathscr{D}$ be a small category. The category of functors $\mathrm{Fun}(\mathscr{D},\mathscr{C})$ inherits a point-wise cocompletely symmetric monoidal structure by [Lur17, Remark 2.1.3.4]. With this monoidal structure, the functor

$$\mathrm{const}\colon \mathscr{C} \longrightarrow \mathrm{Fun}(\mathscr{D},\mathscr{C})$$

becomes limit- and colimit-preserving, and strongly monoidal. This turns $\mathrm{Fun}(\mathscr{D},\mathscr{C})$ into an $\mathscr{C}$-$\mathbb{E}_\infty$-algebra by [Lur17, Corollary 3.4.1.7]; in particular, $\mathrm{Fun}(\mathscr{D},\mathscr{C})$ is left tensored over $\mathscr{C}$ inside $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$.

Note that the symmetric monoidal structure of $\mathrm{Fun}(\mathscr{D},\mathscr{C})$ is closed: for any functor $F\colon \mathscr{D}\to\mathscr{C}$ the action

$$-\otimes F\colon \mathrm{Fun}(\mathscr{D},\mathscr{C}) \longrightarrow \mathrm{Fun}(\mathscr{D},\mathscr{C})$$

admits a right adjoint

$$\underline{\mathrm{Map}}_{\mathrm{Fun}(\mathscr{D},\mathscr{C})}(F,-)\colon \mathrm{Fun}(\mathscr{D},\mathscr{C}) \longrightarrow \mathrm{Fun}(\mathscr{D},\mathscr{C})$$

which can be informally described as the assignment sending a functor G to the functor

$$D \mapsto \underline{\mathrm{Map}}_{\mathscr{C}}(FD, GD).$$

In particular, [Hei20, Theorem 1.1] guarantees that $\mathrm{Fun}(\mathscr{D},\mathscr{C})$ is enriched over itself, hence over $\mathscr{C}$ thanks to Proposition 3.1.1.

In our case $\mathscr{C}$ is $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ and $\mathscr{D}$ is a space X seen as a groupoid. This yields the desired $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$-enrichment of $\mathrm{LocSysCat}(X)$. Following our conventions we denote the resulting 2-category $\mathbf{2LocSysCat}(X)$. Thanks to Remark 3.1.3 this discussion extends also to the case of $\mathscr{A}$-enriched presentable categories, where $\mathscr{A}$ is a presentably symmetric monoidal category. This yields the 2-category $\mathbf{2LocSysCat}(X;\mathscr{A})$. We will call this 2-category the *2-category of $\mathscr{A}$-linear categorical local systems over X*.

Remark 3.1.6. It is possible to define alternatively the 2-category of ($\mathscr{A}$-linear) categorical systems over X as the 2-category of 2-functors between X (seen as a trivial 2-category via the strongly monoidal inclusion $\mathscr{S}\subseteq\widehat{\mathrm{Cat}}_{(\infty,1)}$) and $\mathbf{2Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$. In fact, this approach might seem preferable, as it parallels more closely the definition of ordinary local systems. Indeed, if $\mathscr{C}$ is a presentable category, local systems on X with coefficients in $\mathscr{C}$ are defined precisely as functors from X to $\mathscr{C}$.

However, it is easy to see that the two definitions agree. A straightforward computation shows that the 2-category $2\mathbf{LocSysCat}(X;\mathscr{A})$ is equivalent to the evaluation at $\mathscr{A}$ of the right adjoint to the Cartesian product of 2-categories

$$-\times X\colon 2\widehat{\mathbf{Cat}}_{(\infty,1)} \longrightarrow 2\widehat{\mathbf{Cat}}_{(\infty,1)},$$

which is how the "correct" $\widehat{\mathrm{Cat}}_{(\infty,1)}$-enriched category of $\widehat{\mathrm{Cat}}_{(\infty,1)}$-enriched functors is defined in [GH15]. In particular, $2\mathbf{LocSysCat}(X;\mathscr{A})$ coincides with the internal mapping object in $2\widehat{\mathbf{Cat}}_{(\infty,1)}$ between X and $2\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$.

Construction 3.1.7 (The 2-category of Ω_*X-module categories). Let $\mathscr{A}$ be a presentably symmetric monoidal category and let X be a pointed connected space. Consider the category $\mathrm{LMod}_{\Omega_*X}\left(\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$. By Lemma 2.2 we can replace Ω_*X with $\Omega_*X\otimes\mathscr{A}$, i.e., with the image of Ω_*X under the colimit preserving and symmetric monoidal functor

$$\mathcal{S} \longrightarrow \mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}. \tag{3.1.8}$$

The category $\Omega_*X\otimes\mathscr{A}$ is presentably $\mathbb{E}_1$-monoidal (because Ω_*X is a topological monoid) but it is also a *cocommutative bialgebra* in $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. Indeed, the symmetric monoidal structure on $\mathcal{S}$ is Cartesian ([Lur17, Section 2.4.1]), hence every object is naturally a cocommutative comonoid (this is a dual statement to [Lur17, Proposition 2.4.3.9]). In particular, applying a strongly monoidal functor preserves both the algebra and coalgebra structures. Since we are dealing with a category of left modules over a bialgebra which is cocommutative (i.e., an $\mathbb{E}_\infty$-coalgebra) we can apply the following result due to Beardsley.

Proposition 3.1.9 ([Bea18, Corollary 3.19])**.** *Let $\mathscr{C}$ be a symmetric monoidal category. Let H be a (n,k)-bialgebra in $\mathscr{C}$ – i.e., an $\mathbb{E}_n$-algebra object in the category of $\mathbb{E}_k$-coalgebras in $\mathscr{C}$. Then the category of left H-modules $\mathrm{LMod}_H(\mathscr{C})$ admits an $\mathbb{E}_k$-monoidal structure, and the forgetful functor $\mathrm{oblv}_H\colon \mathrm{LMod}_H(\mathscr{C})\to\mathscr{C}$ is $\mathbb{E}_k$-monoidal.*

Note that this implies that $\mathrm{LMod}_{\Omega_*X}\left(\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$ carries a natural symmetric monoidal structure which is compatible with colimits. Indeed, the action of $\mathcal{S}$ on $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is compatible with colimits; therefore, [Lur17, Corollaries 4.2.3.3 and 4.2.3.5] imply that both limits and colimits of Ω_*X-modules in $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ can be computed after forgetting the Ω_*X-action.

Note also that this symmetric monoidal structure is closed. That is, if $\mathscr{C}$ and $\mathscr{D}$ are two presentably $\mathscr{A}$-linear categories endowed with a Ω_*X-action, then the category of $\mathscr{A}$-linear and colimit-preserving functors $\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathscr{A}}(\mathscr{C},\mathscr{D})$ carries a Ω_*X-action informally described by the rule

$$\begin{aligned} g \cdot F \colon \mathscr{C} &\longrightarrow \mathscr{D} \\ C &\mapsto g \cdot_{\mathscr{D}} F\left(g^{-1} \cdot_{\mathscr{C}} C\right). \end{aligned}$$

In particular, we have a trivial $\Omega_* X$-action functor

$$\mathrm{triv}_{\Omega_* X \otimes \mathscr{A}} \colon \mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{Lin}_{\Omega_* X \otimes \mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}, \tag{3.1.10}$$

induced by pulling back along the functor of symmetric comonoidal categories

$$\Omega_* X \otimes \mathscr{A} \longrightarrow \{*\} \otimes \mathscr{A} \simeq \mathscr{A}$$

The latter, in turn, is induced by the natural map $\Omega_* X \to \{*\}$ after applying the functor (3.1.8). The functor $\mathrm{triv}_{\Omega_* X \otimes \mathscr{A}}$ is a right adjoint which commutes with both limits and colimits. Moreover it is strongly monoidal, in virtue of the description of the symmetric monoidal structure on $\mathrm{LMod}_{\Omega_* X}\left(\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$ provided by Proposition 3.1.9. This turns $\mathrm{LMod}_{\Omega_* X}\left(\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$ into a $\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$-$\mathbb{E}_\infty$-algebra with a closed $\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$-action; hence it promotes it to an 2-category. Following our conventions, we denote this 2-category as $\mathbf{2LMod}_{\Omega_* X}\left(\mathbf{2Lin}_{\mathscr{A}} \mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$.

Construction 3.1.11 (**The** 2**-category of** $\mathrm{LMod}_{\Omega^2_* X}$**-modules**)**.** Let $\mathscr{A}$ be a presentably symmetric monoidal category and let X be a pointed simply connected space. Arguing as in Construction 3.1.7, we see that the double based loop space $\Omega^2_* X$ is both an $\mathbb{E}_2$-monoid and an $\mathbb{E}_\infty$-comonoid in spaces, in a compatible way. Using the terminology of [Bea18], we can say that $\Omega^2_* X$ is a $(2,\infty)$-*bimonoid*.

This implies that the presentable category $\mathrm{LMod}_{\Omega^2_* X}(\mathscr{A})$ is naturally a $(1,\infty)$-bialgebra object in $\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. Indeed, it is the image of the pointed and connected $(2,\infty)$-bimonoid $\Omega_* X$ under the composition of two strongly monoidal functors: namely the functor (2.11), which is proved to be strongly monoidal in Proposition 2.10, and the functor

$$\mathrm{LMod}_{(-)}(\mathscr{A}) \colon \mathrm{Alg}_{\mathbb{E}_1}(\mathscr{A}) \longrightarrow \left(\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)_{\mathscr{A}/} \tag{3.1.12}$$

which is strongly monoidal for any presentably symmetric monoidal category $\mathscr{A}$ ([Lur17, Theorem 4.8.5.16]). Using again Proposition 3.1.9, we see that $\mathrm{Lin}_{\mathrm{LMod}_{\Omega^2_* X}(\mathscr{A})} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ comes equipped with a cocontinuous closed symmetric monoidal structure such that the forgetful functor

$$\mathrm{oblv}_{\mathrm{LMod}_{\Omega^2_* X}(\mathscr{A})} \colon \mathrm{Lin}_{\mathrm{LMod}_{\Omega^2_* X}(\mathscr{A})} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

is strongly monoidal. Let us make a comment on this monoidal structure: the internal mapping category between two objects $\mathscr{C}$ and $\mathscr{D}$ in $\mathrm{Lin}_{\mathrm{LMod}_{\Omega^2_* X}(\mathscr{A})} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is *not* the category of $\mathrm{LMod}_{\Omega^2_* X}(\mathscr{A})$-linear colimit-preserving functors between $\mathscr{C}$ and $\mathscr{D}$. Rather, just like in Construction 3.1.7, it is the category $\underline{\mathrm{Fun}}^{\mathrm{L}}_{\mathscr{A}}(\mathscr{C}, \mathscr{D})$ of $\mathscr{A}$-linear and colimit-preserving functors from $\mathscr{C}$ to $\mathscr{D}$, equipped with its natural $\mathrm{LMod}_{\Omega^2_* X}(\mathscr{A})$-linear structure. Objectwise, this action can be described as follows: for a functor $F\colon \mathscr{C}\to\mathscr{D}$ and an Ω^2_*X-module object M in $\mathscr{A}$, the functor $M\otimes F$ is defined as $F(M\otimes_{\mathscr{C}} -)$

Now note that the forgetful functor

$$\mathrm{oblv}_{\Omega^2_*X}\colon \mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})\longrightarrow \mathscr{A}$$

is lax $\mathbb{E}_1$-monoidal, since it is the right adjoint of the free Ω^2_*X-module functor

$$-\otimes\Omega^2_*X\colon \mathscr{A}\longrightarrow \mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})$$

which is easily seen to be $\mathbb{E}_1$-monoidal. Indeed, the functor $-\otimes\Omega^2_*X$ is the image of the map of $\mathbb{E}_2$-algebras $\mathbb{1}_{\mathscr{A}}\to\Omega^2_*X\otimes\mathbb{1}_{\mathscr{A}}$ under the strongly monoidal functor (3.1.12). In particular, $\mathrm{oblv}_{\Omega^2_*X}$ induces a pullback functor between categories of left modules

$$\mathrm{triv}_{\mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})}\colon \mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\longrightarrow \mathrm{Lin}_{\mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}, \tag{3.1.13}$$

which equips an $\mathscr{A}$-linear presentable category $\mathscr{C}$ with the trivial $\mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})$-action. Arguing as in Construction 3.1.7, we can see that this functor preserves all limits and colimits and is strongly monoidal. As a consequence $\mathrm{Lin}_{\mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is tensored (and hence enriched) over $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. This yields the desired 2-categorical enhancement, which we denote $\mathbf{2Lin}_{\mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$.

Proof of Theorem 3.1.4. The first part of Theorem 3.1.4 follows from the fact that the equivalence of categories proved in Corollary 1.2.8 intertwines the symmetric monoidal structures of $\mathrm{LocSysCat}(X;\mathscr{A})$ and $\mathrm{LMod}_{\Omega_*X}\left(\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$ (see Costructions 3.1.5 and 3.1.7) and hence their $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$-enrichment. For this, it is sufficient to note that the equivalence of Corollary 1.2.8 is compatible with the coaugmentation functors from $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$, i.e., that it takes constant functors to trivial Ω_*X-modules, which is clear.

Let us now move to the second half of Theorem 3.1.4. Arguing as above, we see that it is enough to show that the equivalence of Corollary 2.12 is compatible with the coaugmentation functors from $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$. Recall that the equivalence of these categories of modules arises from the equivalence of presentably $\mathbb{E}_1$-monoidal categories

$$\Omega_*X\otimes\mathscr{A}\simeq \mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})$$

proved in Section 2. In particular, such equivalence does not change the underlying objects and therefore turns trivial Ω_*X-actions into trivial $\mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})$-actions. So we can conclude that the diagram of functors

$$\begin{tikzcd} & \mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \arrow[dl, bend right, "3.1.10"'] \arrow[dr, bend left, "3.1.13"] & \\ \mathrm{LMod}_{\Omega_*X}\left(\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \arrow[rr, "\simeq", "2.12"'] & & \mathrm{Lin}_{\mathrm{LMod}_{\Omega^2_*X}(\mathscr{A})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \end{tikzcd}$$

commutes, and this concludes the proof. □

3.2. **Local systems of n-categories and higher modules.** In this Section, we will switch gears and study local systems of n-categories on a space X. Under suitable connectedness assumptions on X, we will prove an analogue of Theorem 3.1.4 which provides an equivalence of $(n+1)$-categories relating local systems of n-categories and higher monodromy data (see Theorem 3.2.24 below).

We start by fixing notations and by recalling some relevant constructions from [GH15] and [Ste20]. We warn the reader to bear in mind the non-standard notations for higher categories we introduced in Notation 3.0.1.

Notation 3.2.1. Following [GH15, Remark 5.7.13], we define the $(n+1)$-*category of n-categories* inductively as follows. Recall that we denote by $\widehat{\mathrm{Cat}}_{(\infty,1)}$ the very large category of (possibly large) categories. The category $\widehat{\mathrm{Cat}}_{(\infty,1)}$ is symmetric monoidal with the cartesian product, so we can consider the 2-category of $\widehat{\mathrm{Cat}}_{(\infty,1)}$-enriched categories. We set

$$2\widehat{\mathbf{Cat}}_{(\infty,1)} := \mathrm{Lin}_{\widehat{\mathrm{Cat}}_{(\infty,1)}} \widehat{\mathrm{Cat}}_{(\infty,1)}.$$

By [GH15, Corollary 5.7.12] this is again a symmetric monoidal category. Suppose by the inductive hypothesis that we have defined the n-category $n\widehat{\mathbf{Cat}}_{(\infty,n-1)}$ of $(n-1)$-categories. Then we define the $(n+1)$-*category of n-categories* as

$$(n+1)\widehat{\mathbf{Cat}}_{(\infty,n)} := \mathrm{Lin}_{n\widehat{\mathbf{Cat}}_{(\infty,n-1)}} \widehat{\mathrm{Cat}}_{(\infty,1)},$$

i.e., as the $(n+1)$-category of categories enriched over n-categories. This agrees with [GH15, Definition 6.1.5].

Remark 3.2.2. In Notation 3.0.1 we introduced operations which, given an m-category $\mathscr{C}$ and an integer n, allow us to turn $\mathscr{C}$ into an n-category. As we shall clarify here, these operations are in fact functorial. Proposition 3.1.1 guarantees that the limit-preserving (and hence, strongly monoidal) inclusion $\mathcal{S} \subseteq \widehat{\mathrm{Cat}}_{(\infty,1)}$ produces a functor

$$\mathrm{Lin}_{\mathcal{S}} \widehat{\mathrm{Cat}}_{(\infty,1)} = \widehat{\mathrm{Cat}}_{(\infty,1)} \longrightarrow \mathrm{Lin}_{\widehat{\mathrm{Cat}}_{(\infty,1)}} \widehat{\mathrm{Cat}}_{(\infty,1)} =: 2\widehat{\mathbf{Cat}}_{(\infty,1)}$$

which is moreover *lax monoidal* for the Cartesian monoidal structure on both source and target ([GH15, Corollary 5.7.11]). Hence, we can apply again Proposition 3.1.1 to obtain another lax monoidal functor

$$2\widehat{\mathbf{Cat}}_{(\infty,1)} := \mathrm{Lin}_{\widehat{\mathrm{Cat}}_{(\infty,1)}} \widehat{\mathrm{Cat}}_{(\infty,1)} \xrightarrow{\iota_1} \mathrm{Lin}_{2\widehat{\mathbf{Cat}}_{(\infty,1)}} \widehat{\mathrm{Cat}}_{(\infty,1)} =: 3\widehat{\mathbf{Cat}}_{(\infty,2)}.$$

Iterating this argument, we obtain a chain of lax monoidal inclusions

$$\mathcal{S} \subseteq \widehat{\mathrm{Cat}}_{(\infty,1)} \subseteq 2\widehat{\mathbf{Cat}}_{(\infty,1)} \overset{\iota_1}{\subseteq} \ldots \subseteq n\widehat{\mathbf{Cat}}_{(\infty,n-1)} \overset{\iota_{n-1}}{\subseteq} (n+1)\widehat{\mathbf{Cat}}_{(\infty,n)} \subseteq \ldots$$

which allows us to consider any $\mathbb{E}_k$-topological monoid as an $\mathbb{E}_k$-monoidal n-category for every $n \geqslant 1$. This is just the formalization of the natural idea (discussed in Notation 3.0.1

above) that, if $n \geqslant m$, an m-category can be viewed as an n-category such that every k-simplex is an equivalence for $k > m$. In particular, it makes sense to consider *modules* over a topological monoid G in the $(n+1)$-category of n-categories.

Conversely, consider the functor

$$(-)^{\simeq} \colon \widehat{\mathrm{Cat}}_{(\infty,1)} \longrightarrow \mathcal{S}$$

which takes a category to its maximal subgroupoid. This is the right adjoint to the strongly monoidal and colimit preserving inclusion $\mathcal{S} \subseteq \widehat{\mathrm{Cat}}_{(\infty,1)}$; hence it inherits a lax monoidal structure. Another inductive argument using [GH15, Proposition 5.7.17] yields for any positive integer n the adjunction

$$\iota_{n-1} \colon n\widehat{\mathbf{Cat}}_{(\infty,n-1)} \rightleftarrows (n+1)\widehat{\mathbf{Cat}}_{(\infty,n)} \colon \tau_{\leqslant n-1}, \tag{3.2.3}$$

where applying $\tau_{\leqslant n-1}$ amounts to considering an n-category as an $(n-1)$-category by forgetting the non-invertible n-simplices. As explained in Notation 3.0.1, we will mostly drop the symbol ι_{n-1} from our notations.

Remark 3.2.4. For any two n-categories $n\boldsymbol{\mathscr{C}}$ and $n\boldsymbol{\mathscr{D}}$, using [Hin20, §6.1.3] we can construct a category of n-functors (i.e., of $n\widehat{\mathbf{Cat}}_{(\infty,n-1)}$-enriched functors) that we denote as $\mathrm{Fun}_n(n\boldsymbol{\mathscr{C}}, n\boldsymbol{\mathscr{D}})$. Note that if $n\boldsymbol{\mathscr{C}} \simeq \iota_n \cdots \iota_k(k\boldsymbol{\mathscr{C}})$ is just an k-category $k\boldsymbol{\mathscr{C}}$ for some $k \leqslant n$, then the adjunction (3.2.3) implies that we have a chain of equivalences of categories

$$\mathrm{Fun}_n(k\boldsymbol{\mathscr{C}}, n\boldsymbol{\mathscr{D}}) \simeq \mathrm{Fun}_{n-1}\big(k\boldsymbol{\mathscr{C}}, \tau_{\leqslant n-1}n\boldsymbol{\mathscr{D}}\big) \simeq \cdots \simeq \mathrm{Fun}_k\big(k\boldsymbol{\mathscr{C}}, \tau_{\leqslant k}n\boldsymbol{\mathscr{D}}\big).$$

If $k=1$, i.e., $n\boldsymbol{\mathscr{C}} \simeq \iota_k \cdots \iota_1(\mathscr{C})$ is an ordinary category $\mathscr{C}$, using the $n\widehat{\mathbf{Cat}}_{(\infty,n-1)}$-enrichment of $\tau_{\leqslant 1}\mathscr{D}$ one can produce an n-enhancement of the category

$$\mathrm{Fun}_n(\mathscr{C}, n\boldsymbol{\mathscr{D}}) \simeq \mathrm{Fun}\big(\mathscr{C}, \tau_{\leqslant 1}n\boldsymbol{\mathscr{D}}\big)$$

which recovers the internal mapping object $n\mathbf{Fun}(\mathscr{C}, n\boldsymbol{\mathscr{D}})$ for the Cartesian symmetric monoidal structure on $n\widehat{\mathbf{Cat}}_{(\infty,n-1)}$.

The theory of presentable n-categories was only recently introduced by Stefanich. As it will play a key role in the following, we shall revisit its basic definitions and constructions. For more details, the reader can consult [Ste20, Section 5]. We warn the reader that the definition of presentable n-categories due to Stefanich that we use in this section is just one of the available definitions for presentable n-categories. In [MS21], the authors propose yet another definition of presentable n-categories, which is incompatible with the one by Stefanich. For example, the underlying n-category of a presentable $(n+1)$-category in the sense of Mazel-Gee and Stern is again presentable; however, the theory of presentable n-categories developed by Stefanich is built in such a way that the $(n+1)$-category $(n+1)\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ of presentable n-categories is a presentable $(n+1)$-category. In particular, $2\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$ is a presentable 2-category in the sense of Stefanich, but cannot be presentable in the sense of Mazel-Gee and Stern since $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is known to be not presentable.

Consider the functor

$$\mathrm{LMod}_{(-)}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \colon \mathrm{Alg}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \longrightarrow \widehat{\mathrm{CAT}}^{\mathrm{rex}}_{(\infty,1)} \tag{3.2.5}$$

which takes a cocomplete monoidal category $\mathscr{A}$ and sends it to the (very large) cocomplete category $\mathrm{LMod}_{\mathscr{A}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right)$ of cocomplete $\mathscr{A}$-modules. It is a symmetric monoidal functor, so if $\mathscr{A}$ is a cocomplete $\mathbb{E}_k$-monoidal category for some $k \geqslant 2$ then the category $\mathrm{LMod}_{\mathscr{A}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right)$ inherits a cocomplete $\mathbb{E}_{k-1}$-monoidal structure given by the relative Lurie tensor product over $\mathscr{A}$. In particular, if $k = +\infty$ (i.e., if we start from the category of cocomplete *symmetric* monoidal categories), we obtain a functor

$$\mathrm{Mod}_{(-)}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \colon \mathrm{CAlg}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \longrightarrow \mathrm{CAlg}\left(\widehat{\mathrm{CAT}}^{\mathrm{rex}}_{(\infty,1)}\right). \tag{3.2.6}$$

For a cocomplete symmetric monoidal category $\mathscr{A}$, we would like to define an n-category of $\mathscr{A}$-modules by iterating n times the functor (3.2.6). However, in order to have a consistent theory, we would need a chain of nested universes. We can fix this issue as follows. Let κ_0 be the first large cardinal with respect to our initial choice of universes $\mathscr{U}$ and $\mathscr{V}$. This means that κ_0-small spaces and categories are what we call small spaces and categories.

Definition 3.2.7. Let

$$\mathrm{LMod}^{\mathrm{pr}}_{(-)} := \mathrm{LMod}_{(-)}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right)^{\kappa_0} \colon \mathrm{Alg}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \longrightarrow \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)} \tag{3.2.8}$$

denote the functor that sends a cocomplete monoidal category $\mathscr{A}$ to the category of κ_0-compact objects inside $\mathrm{LMod}_{\mathscr{A}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right)$. We say that an object of $\mathrm{LMod}_{\mathscr{A}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right)^{\kappa_0}$ is a *presentable left $\mathscr{A}$-module*.

The functor (3.2.8) admits a lax monoidal structure ([Ste20, Remark 5.1.11]), hence it sends cocompletely $\mathbb{E}_k$-monoidal categories to cocompletely $\mathbb{E}_{k-1}$-monoidal categories. For $n \leqslant k$ we set

$$\mathrm{LMod}^{\mathrm{pr},n}_{(-)} \colon \mathrm{Alg}_{\mathbb{E}_k}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \longrightarrow \mathrm{Alg}_{\mathbb{E}_{k-n}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \tag{3.2.9}$$

to be the n-fold iteration of the functor (3.2.8).

Definition 3.2.10. Let $k \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}$ and $n \leqslant k$ be integers, and let $\mathscr{A}$ be a presentably $\mathbb{E}_k$-monoidal category. The *category of presentable $\mathscr{A}$-linear n-categories*

$$\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

is the cocompletely symmetric monoidal category obtained by applying functor (3.2.9) to $\mathscr{A}$. In the case when $\mathscr{A} = \mathcal{S}$ is the category of spaces, we shall simply write $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ and call it the *category of presentable n-categories*. This choice of notation is justified by Remark 3.2.11.

Remark 3.2.11. Note that when $n = 1$ the category $\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ of Definition 3.2.10 agrees with the usual category of $\mathscr{A}$-linear presentable categories. This is a consequence of [Ste20, Proposition 5.1.4 and Corollary 5.1.5]: κ_0-compact left $\mathscr{A}$-modules in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ are *precisely* $\mathscr{A}$-linear presentable categories.

Remark 3.2.12. Our Definition 3.2.10 differs slightly from Stefanich's conventions. Stefanich denotes the category $\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ by $\mathrm{LMod}^{\mathrm{pr},n}_{\mathscr{A}}$ and refers to it as the *category of presentable categorical n-fold $\mathscr{A}$-modules* (see [Ste20, Definition 5.2.2]). We opted for the conventions in Definition 3.2.10 as this highlights the fact that objects in $\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ should be viewed as ($\mathscr{A}$-linear) presentable n-categories; also, by Remark 3.2.11, the notation $\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ has the advantage of being coherent with the notation we use in the ordinary case $n = 1$.

Construction 3.2.13. Let $k \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}$ and $n \leqslant k$ be integers, and let $\mathscr{A}$ be a presentably $\mathbb{E}_k$-monoidal category. We define the category

$$\mathrm{Lin}_{\mathscr{A}} \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)} \coloneqq \mathrm{Mod}_{\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}\right)$$

of cocompletely $\mathscr{A}$-linear n-categories as the image of $\mathscr{A}$ under the composition

$$\mathrm{Alg}_{\mathbb{E}_k}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \xrightarrow{3.2.9} \mathrm{Alg}_{\mathbb{E}_{k-n+1}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right) \xrightarrow{3.2.6} \mathrm{Alg}_{\mathbb{E}_{k-n}}\left(\widehat{\mathrm{CAT}}^{\mathrm{rex}}_{(\infty,1)}\right)$$

where the first functor is the $(n-1)$-fold iteration of the functor (3.2.8) and the second is simply the functor (3.2.6).

Note that, by definition, there is an inclusion

$$\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \hookrightarrow \mathrm{Lin}_{\mathscr{A}} \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}.$$

In general, objects in $\mathrm{Lin}_{\mathscr{A}} \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n-1)}$ might not be presentable, but they are always cocompletely tensored and enriched over $\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$. Thus, we can view them as n-categories as follows. Consider the composition of lax monoidal functors

$$\mathrm{Lin}_{\mathscr{A}} \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)} \xrightarrow{(a)} \mathrm{Lin}_{n\,\mathrm{Lin}_{\mathscr{A}} \mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} \widehat{\mathrm{Pr}}^{\mathrm{L}}_{(\infty,1)} \xhookrightarrow{(b)} \mathrm{Lin}_{m\,\mathrm{Lin}_{\mathscr{A}} \mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} \widehat{\mathrm{CAT}}_{(\infty,1)} \xrightarrow{(c)} \mathrm{Lin}_{n\widehat{\mathbf{Cat}}_{(\infty,n-1)}} \widehat{\mathrm{CAT}}_{(\infty,1)}.$$

where the functor (a) is the Ind-completion functor Ind_{κ_0}, the functor (b) is induced by the forgetful functor

$$\widehat{\mathrm{Pr}}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \widehat{\mathrm{CAT}}_{(\infty,1)}$$

and the functor (c) is the change of enrichment along the lax monoidal forgetful functor

$$\mathrm{Lin}_{\mathscr{A}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \longrightarrow (n+1)\widehat{\mathbf{Cat}}_{(\infty,n)}.$$

In this way, one obtains from an arbitrary $\mathscr{A}$-linear cocomplete n-category $n\mathscr{C}$ an associated very large n-category. Taking its sub-n-category of κ_0-compact objects, we get a well defined lax monoidal functor

$$\mathrm{Lin}_{\mathscr{A}} \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)} \longrightarrow \mathrm{Lin}_{n\widehat{\mathbf{Cat}}_{(\infty,n-1)}} \widehat{\mathrm{Cat}}_{(\infty,1)} =: (n+1)\widehat{\mathbf{Cat}}_{(\infty,n)}.$$

In particular, for any $n \geqslant 1$ we have an induced lax monoidal change-of-enrichment functor

$$\begin{aligned} \operatorname{Lin}_{(n+1)\operatorname{Lin}_{\mathscr{A}} \widehat{\operatorname{Cat}}^{\operatorname{rex}}_{(\infty,n)}} \widehat{\operatorname{Pr}}^{\operatorname{L}}_{(\infty,1)} &\longrightarrow \operatorname{Lin}_{\operatorname{Lin}_{\mathscr{A}} \widehat{\operatorname{Cat}}^{\operatorname{rex}}_{(\infty,n)}} \widehat{\operatorname{Cat}}_{(\infty,1)} \\ &\longrightarrow \operatorname{Lin}_{(n+1)n\widehat{\mathbf{Cat}}_{(\infty,n)}} \widehat{\operatorname{Cat}}_{(\infty,1)} =: (n+2)\widehat{\mathbf{Cat}}_{(\infty,n+1)} \end{aligned} \tag{3.2.14}$$

Definition 3.2.15. We denote by $(n+1)\mathbf{Lin}_{\mathscr{A}}\widehat{\mathbf{Cat}}^{\mathrm{L}}_{(\infty,n)}$ the $(n+1)$-category which is the image of the unit object in $\operatorname{Lin}_{\operatorname{Lin}_{\mathscr{A}} \widehat{\operatorname{Cat}}^{\operatorname{rex}}_{(\infty,n)}} \widehat{\operatorname{Pr}}^{\operatorname{L}}_{(\infty,1)}$ under the functor (3.2.14).

The category $(n+1)\mathbf{Lin}_{\mathscr{A}}\widehat{\mathbf{Cat}}^{\mathrm{L}}_{(\infty,n)}$ is a symmetric monoidal $(n+1)$-category whose underlying symmetric monoidal category is equivalent to $\operatorname{Lin}_{\mathscr{A}} \widehat{\operatorname{Cat}}^{\operatorname{rex}}_{(\infty,n)}$ ([Ste20, Remark 5.3.5]).

Warning 3.2.16. When $n \geqslant 1$, if $n\boldsymbol{\mathscr{A}}$ is a presentable monoidal n-category it is not obvious that an $n\boldsymbol{\mathscr{A}}$-module $n\boldsymbol{\mathscr{C}}$ is also enriched over $n\boldsymbol{\mathscr{A}}$, essentially because it it not known whether the monoidal structure on $(n+1)\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ is closed (see the proof of Theorem 3.2.24 and Conjecture 3.3.1). Therefore, in the following, we shall write $(n+1)\mathbf{LMod}_{n\boldsymbol{\mathscr{A}}}\big((n+1)\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\big)$ for the $(n+1)$-category of presentable n-categories which are left tensored over $n\boldsymbol{\mathscr{A}}$, instead of $(n+1)\mathbf{Lin}_{n\boldsymbol{\mathscr{A}}}\big(\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}\big)$.

Definition 3.2.17. The $(n+1)$-*category of $\mathscr{A}$-linear presentable n-categories* is the full sub-$(n+1)$-category

$$(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)} \subseteq (n+1)\widehat{\mathbf{Cat}}^{\mathrm{L}}_{(\infty,n)}$$

spanned by presentable n-categories (in the sense of Definition 3.2.10). In particular, the underlying symmetric monoidal category of $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ is equivalent to $\operatorname{Lin}_{\mathscr{A}} \operatorname{Pr}^{\mathrm{L}}_{(\infty,n)}$ ([Ste20, Remark 5.3.7]).

Remark 3.2.18. Unraveling all constructions, and using [Ste20, Remark 5.3.5 and 5.3.7], we see that the 2-categorical enhancement $2\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$ relies on considering $\operatorname{Lin}_{\mathscr{A}} \operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}$ as enriched over itself via its closed symmetric monoidal structure. In particular, thanks to Remark 3.2.11, the category $2\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$ as defined in Definition 3.2.17 is equivalent to the 2-categorical enhancement of $\operatorname{Lin}_{\mathscr{A}} \operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}$ we described earlier in Remark 3.1.3.

Having defined the $(n+1)$-category of $\mathscr{A}$-linear presentable n-categories, the definition of the $(n+1)$-category of local systems of $\mathscr{A}$-linear presentable n-categories on a space X is straightforward.

Definition 3.2.19. The $(n+1)$-*category of $\mathscr{A}$-linear presentable n-categories on X* is defined as

$$(n+1)\mathbf{LocSysCat}^{n}(X;\mathscr{A}) := (n+1)\mathbf{Fun}_{n+1}\Big(X,\,(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\Big).$$

Remark 3.2.20. Note that, as in Remark 3.1.6, this $(n+1)$-category is equivalent to the internal mapping object in $(n+2)\widehat{\mathbf{Cat}}_{(\infty,n+1)}$ between X (seen as an $(n+1)$-category) and $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$.

We introduce the last bit of notations that we need in order to further categorify Corollary 2.12.

Construction 3.2.21 ([Ste20, Section 5.2])**.** Let $k \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}$ and $n \leqslant k$ be integers, and let $\mathscr{A}$ be a presentably $\mathbb{E}_k$-monoidal category. Let A be an $\mathbb{E}_n$-algebra object in $\mathscr{A}$. We can give an inductive definition of the presentable $(n+1)$-category $(n+1)\mathbf{LMod}^n_A$ of higher A-modules as follows.

(1) For $n = 0$, we simply define the category $\mathrm{LMod}^0_A(\mathscr{A})$ to be $\mathrm{LMod}_A(\mathscr{A})$ and

$$\mathrm{LMod}^0_{(-)}(\mathscr{A}) := \mathrm{LMod}_{(-)}(\mathscr{A}) \colon \mathrm{Alg}_{\mathbb{E}_k}(\mathscr{A}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_{k-1}}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big)$$

to be the functor induced at the level of $\mathbb{E}_k$-algebras by the strongly monoidal functor (3.1.12).

(2) For any $1 \leqslant n \leqslant k$, we define the $(n+1)$-category $(n+1)\mathbf{LMod}^n_A(\mathscr{A})$ to be the image of A under the functor

$$\mathrm{Alg}_{\mathbb{E}_k}(\mathscr{A}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_{k-n-1}}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n+1)}\Big)$$

defined inductively as follows. It is the composition of the functor

$$n\mathbf{LMod}^{n-1}_{(-)}(\mathscr{A}) \colon \mathrm{Alg}_{\mathbb{E}_k}(\mathscr{A}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_{k-n}}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\Big)$$

with the functor

$$(n+1)\mathbf{LMod}_{(-)}(\mathscr{A})^{\kappa_0} \colon \mathrm{Alg}_{\mathbb{E}_{k-n}}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\Big) \longrightarrow \mathrm{Alg}_{\mathbb{E}_{k-n-1}}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n+1)}\Big).$$

The latter functor is induced by the functor (3.2.8), since for every cocompletely $\mathbb{E}_k$-monoidal category $\mathscr{A}$ the assignment $\mathscr{A} \mapsto \mathrm{LMod}^{\mathrm{pr}}_{\mathscr{A}}$ is functorial and strongly monoidal ([Ste20, Remark 5.1.13]). This yields a strongly monoidal functor

$$(n+1)\mathbf{LMod}_{(-)}(\mathscr{A}) \colon \mathrm{Alg}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\Big) \longrightarrow \mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n+1)}.$$

In particular, for any $\mathbb{E}_n$-algebra A the $(n+1)$-category $(n+1)\mathbf{LMod}^n_A(\mathscr{A})$ is a presentably $\mathbb{E}_{k-n-1}$-monoidal $\mathscr{A}$-linear $(n+1)$-category. If $\mathscr{A}$ is a symmetric monoidal category and A is a *commutative* algebra in $\mathscr{A}$, we shall write simply $(n+1)\mathbf{Mod}^n_A(\mathscr{A})$. Note that the latter is a presentably *symmetric* monoidal $\mathscr{A}$-linear $(n+1)$-category.

Definition 3.2.22. We call $(n+1)\mathbf{LMod}^n_A$ the $(n+1)$*-category of n-fold A-modules*. When $\mathscr{A} = \mathcal{S}\mathrm{p}$ is the category of spectra and A is a commutative ring spectrum, we set

$$(n+1)\mathbf{Mod}^n_A := (n+1)\mathbf{Mod}^n_A(\mathcal{S}\mathrm{p}).$$

and call it the $(n+1)$*-category of n-fold A-modules*.

Remark 3.2.23. For $n \geqslant 1$, the $(n+1)$-category $(n+1)\mathbf{LMod}^n_A(\mathscr{A})$ is equivalent to the $(n+1)$-category $(n+1)\mathbf{LMod}_{n\mathbf{LMod}^{n-1}_A(\mathscr{A})}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$. So we can think of $(n+1)\mathbf{LMod}^n_A(\mathscr{A})$ as the $(n+1)$-category of A-linear presentable n-categories. In particular, when $\mathscr{A} = \mathcal{S}\mathrm{p}$ is the category of spectra and A is an $\mathbb{E}_n$-ring spectrum, we will denote the $(n+1)\mathbf{LMod}^n_A(\mathscr{A})$ as $(n+1)\mathbf{Lin}_A\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$.

The previous discussions provides all the ingredients to state the n-categorical generalization of Theorem 3.1.4.

Theorem 3.2.24. *Let $n \geqslant 1$ be an integer, let X be a pointed n-connected space (i.e., $\pi_k(X) \cong 0$ for every $k \leqslant n$), and let $\mathscr{A}$ be a presentably symmetric monoidal category. Then there exist equivalences of $(n+1)$-categories*

$$\begin{aligned}(n+1)\mathbf{LocSysCat}^n(X;\mathscr{A}) &\simeq (n+1)\mathbf{LMod}_{\Omega_* X}\Big((n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr^L}_{(\infty,n)}\Big)\\ &\simeq (n+1)\mathbf{LMod}_{n\mathbf{LMod}^{n-1}_{\Omega_*^{n+1}X}(\mathscr{A})}\Big((n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr^L}_{(\infty,n)}\Big).\end{aligned}$$

Remark 3.2.25. For simplicity, in the statement of Theorem 3.2.24, we assume $\mathscr{A}$ to be a presentably symmetric monoidal category. This is also the case that arises more naturally in our intended applications. We remark however that it is sufficient to assume that $\mathscr{A}$ is a presentably $\mathbb{E}_n$*-monoidal category*. The proof of Theorem 3.2.24 we shall give below applies equally well to this more general setting.

The proof strategy is essentially the same we followed in Theorem 3.1.4, except for one additional technical subtlety (see the proof of Theorem 3.2.24 below). Namely we shall deduce our n-categorical statement from the fact that the underlying 1-categories are equivalent in a way that is compatible with the enrichment.

Since $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is cocomplete ([Ste20, Section 5.1]), Proposition 1.2.7 allows us to deduce immediately the following result.

Lemma 3.2.26. *Let X be a pointed connected space, and let $\mathscr{A}$ be a presentably symmetric monoidal category. Let $\mathrm{LocSysCat}^n(X;\mathscr{A})$ be the underlying category of $(n+1)\mathbf{LocSysCat}^n(X;\mathscr{A})$. Then there exists an equivalence of categories*

$$\mathrm{LocSysCat}^n(X;\mathscr{A}) \simeq \mathrm{LMod}_{\Omega_* X}\Big(\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\Big).$$

As $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is cocomplete and symmetric monoidal we have a canonical strongly monoidal and colimit-preserving functor

$$\mathscr{S} \longrightarrow \mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}. \tag{3.2.27}$$

Recall that this functor sends a space X to the colimit over the constant diagram with shape X with values in the monoidal unit of $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ (that is, $n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$), i.e.,

$$X \mapsto \operatorname*{colim}_X n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}.$$

Lemma 3.2.28. *The functor (3.2.27) is equivalent to the functor*

$$n\mathbf{LocSysCat}^{n-1}(-;\mathscr{A}) := n\mathbf{Fun}_n\Big(-, n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\Big).$$

Proof. This is a higher categorical generalization of Lemma 2.6. Assume first for ease of exposition that we are in the absolute case, where $\mathscr{A} = \mathscr{S}$ is the category of spaces. The core ingredient of the proof of Lemma 2.6 was the so-called *passage to the adjoints property* of

$\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$: a colimit over a diagram of presentable categories can be computed as the limit over the opposite diagram obtained after passing to the right adjoints. It is still unknown whether $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ admits all small limits. However, Stefanich proves that it admits all limits of *left adjointable diagrams* – i.e., diagrams $K \to \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ arising from an opposite diagram $K^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ by taking left adjoints. In this case, moreover, the limit over K agrees with the colimit over K^{op} ([Ste20, Theorem 5.5.14]).

It is clear that every diagram over a small space X is left adjointable, and that it is canonically equivalent to its opposite diagram via the involution $X \simeq X^{\mathrm{op}}$. So, we have

$$\operatorname*{colim}_X n\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \simeq \lim_X n\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \simeq n\mathbf{Fun}_n\Big(X,\, n\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\Big).$$

The statement for the case of coefficients in an arbitrary presentably symmetric monoidal category $\mathscr{A}$ is obtained by the previous one simply by base change. $\square$

Lemma 3.2.29. *Let $n \geqslant 1$ be an integer, let X be a pointed n-connected space and let $\mathscr{A}$ be a presentably symmetric monoidal category. Let $\Omega^{n+1}_* X$ denote the $(n+1)$-fold loop space of X, and let $\Omega^{n+1}_* X \otimes \mathbb{1}_{\mathscr{A}}$ denote the $\mathbb{E}_{n+1}$-algebra in $\mathscr{A}$ obtained by applying to $\Omega^{n+1}_* X$ the unique strongly monoidal and colimit preserving functor $\mathscr{S} \to \mathscr{A}$. Then, there is an equivalence of categories*

$$\mathrm{LMod}_{\Omega_* X}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\Big) \simeq \mathrm{LMod}_{n\mathbf{LMod}^{n-1}_{\Omega^{n+1}_* X \otimes \mathbb{1}_{\mathscr{A}}}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\,.$$

Proof. Combining Lemma 3.2.26 and Lemma 3.2.28, we obtain equivalences of categories

$$\mathrm{LocSysCat}^n(X;\mathscr{A}) \simeq \mathrm{LMod}_{\Omega_* X}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\Big) \simeq \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\mathscr{A})}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\,.$$

Note that in the right hand side of the above chain of equivalences we can replace $n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\mathscr{A})$ with $n\mathbf{LMod}_{\Omega^2_* X}\Big(n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\Big)$. Indeed, arguing as in Proposition 2.10, we can deduce that for any integer $n \geqslant 1$ there is an equivalence

$$(n+1)\mathbf{LMod}_{\Omega_*(-)}\Big((n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\Big) \simeq (n+1)\mathbf{LocSysCat}^n(-;\mathscr{A})$$

of strongly monoidal functors from $\mathscr{S}^{\geqslant 1}_*$ to $\mathrm{Lin}_{(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$. This implies that $\mathrm{LocSysCat}^{n-1}(\Omega_* X;\mathscr{A})$ and $\mathrm{LMod}_{\Omega^2_* X}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}\Big)$ are equivalent *as $\mathbb{E}_1$-monoidal categories*. Since $\Omega^k_* X$ is always $(n-k)$-connected for every $0 \leqslant k \leqslant n$, we can iterate this argument and obtain an equivalence of categories between $\mathrm{LocSysCat}^n(X;\mathscr{A})$ and the category of $\mathscr{A}$-linear presentable n-categories which are presentably left tensored over the presentable n-category of iterated left modules over $\Omega^n_* X$. Unraveling all definitions, we see that this n-category agrees precisely with the object $\mathrm{LMod}^{n-1}_{\Omega^n_* X}$ constructed in Construction 3.2.21, hence we can conclude as desired. $\square$

Proof of Theorem 3.2.24. We would like to conclude as in the proof of Theorem 3.1.4 by combining the following two facts: first, the fact that the underlying 1-categories are equivalent, as proved in Lemmas 3.2.26 and 3.2.29; second, the observation that these equivalences are compatible with the enrichment over $n\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$. There is however one subtlety that arises when $n \geqslant 3$ and that requires some extra care. Namely, as mentioned in [Ste20, Remark 1.1.3], it is not known whether the mapping objects in a presentable n-category $\mathscr{C}$ are *presentable* $(n-1)$-categories. In particular, it is not known whether the symmetric monoidal structure of $n\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ is closed.

We can fix the issue as follows. The category $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is a symmetric monoidal subcategory of the category $\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}$, which on the other hand is cocompletely *closed* symmetric monoidal. So, up to enlarging appropriately our universe, Proposition 1.2.7 and Lemma 2.2 imply analogous equivalences

$$\begin{aligned}\mathrm{Fun}_n\Big(X, \mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}\Big) &\simeq \mathrm{LMod}_{\Omega_* X}\Big(\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}\Big)\\ &\simeq \mathrm{LMod}_{\Omega_* X \otimes n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}}\Big(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}\Big)\end{aligned}$$

in the more general setting of *cocomplete* $\mathscr{A}$-linear n-categorical local systems over X. Now, all these objects inherit a *closed* symmetric monoidal structure providing them an enrichment over themselves, hence over $\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}$ via a symmetric monoidal functor out of $\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}$: this is done exactly as in the 2-categorical case presented in Constructions 3.1.5, 3.1.7 and 3.1.11. Then, it is straightforward to see that the above equivalences commute with the coaugmentation functors coming from $\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}$. In particular, they preserve the $\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}$-enrichment and hence can be promoted to $(n+1)$-categorical equivalences.

Since these equivalences obviously preserve objects whose underlying cocomplete $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$-module category is κ_0-compact, the $(n+1)$-categorical equivalences nicely restrict to the sub-$(n+1)$-categories appearing in the statement of Theorem 3.2.24. This concludes the proof that the 1-categorical equivalence provided by Lemmas 3.2.26 and 3.2.29 can be enhanced to an $(n+1)$-categorical equivalence.

□

3.2.30. Before proceeding, we would like to comment on the role played by the connectedness assumptions in the statement of Theorem 3.2.24. In fact, it is possible to formulate a general dictionary relating n-categorical local systems and monodromy data for any space X. We will not do so explicitly in this article, as the statement is more cumbersome than Theorem 3.2.24, and it is not relevant for our intended applications to Koszul duality.

However, we shall try to clarify what is involved in such an extension. To this effect we will prove, first, Proposition 3.2.31 below. Equipped with Proposition 3.2.31 it is possible to state and prove a generalization of Theorem 3.2.24 which applies to all spaces X independently on connectedness assumptions. However, formulating the correct statement is somewhat awkward. We give an idea of these subtleties in the simple case of $X = S^1$ in Example 3.2.33.

Proposition 3.2.31. *Let $n \geqslant 1$ be a positive integer and let $n\boldsymbol{\mathscr{A}} := \prod_{\alpha\in A} n\boldsymbol{\mathscr{A}}_\alpha$ be a small product of presentably symmetric monoidal n-categories. Then we have an equivalence of categories*

$$\mathrm{Mod}_{n\boldsymbol{\mathscr{A}}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \simeq \prod_{\alpha\in A}\mathrm{Mod}_{n\boldsymbol{\mathscr{A}}_\alpha}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}.$$

Remark 3.2.32. Note that in the de-categorified setting, the analogue of Proposition 3.2.31 is false as soon as the set of indices A is not finite. This boils down to the well-known difference between the spectrum of an infinite product of commutative rings, and the infinite disjoint union of spectra of commutative rings.

Proof of Proposition 3.2.31. There is a natural functor

$$\mathrm{Mod}_{n\boldsymbol{\mathscr{A}}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \longrightarrow \prod_{\alpha} \mathrm{Lin}_{n\boldsymbol{\mathscr{A}}_\alpha}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

induced by base change along the obvious projections $n\boldsymbol{\mathscr{A}} \to n\boldsymbol{\mathscr{A}}_\alpha$. Moreover, on both sides we have forgetful functors

$$\mathrm{oblv}_{n\mathscr{A}} : \mathrm{Mod}_{n\boldsymbol{\mathscr{A}}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

and

$$\prod \circ \langle \mathrm{oblv}_{n\boldsymbol{\mathscr{A}}_\alpha} \rangle_\alpha : \prod_{\alpha} \mathrm{Mod}_{n\boldsymbol{\mathscr{A}}_\alpha}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \longrightarrow \prod_{\alpha} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}.$$

It is not difficult to see that both forgetful functors are monadic over $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$. In the first case, this is obvious; in the second case, we use the following facts together with Barr–Beck–Lurie monadicity.

- The functor $\prod \circ \langle \mathrm{oblv}_{n\boldsymbol{\mathscr{A}}_\alpha} \rangle_\alpha$ admits obviously a left adjoint, given by the assignment $n\boldsymbol{\mathscr{C}} \mapsto \{n\boldsymbol{\mathscr{C}} \otimes n\boldsymbol{\mathscr{A}}_\alpha\}_{\alpha \in A}$.
- It follows from the proof of Lemma 3.2.28 that products are the same as coproducts inside $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$, so they commute straightforwardly with *all colimits*.
- Since $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is pointed, the operation of taking products of term-wise conservative functors is again conservative. (The proof of this fact is provided in Lemma 4.1.14 below.)

Now note that the diagram obtained by passing to the right adjoints commute straightforwardly: indeed, for every index α the natural n-functor

$$n\boldsymbol{\mathscr{C}} \otimes n\boldsymbol{\mathscr{A}} \otimes_{n\mathscr{A}} n\boldsymbol{\mathscr{A}}_\alpha \longrightarrow n\boldsymbol{\mathscr{C}} \otimes n\boldsymbol{\mathscr{A}}_\alpha$$

is clearly an equivalence. So we conclude by [Lur17, Corollary 4.7.3.16]. □

As we mentioned, using Proposition 3.2.31 it is possible to remove all connectedness assumptions from the statement of Theorem 3.2.24. This requires fixing base points on each connected component of X and of $\Omega^k_* X$ for all k's ranging from 1 to n; and then transferring the Day convolution monoidal structure from $\mathrm{LocSysCat}^{n-k+1}(\Omega^{k-1}_* X; \mathscr{A})$ to $\prod_\alpha \mathrm{LMod}_{\Omega^k_* X_\alpha}(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-k+1)})$ under the equivalence of categories given by Proposition 3.2.31. This is the most delicate point: as Example 3.2.33 makes apparent the statement, though not more difficult, becomes more involved.

Example 3.2.33. Consider the case $n=2$ and $X=S^1$. Then $\Omega_* S^1 \simeq \mathbb{Z}$ and we would like to obtain equivalences

$$\mathbf{3LocSysCat}^2(X;\mathscr{A}) \simeq \mathbf{3Mod}_{\mathbb{Z}}\left(\mathbf{3Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,2)}\right) \simeq \prod_{n\in\mathbb{Z}} \mathbf{3Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,2)}.$$

Note that we have a chain of equivalences

$$\begin{aligned}
\mathbf{3Mod}_{\mathbb{Z}}\left(\mathbf{3Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,2)}\right) &\overset{2.8}{\simeq} \mathbf{3Mod}_{\mathbf{2LocSysCat}(\mathbb{Z},\mathscr{A})}\left(\mathbf{3Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,2)}\right) \\
&\simeq \mathbf{3Mod}_{\prod_n \mathbf{2LocSysCat}(\{n\};\mathscr{A})}\left(\mathbf{3Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,2)}\right) \\
&\simeq \mathbf{3Mod}_{\prod_n \mathbf{2Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}}\left(\mathbf{3Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,2)}\right) \\
&\simeq \prod_{n\in\mathbb{Z}} \mathbf{3Mod}_{\mathbf{2Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}}\left(\mathbf{3Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,2)}\right) \simeq \prod_{n\in\mathbb{Z}} \mathbf{3Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,2)}.
\end{aligned}$$

However, note that in the second equivalence we replaced $\mathbf{2LocSysCat}(\mathbb{Z};\mathscr{A})$, equipped with the Day convolution monoidal structure (which corresponds to the Künneth-like monoidal structure on the category of graded objects of $\mathbf{2Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$), with the product $\prod_n \mathbf{2Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$, which is naturally endowed with a *point-wise* monoidal structure.

In general, if G is a topological monoid, the equivalence of categories

$$\mathrm{LocSysCat}^n(G;\mathscr{A}) \simeq \prod_{\substack{G_\alpha \text{ connected} \\ \text{component}}} \mathrm{LocSysCat}^n(G_\alpha;\mathscr{A}) \simeq \prod_{\substack{G_\alpha \text{ connected} \\ \text{component}}} \mathrm{LMod}_{\Omega_* G_\alpha}\left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\right)$$

is not monoidal: on the right hand side we have the monoidal structure induced component-wise by the relative tensor product over the $\mathbb{E}_2$-monoid $n\mathbf{LocSysCat}^{n-1}(\Omega_* G_\alpha;\mathscr{A})$, while on the left hand side we have the Day convolution monoidal structure – which is the one we have to consider in order to obtain the correct $(n+1)$-category of presentable n-categories with an action of G. So one has to impose the latter monoidal structure on the product of $\mathrm{LocSysCat}^n(G_\alpha;\mathscr{A})$ in order to obtain the desired generalization of Theorem 3.2.24. Further, for higher n, this issue gets compounded, as it arises at each iterated application of the (component-wise) based loop space: unless, of course, the space X is n-connected. This is the reason we assumed n-connectedness in Theorem 3.2.24: it allows us to bypass these issues, and yields a much cleaner statement.

3.3. **Topological actions on n-categories and higher Hochschild cohomology.** In this Section we will generalize Proposition 2.16, and hence Teleman's Theorem 2.17, to the n-categorical setting. The possibility of proving such a statement was already suggested by Teleman in [Tel14, Remark 2.8]. We stress however that our main result in this section (Proposition 2.16) is conditional on the validity of an expected property of presentable n-categories which is as yet conjectural. Namely, as we discussed in the proof of Theorem 3.2.24, it is not known whether the symmetric monoidal structure on $n\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ is closed when

$n \geqslant 3$. Our proof strategy depends in a crucial way on the assumption that this claim holds. For clarity we formulate it explicitly as the following conjecture.

Conjecture 3.3.1. *Let $\mathscr{A}$ be presentably symmetric monoidal category, and let $n \geqslant 2$ be an integer. If $n\mathbf{\mathscr{C}}$ and $n\mathbf{\mathscr{D}}$ are two presentably $\mathscr{A}$-linear n-categories, then the $\mathscr{A}$-linear n-category*

$$n\mathbf{Fun}^{\mathrm{L}}_{(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}(n\mathbf{\mathscr{C}}, n\mathbf{\mathscr{D}})$$

of $\mathscr{A}$-linear colimit-preserving n-functors between $n\mathbf{\mathscr{C}}$ and $n\mathbf{\mathscr{D}}$ is presentable. In particular, it serves as a mapping object in $n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$.

Note that when $n = 1$ Conjecture 3.3.1 holds, as the symmetric monoidal structure on $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is closed and therefore $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is enriched over itself. It is natural to expect that this holds for all n, however this has not been established yet. As we discussed, the category $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ is closed symmetric monoidal and thus $\mathrm{Mod}_{n\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right)$ admits morphism objects for every positive integer n. However, in general, these are only categories tensored over $(n+1)\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$. Conjecture 3.3.1 claims that the morphism object between two presentable n-categories is not only $(n+1)\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$-tensored but is in fact presentable.

Let $\mathscr{A}$ be a presentably symmetric monoidal category, and let n be a positive integer. Following [GH15] we can define inductively the category of $\mathscr{A}$-enriched n-categories as

$$\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,n)} := \mathrm{Lin}_{\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,n-1)}}\widehat{\mathrm{Cat}}_{(\infty,1)}.$$

In virtue of [GH15, Theorem 6.3.2 and Corollary 6.3.11], we have a lax monoidal functor

$$\Omega^n\colon (\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,n)})_* \longrightarrow \mathrm{Alg}(\mathscr{A})$$

from the category of pointed $\mathscr{A}$-enriched n-categories to the category of algebras in $\mathscr{A}$, given by taking the algebra of endomorphisms of the object determined by the pointing.

Definition 3.3.2. Let $\mathscr{A}$ be a presentably symmetric monoidal category and let $n \geqslant 1$ be an integer. Let $n\mathbf{\mathscr{C}}$ be an $n\mathbf{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,n-1)}$-enriched category (in particular, it is a n-category). Let

$$n\underline{\mathbf{End}}_{n\mathbf{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,n-1)}}(n\mathbf{\mathscr{C}}, n\mathbf{\mathscr{C}})$$

be the $n\mathbf{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,n-1)}$-enriched category of $n\mathbf{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,n-1)}$-linear endofunctors of $n\mathbf{\mathscr{C}}$, seen as naturally pointed at the identity. The $\mathbb{E}_n$*-Hochschild cohomology of* $n\mathbf{\mathscr{C}}$ is the algebra object in $\mathscr{A}$ defined as

$$\mathrm{HH}^{\bullet}_{\mathbb{E}_n}(n\mathbf{\mathscr{C}}) := \Omega^n\left(n\underline{\mathbf{End}}_{n\mathbf{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,n-1)}}(n\mathbf{\mathscr{C}}, n\mathbf{\mathscr{C}})\right).$$

Remark 3.3.3. Note that, in the case $n = 1$, Definition 3.3.2 agrees with Definition 2.15. Indeed, the $\mathbb{E}_1$-Hochschild cohomology of a presentably $\mathscr{A}$-linear category $\mathscr{C}$ is just the object of $\mathscr{A}$ classifying endomorphisms of the identity functor of $\mathscr{C}$ – i.e., it is just the ordinary Hochschild cohomology of $\mathscr{C}$.

Following [GH15, Remark 6.3.16], it is possible to describe $\mathrm{HH}^\bullet_{\mathbb{E}_n}(n\mathscr{C})$ in fairly concrete terms. The n-category of n-endofunctors of $n\mathscr{C}$ is an $\mathbb{E}_1$-algebra, so it is naturally pointed at the unit (i.e., the identity endofunctor $\mathrm{id}_{n\mathscr{C}}$ of $n\mathscr{C}$). There is an $\mathscr{A}$-linear $(n-1)$-category of natural transformations between $\mathrm{id}_{n\mathscr{C}}$ and itself, which is again naturally pointed at the identity. Again, the higher natural transformations between such natural transformations form an $\mathscr{A}$-linear $(n-2)$-category, which is itself pointed. Iterating this procedure, after n steps we obtain an algebra object of $\mathscr{A}$ which parametrizes the (possibly non-invertible) natural transformations of n-simplices between $\mathrm{id}_{n\mathscr{C}}$ and itself.

Remark 3.3.4. If A is an $\mathbb{E}_n$-algebra object inside a presentably symmetric monoidal category $\mathscr{A}$, the $\mathbb{E}_n$-Hochschild cohomology of A is defined in [BFN10, Definition 5.8] as the mapping object

$$\mathrm{HH}^\bullet_{\mathbb{E}_n}(A) := \underline{\mathrm{Map}}_{\mathrm{Mod}_A^{\mathbb{E}_n}(\mathscr{A})}(A, A).$$

We can also consider the $\mathscr{A}$-linear n-category $\mathbf{B}^n A$ defined in [GH15, Corollary 6.3.11], which is the $\mathscr{A}$-enriched n-category having one single object, one single morphism for all $1 \leqslant k < n$, and an $\mathbb{E}_n$-algebra of n-morphisms equivalent to A itself. It is straightforward to see that

$$\mathrm{HH}^\bullet_{\mathbb{E}_n}(A) \simeq \mathrm{HH}^\bullet_{\mathbb{E}_n}(\mathbf{B}^n A).$$

We are now ready to state our n-categorical generalization of Proposition 2.16 and Theorem 2.17.

Proposition 3.3.5. *Let $\mathscr{A}$ be a presentably symmetric monoidal category with monoidal unit $\mathbb{1}_{\mathscr{A}}$, and let $n \geqslant 1$ be an integer. Let $n\mathscr{C}$ be an $\mathscr{A}$-linear presentable n-category, and let G be an $(n-1)$-connected topological group. Let G-* $\mathrm{ModStr}(n\mathscr{C})$ *denote the space of all possible left G-module structures on $n\mathscr{C}$. If Conjecture 3.3.1 holds, there is an equivalence of spaces*

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}(\mathscr{A})}\Big(\Omega^n_* G \otimes \mathbb{1}_{\mathscr{A}},\, \mathrm{HH}^\bullet_{\mathbb{E}_n}(\mathscr{C})\Big) \simeq G\text{-}\,\mathrm{ModStr}(n\mathscr{C}).$$

Remark 3.3.6. For $n = 1$, this is just Theorem 2.17. Noting that (-1)-connected spaces are just non-empty spaces, using the notational trick of interpreting objects of $\mathscr{A}$ as "$\mathscr{A}$-linear 0-categories" we can make the statement of Proposition 3.3.5 meaningful also for $n = 0$: indeed, a G-action on an object M of a presentably symmetric monoidal category $\mathscr{A}$ is just a $\mathbb{1}_{\mathscr{A}}[G] := (G \otimes \mathbb{1}_{\mathscr{A}})$-module structure on M, which is the same as an $\mathbb{E}_1$-algebra morphism

$$\mathbb{1}_{\mathscr{A}}[G] \longrightarrow \underline{\mathrm{End}}_{\mathscr{A}}(M).$$

Proof of Proposition 3.3.5. The proof of Theorem 2.17 works verbatim in the categorified setting once we know that, as in the 1-categorical case, the $\mathbb{E}_n$-Hochschild cohomology in the presentable setting can be expressed as a right adjoint operation to taking the n-category of left modules. More precisely, let

$$n\mathbf{LMod}_{(-)} \colon \mathrm{Alg}\Big(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}\Big) \longrightarrow \Big(\mathrm{LMod}_{n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}}\Big(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\Big)\Big)_{n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}/}$$

be the functor taking a presentably monoidal $\mathscr{A}$-linear $(n-1)$-category $(n-1)\mathscr{M}$ and sending it to the n-category of left $(n-1)\mathscr{M}$-modules $n\mathbf{LMod}_{(n-1)\mathscr{M}}\left(n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\right)$ pointed at the unit $(n-1)\mathscr{M}$. Taking the κ_0-compact objects, this functor lands in

$$\left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\right)_{n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}/}\subseteq\left(\mathrm{LMod}_{n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}\right)\right)_{n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}/}.$$

Next, we need to show that the functor

$$\mathrm{Alg}\left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}\right)\longrightarrow\left(\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\right)_{n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}/}$$

admits a right adjoint Φ_n, explicitly described by taking the endomorphisms of the object determined by the pointing from $n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$. For this, we can apply the same strategy used in the proof of [Lur17, Theorem 4.8.5.11], since the only caveat for the existence of such a right adjoint is the existence of internal morphism objects in $\mathrm{Lin}_{\mathscr{A}}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$. This is precisely the content of Conjecture 3.3.1.

Using Theorem 3.2.24, we can express the action of G (which, being $(n-1)$-connected, is the based loop space of its n-connected classifying space $\mathbf{B}G$) on a presentably $\mathscr{A}$-enriched n-category $n\mathscr{C}$ as an action of the n-category of iterated modules $n\mathbf{LMod}^{n-1}_{\Omega^n_*G}(\mathscr{A})$ on $n\mathscr{C}$. This is the same as a $\mathbb{E}_1$-monoidal functor

$$n\mathbf{LMod}^{n-1}_{\Omega^n_*G}(\mathscr{A})\longrightarrow n\underline{\mathbf{End}}_{(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}(n\mathscr{C},n\mathscr{C}),$$

which by adjunction is the same as a $\mathbb{E}_2$-monoidal functor

$$(n-1)\mathbf{LMod}^{n-1}_{\Omega^n_*G}(\mathscr{A})\longrightarrow\Phi_n\left(n\underline{\mathbf{End}}_{(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}(n\mathscr{C},n\mathscr{C})\right).$$

Iterating this construction, we obtain an $\mathbb{E}_{n+1}$-algebra morphism

$$\Omega^n_*G\otimes\mathbb{1}_{\mathscr{A}}\longrightarrow\Phi_1\Phi_2\cdots\Phi_n\left(n\underline{\mathbf{End}}_{(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}(n\mathscr{C},n\mathscr{C})\right).$$

In virtue of our description of the right adjoints Φ_n, this is immediately seen to match our Definition 3.3.2 of the $\mathbb{E}_n$-Hochschild cohomology of $n\mathscr{C}$. □

Remark 3.3.7. Another way to interpret Proposition 3.3.5 is the following. In virtue of [GH15, Corollary 6.3.11], if $\mathscr{A}$ is a presentably symmetric monoidal category then the delooping functor

$$\Omega^n\colon\left(\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,1)}\right)_*\longrightarrow\mathrm{Alg}(\mathscr{A})$$

is the lax monoidal right adjoint to the looping functor

$$\mathbf{B}^n\colon\mathrm{Alg}(\mathscr{A})\longrightarrow\left(\mathrm{Lin}_{\mathscr{A}}\widehat{\mathrm{Cat}}_{(\infty,1)}\right)_*$$

described in Remark 3.3.4. In particular, if A is an $\mathbb{E}_{n+1}$-algebra in $\mathscr{A}$, it is immediate to conclude that a monoidal n-functor of n-categories

$$\mathbf{B}^nA\longrightarrow n\underline{\mathbf{End}}_{(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}(n\mathscr{C},n\mathscr{C})$$

which equips an $\mathscr{A}$-enriched n-category $n\mathscr{C}$ of a $\mathbf{B}^n A$-module structure is the same as an $\mathbb{E}_{n+1}$-algebra morphism

$$A \longrightarrow \mathrm{HH}^{\bullet}_{\mathbb{E}_n}(n\mathscr{C}).$$

Thus, Proposition 3.3.5 suggests an n-categorical Morita equivalence: in $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$, modules for the $\mathscr{A}$-enriched n-category $\mathbf{B}^n A$ are the same as modules for the presentable $\mathscr{A}$-linear n-category of $(n-1)$-fold A-modules $n\mathbf{LMod}^{n-1}_A(\mathscr{A})$ of Definition 3.2.22.

We conclude this section with an immediate consequence of Proposition 3.3.5, which may be relevant in the context of higher Brauer groups and invertible objects in higher categories of $\Bbbk$-linear presentable n-categories.

Corollary 3.3.8. *Let G be an $(n-1)$-connected topological group, and let $\Bbbk$ be an algebraically closed field. Assume Conjecture 3.3.1 to hold. Then the equivalence classes of all possible actions of G on $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ are in bijective correspondence with the group of multiplicative characters of $\pi_n(X)$.*

Proof. The assumption that Conjecture 3.3.1 holds allows us to assume that Proposition 3.3.5 holds as well. Hence, the proof is completely analogous to the one of Proposition 2.21. □

4. Betti stacks and n-affineness

4.1. **Betti stacks and 1-affineness.** In this Section we study the question of 1-*affineness*, for Betti stacks. We shall use our result in this section as a stepping stone for our study of general n-affineness properties of Betti stacks in Section 4.2 below. We refer the reader to the Introduction for a thorough discussion of the relationship between n-affineness and higher Koszul duality. We start by reviewing basic definitions and results from [Gai15].

Construction 4.1.1 ([Gai15, Section 1.1])**.** Let $\Bbbk$ be an $\mathbb{E}_\infty$-ring spectrum, and denote by $\mathrm{Aff}_{\Bbbk}$ the category of affine schemes over $\Bbbk$, i.e., the opposite category of the category $\mathrm{CAlg}^{\geqslant 0}_{\Bbbk}$ of connective and commutative $\Bbbk$-algebras. Let $\mathrm{PSt}_{\Bbbk}$ be the category of *prestacks over* $\Bbbk$, i.e., the category of accessible presheaves over the category $\mathrm{Aff}_{\Bbbk}$. The functor

$$\mathrm{ShvCat}\colon \mathrm{PSt}_{\Bbbk} \longrightarrow \mathrm{Lin}_{\Bbbk}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$$

is by definition the right Kan extension of the functor

$$\mathrm{Lin}_{(-)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\colon \mathrm{Aff}^{\mathrm{op}}_{\Bbbk} \simeq \mathrm{CAlg}_{\Bbbk} \longrightarrow \mathrm{Lin}_{\Bbbk}\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$$

along the Yoneda embedding $よ\colon \mathrm{Aff}^{\mathrm{op}}_{\Bbbk} \to \mathrm{PSt}^{\mathrm{op}}_{\Bbbk}$.

Definition 4.1.2. Let $\mathscr{X}$ be a prestack. Then the category $\mathrm{ShvCat}(\mathscr{X})$ is the *category of quasi-coherent sheaves of ($\Bbbk$-linear) categories over* $\mathscr{X}$.

If $\mathscr{F}$ is a quasi-coherent sheaf of categories over $\mathscr{X}$, we have a well defined functor

$$\Gamma(-,\mathscr{F})\colon \left(\mathrm{Aff}_{\Bbbk/\mathscr{X}}\right)^{\mathrm{op}} \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

which we can right Kan extend to get the functor

$$\Gamma(-,\mathscr{F})\colon \mathrm{PSt}^{\mathrm{op}}_{\Bbbk/\mathcal{X}} \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

By fixing the prestack to be $\mathcal{X}$ itself, for any quasi-coherent sheaf of categories over $\mathcal{X}$ the $\Bbbk$-linear category of its global section is actually acted on by the stable category $\mathrm{QCoh}(\mathcal{X})$. Hence, we deduce the existence of a *global section functor*

$$\Gamma^{\mathrm{enh}}(\mathcal{X},-)\colon \mathrm{ShvCat}(\mathcal{X}) \longrightarrow \mathrm{Lin}_{\mathrm{QCoh}(\mathcal{X})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \tag{4.1.3}$$

which is right adjoint to the sheafification functor

$$\mathrm{Loc}_{\mathcal{X}}\colon \mathrm{Lin}_{\mathrm{QCoh}(\mathcal{X})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{ShvCat}(\mathcal{X}). \tag{4.1.4}$$

The latter acts on objects by sending a presentably $\mathrm{QCoh}(\mathcal{X})$-linear category $\mathscr{C}$ to the quasi-coherent sheaf of categories obtained by sheafifying the assignment

$$\mathrm{Spec}(S) \mapsto \mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(\mathcal{Y})} \mathscr{C}.$$

Definition 4.1.5 ([Gai15, Definition 1.3.7])**.** A prestack $\mathcal{X}$ is 1-*affine* if $\Gamma^{\mathrm{enh}}(\mathcal{X},-)$ and $\mathrm{Loc}_{\mathcal{X}}$ are mutually inverse equivalences.

Remark 4.1.6. Definition 4.1.5 has to be interpreted as a generalization of affineness, in the following sense. When $X=\mathrm{Spec}(R)$ is an affine scheme, then there is a canonical equivalence of stable categories

$$\mathrm{QCoh}(X) \simeq \mathrm{Mod}_{\Gamma(X,\mathcal{O}_X)}. \tag{4.1.7}$$

In [Gai15], stacks for which the equivalence (4.1.7) holds are called *weakly* 0-*affine*. Let us remark that, actually, the class of weakly 0-affine stacks (in this sense) sits between the class of affine schemes and an even weaker notion of 0-affineness. Indeed, if $\mathcal{X}$ is an arbitrary stack one could define a notion of 0-affineness by asking that the global sections functor

$$\Gamma(\mathcal{X},-)\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Mod}_{\Bbbk}$$

is monadic. If $\mathcal{X}$ is a weakly 0-affine stack in the sense of Gaitsgory, then the global sections are trivially monadic over $\mathrm{Mod}_{\Bbbk}$. However, this latter condition is *weaker*. For the rest of this Remark, we shall call a stack $\mathcal{X}$ that satisfies this condition *almost 0-affine*.

Let us explain the difference between Gaitsgory's weak 0-affineness, and our notion of almost 0-affineness. In virtue of the Schwede–Shipley recognition principle for stable categories of modules in spectra ([Lur17, Proposition 7.1.2.6]), if $\mathcal{X}$ is weakly 0-affine then $\mathcal{O}_{\mathcal{X}}$ has to be a compact generator of $\mathrm{QCoh}(\mathcal{X})$. This means that the functor

$$\Gamma(\mathcal{X},-) \simeq \underline{\mathrm{Map}}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{O}_{\mathcal{X}},-)$$

has to reflect equivalences and preserve all colimits. But for $\Gamma(\mathcal{X},-)$ to be monadic it is sufficient that it reflects equivalences and preserves only a special class of colimits – namely, colimits of $\Gamma(\mathcal{X},-)$-split simplicial objects ([Lur17, Theorem 4.7.3.5]). This implies that if $\mathcal{X}$

is almost 0-affine then $\mathcal{O}_{\mathfrak{X}}$ has to be a generator (although this is not a sufficient condition): but it might very well fail to be a *compact* generator.

More generally, if $\mathscr{C}$ is a stable symmetric monoidal category, it can happen that the monoidal unit $\mathbb{1}$ is not compact, but the functor it corepresents preserves colimits of $\underline{\mathrm{Map}}_{\mathscr{C}}(\mathbb{1}_{\mathscr{C}}, -)$-split simplicial objects. A particularly easy example is the following: take $\mathscr{C}$ to be a countable product of copies of $\mathrm{Mod}_{\Bbbk}$ (which can be interpreted as the category of local systems over the discrete space $\mathbb{Z}$). We will deduce from Lemma 4.1.19 that the functor of global sections is monadic, yet the monoidal unit $\mathbb{1}_{\mathbb{Z}}$ (which consists of the constant sequence $(\Bbbk)_{n\in\mathbb{Z}}$) is not compact. Indeed, the global sections functor is equivalent to the functor

$$(M_n)_{n\in\mathbb{Z}} \mapsto \underline{\mathrm{Map}}_{\mathscr{C}}(\mathbb{1}_{\mathbb{Z}}, M_n) \simeq \prod_{n\in\mathbb{Z}} \mathrm{Map}_{\Bbbk}(\Bbbk, M_n) \simeq \prod_{n\in\mathbb{Z}} M_n$$

and in general infinite colimits do not commute with infinite products. Another, highly non-trivial example of the difference between these two notions is provided by $\mathbb{CP}^{\infty}$ (Corollary 5.41).

This issue presists in the categorified setting. This means that Definition 4.1.5 has to be interpreted as a "strong" notion of 1-affineness, as is a direct categorification of Gaitsgory's weak 0-affineness. By the same token, we could define *almost* 1-affineness by requiring the mere monadicity of the global sections functor. These two notions are, in general, genuinely different. However, as we shall explain in Porism 4.1.13 below, for Betti stacks the situation is simpler, as almost 1-affineness implies 1-affineness. This will entail a significant simplification of some of our arguments.

Let $\mathrm{St}_{\Bbbk}$ be the category of *stacks over* $\Bbbk$, i.e., the full subcategory of $\mathrm{PSt}_{\Bbbk}$ spanned by those prestacks which are hypercomplete sheaves with respect to the étale topology over $\mathrm{Aff}_{\Bbbk}$. The natural functor $\mathrm{Aff}_{\Bbbk} \longrightarrow \{*\}$ induces a functor

$$(-)_{\mathrm{B}} \colon \mathrm{Shv}(\{*\}) \simeq \mathcal{S} \longrightarrow \mathrm{St}_{\Bbbk}$$

which corresponds to sending a space X to the sheafification of the constant prestack

$$\mathrm{Aff}_{\Bbbk}^{\mathrm{op}} \longrightarrow \{*\} \xrightarrow{X} \mathcal{S}.$$

Definition 4.1.8. The functor $(-)_{\mathrm{B}} \colon \mathcal{S} \to \mathrm{St}_{\Bbbk}$ is the *Betti stack functor*.

4.1.9. Betti stacks are intimately linked to the theory of local systems over spaces. Indeed, for every space X we can consider the category $\mathrm{QCoh}(X_{\mathrm{B}})$ of quasi-coherent sheaves over its associated Betti stack, and for every affine scheme $\mathrm{Spec}(R)$ over $\Bbbk$ one has a symmetric monoidal equivalence of stable categories

$$\mathrm{QCoh}(X_{\mathrm{B}} \times \mathrm{Spec}(R)) \simeq \mathrm{LocSys}(X; R),$$

where on the right hand side we are considering the point-wise tensor product, as proved in [PS20, Proposition 3.1.1]. In particular, taking R to be $\Bbbk$ in the above formula yields

$$\mathrm{QCoh}\,(X_{\mathrm{B}} \times \mathrm{Spec}(\Bbbk)) \simeq \mathrm{QCoh}\,(X_{\mathrm{B}}) \simeq \mathrm{LocSys}(X;\Bbbk).$$

Analogously, at a categorified level, it turns out that quasi-coherent sheaves of categories over X_{B} recover categorical local systems over X.

Lemma 4.1.10. *For any space X and for any base $\mathbb{E}_\infty$-ring spectrum $\Bbbk$, we have an equivalence of categories*

$$\mathrm{ShvCat}\,(X_{\mathrm{B}}) \simeq \mathrm{LocSysCat}\,(X;\Bbbk),$$

where X_{B} is seen as a stack over $\Bbbk$.

Proof. The functor $(-)_{\mathrm{B}} \colon \mathcal{S} \to \mathrm{St}_\Bbbk$ is a pullback functor between categories of sheaves, hence it obviously commutes with colimits. Presenting X as a colimit of its contractible cells yields hence equivalences

$$X_{\mathrm{B}} \simeq \operatorname*{colim}_{\{*\}\to X} \{*\}_{\mathrm{B}} \simeq \operatorname*{colim}_{\mathrm{Spec}(\Bbbk)\to X_{\mathrm{B}}} \mathrm{Spec}(\Bbbk),$$

where the last equivalence is due to [PS20, Proposition 3.1.1]. Since the functor ShvCat sends colimits of prestacks to limits of categories ([Gai15, Lemma 1.1.3]), we would like to conclude that

$$\mathrm{ShvCat}\,(X_{\mathrm{B}}) \simeq \mathrm{ShvCat}\left(\operatorname*{colim}_{\mathrm{Spec}(\Bbbk)\to X_{\mathrm{B}}} \mathrm{Spec}(\Bbbk)\right) \simeq \lim_{\mathrm{Spec}(\Bbbk)\to X_{\mathrm{B}}} \mathrm{ShvCat}\,(\mathrm{Spec}(\Bbbk)),$$

but the colimit inside the brackets is a colimit of *stacks*, which is in general different from the colimit of *prestacks*: the former is computed by sheafifying the latter, i.e., by applying the left adjoint to the inclusion $\mathrm{St}_\Bbbk \subseteq \mathrm{PSt}_\Bbbk$. However, the functor ShvCat is a sheaf for the fppf topology ([Gai15, Theorem 1.5.7]), hence for the étale topology; in particular, it factors through the sheafification functor $\mathrm{PSt}_\Bbbk \to \mathrm{St}_\Bbbk$. So, we indeed deduce that

$$\mathrm{ShvCat}\,(X_{\mathrm{B}}) \simeq \lim_X \mathrm{ShvCat}\,(\mathrm{Spec}(\Bbbk)) \simeq \lim_X \mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)},$$

where in the second equivalence we used the fact that every affine scheme is tautologically 1-affine. On the other hand, we already know that

$$\mathrm{LocSysCat}\,(X;\Bbbk) \simeq \lim_X \mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)},$$

hence the two expressions match. □

Combining Lemma 4.1.10 with Corollary 2.12, we obtain the following.

Corollary 4.1.11. *For any simply connected space X and any $\mathbb{E}_\infty$-ring spectrum $\Bbbk$ we have an equivalence of categories*

$$\mathrm{ShvCat}\,(X_{\mathrm{B}}) \simeq \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \simeq \mathrm{Lin}_{\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega^2_* X;\Bbbk)}} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

Our main result in this section is a characterization of 1-affine Betti stacks. We start by proving, in the next Proposition, that for Betti stacks X_{B} the functor $\mathrm{Loc}_{X_{\mathrm{B}}}$ is always fully faithful.

Proposition 4.1.12. *Let X be a space. Then*

$$\mathrm{Loc}_{X_{\mathrm{B}}}\colon \mathrm{Lin}_{\mathrm{QCoh}(X_{\mathrm{B}})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{ShvCat}(X_{\mathrm{B}})$$

is fully faithful.

Proof. Consider the following diagram of categories.

$$\begin{tikzcd}
\mathrm{ShvCat}(X_{\mathrm{B}}) \simeq \mathrm{LocSysCat}(X;\Bbbk) \arrow[rr, "\Gamma^{\mathrm{enh}}(X_{\mathrm{B}},-)"] \arrow[dr, bend right, "\Gamma(X,-)"'] & & \mathrm{Lin}_{\mathrm{QCoh}(X_{\mathrm{B}})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \simeq \mathrm{Lin}_{\mathrm{LocSys}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \arrow[dl, bend left, "\mathrm{oblv}_{\mathrm{LocSys}(X;\Bbbk)}"] \\
& \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} &
\end{tikzcd}$$

Here, $\Gamma(X,-)\colon \mathrm{LocSysCat}(X) \to \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is the functor which takes global sections of a local system of $\Bbbk$-linear presentable categories: this amounts to taking the limit over the diagram of presentable categories of shape X defined by a local system of categories. Since limits and colimits over spaces in $\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ agree (Lemma 2.6), this functor is both a right and left adjoint to the constant local system functor. In particular, it is a right adjoint that preserves *all* geometric realizations.

Analogously,

$$\mathrm{oblv}_{\mathrm{LocSys}(X;\Bbbk)}\colon \mathrm{Lin}_{\mathrm{LocSys}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

simply forgets the $\mathrm{LocSys}(X;\Bbbk)$-module structure of a $\Bbbk$-linear presentable category. Again, this obviously commutes with both limits and colimits (hence, with geometric realizations) and it is a right adjoint to the base change functor

$$-\otimes_{\mathrm{Mod}_{\Bbbk}}\mathrm{LocSys}(X;\Bbbk)\colon \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{Lin}_{\mathrm{LocSys}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

Moreover, such functor is conservative (it is actually monadic).

Notice that this diagram does commute. Indeed, under the equivalence of Lemma 4.1.10, the global sections of a quasi-coherent sheaf of categories $\mathcal{F}$ over the Betti stack X_{B} correspond to the "enhanced" global sections of a local system of categories over X, taking into account the natural tensor action of $\mathrm{LocSys}(X;\Bbbk)$ over them. This is simply a categorification of the fact that global sections of local systems of $\Bbbk$-modules over a space are endowed with an action of the algebra of its $\Bbbk$-cochains. In particular, forgetting such action recovers the underlying $\Bbbk$-linear presentable category $\Gamma(X,\mathcal{F})$.

Summarizing, the above diagram is a commutative diagram of categories where the arrow on the left is a right adjoint which preserves geometric realizations and the arrow on the right is a *conservative* right adjoint which preserves geometric realizations. Moreover, for any

$\Bbbk$-linear presentable category $\mathscr{C}$ we have that the unit map for the adjunction const $\dashv \Gamma(X,-)$

$$\mathscr{C} \longrightarrow \Gamma(X, \operatorname{const}(\mathscr{C})) \simeq \operatorname{oblv}_{\operatorname{LocSys}(X;\Bbbk)} \Gamma^{\operatorname{enh}}(X_{\mathrm{B}}, \operatorname{const}(\mathscr{C}))$$

produces by adjunction the map

$$\mathscr{C} \otimes_{\operatorname{Mod}_{\Bbbk}} \operatorname{LocSys}(X;\Bbbk) \longrightarrow \Gamma^{\operatorname{enh}}(X_{\mathrm{B}}, \operatorname{const}(\mathscr{C})).$$

If this was an equivalence, then invoking [Lur17, Corollary 4.7.3.16] one would deduce the existence of a left adjoint to $\Gamma^{\operatorname{enh}}(X_{\mathrm{B}},-)$ (which of course has to be $\operatorname{Loc}_{X_{\mathrm{B}}}$) and this left adjoint is moreover *fully faithful*. Since forgetting the $\operatorname{LocSys}(X;\Bbbk)$-module structure is conservative, we can check whether this map is an equivalence at the level of the underlying $\Bbbk$-linear categories. On one side, we can write the domain of the above map as

$$\begin{aligned}\mathscr{C} \otimes_{\operatorname{Mod}_{\Bbbk}} \operatorname{LocSys}(X;\Bbbk) &\simeq \mathscr{C} \otimes_{\operatorname{Mod}_{\Bbbk}} \lim_X \operatorname{Mod}_{\Bbbk} \simeq \mathscr{C} \otimes_{\operatorname{Mod}_{\Bbbk}} \operatorname*{colim}_X \operatorname{Mod}_{\Bbbk} \\ &\simeq \operatorname*{colim}_X \mathscr{C} \otimes_{\operatorname{Mod}_{\Bbbk}} \operatorname{Mod}_{\Bbbk} \simeq \operatorname*{colim}_X \mathscr{C} \simeq \lim_X \mathscr{C},\end{aligned}$$

thanks to the fact that one can swap limits and colimits of presentable categories indexed by groupoids thanks to Lemma 2.6. On the other hand, the codomain of this map is

$$\operatorname{oblv}_{\operatorname{LocSys}(X)} \Gamma^{\operatorname{enh}}(X_{\mathrm{B}}, \operatorname{const}(\mathscr{C})) \simeq \lim_X \Gamma(\{*\}_{\mathrm{B}}, \operatorname{const}(\mathscr{C})) \simeq \lim_X \mathscr{C},$$

so we immediately deduce our claim. □

Porism 4.1.13. The proof of Proposition 4.1.12 relies crucially on [Lur17, Corollary 4.7.3.16]. As an immediate consequence of that result, it follows that the fully faithful left adjoint $\operatorname{Loc}_{X_{\mathrm{B}}}$ is an equivalence precisely if the global sections functor is conservative, because this is the only obstruction to its monadicity.

Moreover, the proof of Proposition 4.1.12 identifies the global sections functor $\Gamma(X_{\mathrm{B}},-)$ with the global sections functor for local systems of categories over X, which in turn is realized as a limit of presentable stable categories indexed by the space X. Since limits and colimits of presentable categories over diagrams indexed by spaces are canonically equivalent (Lemma 2.6) it follows that in this case the global sections *always* commute with *all* colimits. This simplifies greatly the discussion in Remark 4.1.6 and implies that for Betti stacks the weak and strong notions of 1-affineness agree.

In order to give a characterization of 1-affine Betti stacks we need a few preliminary results. First, we show that we can reduce to consider connected spaces.

Lemma 4.1.14. *Let X be a disjoint union of (possibly infinitely many) connected components*

$$X = \bigcup_{\alpha \in A} X_\alpha$$

such that the Betti stack of each connected component is 1-affine. Then X_{B} is 1-affine as well.

Proof. Under our assumptions, the global sections functor

$$\Gamma(X,-)\colon \operatorname{LocSysCat}(X;\Bbbk) \longrightarrow \operatorname{Lin}_{\Bbbk}\operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}$$

is simply the product of all the global sections functors

$$\Gamma(X_\alpha,-)\colon \operatorname{LocSysCat}(X_\alpha;\Bbbk) \longrightarrow \operatorname{Lin}_{\Bbbk}\operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

By Porism 4.1.13, it follows that we only need to check that taking products of conservative functors is conservative. So suppose

$$F \simeq (F_\alpha)\colon \mathscr{F} \simeq (\mathscr{F}_\alpha) \longrightarrow \mathscr{G} \simeq (\mathscr{G}_\alpha)$$

is a map between categorical local systems on X, and suppose it becomes an equivalence after applying $\Gamma(X,-)$. Since $\operatorname{Lin}_{\Bbbk}\operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}$ is pointed, we have a commutative diagram

$$\begin{tikzcd}
\Gamma(X,\mathscr{F}_\alpha) \arrow[r, "\Gamma(X_\alpha, F_\alpha)"] \arrow[d, "\iota_\alpha"'] & \Gamma(X,\mathscr{G}_\alpha) \arrow[d, "\iota_\alpha"] \\
\prod_\alpha \Gamma(X,\mathscr{F}_\alpha) \arrow[r, "\prod_\alpha \Gamma(X_\alpha, F_\alpha)"] \arrow[d, "\pi_\alpha"'] & \prod_\alpha \Gamma(X,\mathscr{G}_\alpha) \arrow[d, "\pi_\alpha"] \\
\Gamma(X,\mathscr{F}_\alpha) \arrow[r, "\Gamma(X_\alpha, F_\alpha)"] & \Gamma(X,\mathscr{G}_\alpha)
\end{tikzcd}$$

that exhibits each $\Gamma(X_\alpha, F_\alpha)$ as a retract of the equivalence $\Gamma(X,F) \simeq \prod \Gamma(X_\alpha, F_\alpha)$; in particular, they are equivalences as well. But since each $(X_\alpha)_{\mathrm{B}}$ is 1-affine, it follows that each $\Gamma(X_\alpha,-)$ is conservative; in particular, F_α is an equivalence for each α. So, F must be an equivalence as well. □

Using Lemma 4.1.14, we can restrict attention to connected spaces. The next result give a complete characterization of Betti stacks that are 1-affine. A drawback of our result is that the necessary and sufficient condition we find is not very easy to verify in practice. In the remainder of the Section we shall complement this result by providing more explicit criteria for 1-affineness and its failure.

Theorem 4.1.15. *Let X be a pointed and connected space. Then X_{B} is 1-affine if and only if the global section functor*

$$\Gamma(\Omega_* X,-) : \operatorname{LocSys}(\Omega_* X;\Bbbk) \longrightarrow \operatorname{Mod}_{\Bbbk}$$

is monadic.

Remark 4.1.16. Before proceeding with the proof of Theorem 4.1.15 let us comment on its statement. Under what conditions is the global section functor monadic? In Theorem 4.1.15 we look at the loop space of a connected space X, but this question makes sense for a general space Y. We touched upon this question in the general setting of stacks in Remark 4.1.6

above, where we called a stack for which this property holds *almost 0-affine*. We can say something more in the case of Betti stacks. It turns out that the answer is quite subtle. As explained in Remark 4.1.6, we have the following simple facts:

(1) $\Gamma(Y,-)$ is monadic, i.e. Y_{B} is almost 0-affine, *only if* the trivial local system $\underline{\Bbbk}_Y$ is a generator of $\mathrm{LocSys}(Y;\Bbbk)$
(2) $\Gamma(Y,-)$ is monadic *if* the trivial local system $\underline{\Bbbk}_Y$ is a compact generator of $\mathrm{LocSys}(Y;\Bbbk)$. Indeed, in this case Y_{B} is weakly 0-affine in the sense of Gaitsgory, and therefore in particular almost 0-affine. In turn, by Schwede–Shipley ([Lur17, Proposition 7.1.2.6]) $\underline{\Bbbk}_Y$ is a compact generator if and only if the global section functor induces an equivalence

$$\mathrm{LocSys}(Y;\Bbbk) \simeq \mathrm{Mod}_{\mathrm{C}^\bullet(Y;\Bbbk)}. \tag{4.1.17}$$

In [BN12, Corollary 3.18] it is stated that equivalence (4.1.17) holds for all simply connected and finite spaces. However, as pointed out to us by Y. Harpaz and confirmed by D. Nadler in private communication, this statement is wrong. Consider for example $Y := S^2$ to be the sphere. The algebra of $\Bbbk$-chains on its based loop space $\Omega_* Y$ is a free associative algebra $\Bbbk\langle u\rangle$ generated by a variable u lying in homological degree 1. Then the functor $\mathrm{Map}_{\mathrm{C}_\bullet(\Omega_* Y;\Bbbk)}(\Bbbk,-)$ cannot be conservative, because the non-trivial $\Bbbk\langle u\rangle$-module $\Bbbk\langle u,u^{-1}\rangle$ is right orthogonal to $\Bbbk$. In particular, $\Bbbk$ cannot be a *generator* of $\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* Y;\Bbbk)}$: and so in particular it cannot be a *compact generator*.

Equivalence (4.1.17) holds for 0-truncated spaces, and we believe that in fact this might be a necessary condition. We do not know of any non-trivial space satisfying (4.1.17). On the other hand a characterization of almost 0-affine Betti stacks would be very interesting, but it seems difficult to achieve, and we have only partial results in this direction. In Lemma 4.1.24, we show that the non-triviality of $\pi_1(Y,y)$ at any base point obstructs the monadicity of the global sections functor of local systems over Y, which is perhaps an expected result. In Corollary 5.41 we will also show that the Betti stack of $\mathbb{CP}^\infty$ is almost 0-affine, so we do have non-trivial examples. We leave the further exploration of these questions to future work.

Let us now proceed with the proof of Theorem 4.1.15. The key ingredient in the proof is the following result of Gaitsgory.

Proposition 4.1.18 ([Gai15, Proposition 11.2.1]). *Let G be a group prestack. Assume that Loc_G is fully faithful, that $\mathrm{QCoh}(G)$ is dualizable as a $\Bbbk$-linear presentable category, and that the convolution tensor product on $\mathrm{QCoh}(G)$ turns it into a rigid monoidal category. The following are equivalent.*

(1) The stack $\mathbf{B}G$ is 1-affine.
(2) The global sections functor $\mathrm{QCoh}(G) \to \mathrm{Mod}_\Bbbk$ is monadic.

Proof of Theorem 4.1.15. Let $G := \Omega_* X$ be the based loop space of X. Since the Betti stack functor is the left adjoint in a geometric morphism between categories of sheaves and as such it preserves products, we have that $(\Omega_* X)_{\mathrm{B}}$ is still a group stack (hence a group prestack, since products are preserved by the inclusion $\mathrm{St}_{\Bbbk} \subseteq \mathrm{PSt}_{\Bbbk}$ and

$$\Omega_*(X_{\mathrm{B}}) \simeq (\Omega_* X)_{\mathrm{B}}\,.$$

We set $G := \Omega_*(X_{\mathrm{B}}) \simeq (\Omega_* X)_{\mathrm{B}}$ and note that X_{B} is realized as the delooping of G in the category $\mathrm{PSt}_{\Bbbk}$. In formulas, we can write

$$X_{\mathrm{B}} \simeq \mathbf{B}G.$$

Let us check that also the other assumptions of Proposition 4.1.18 hold in our situation. Observe first that

$$\mathrm{QCoh}(G) \simeq \mathrm{LocSys}(\Omega_* X; \Bbbk)$$

is compactly generated. If $\Omega_* X$ is connected this follows because $\mathrm{LocSys}(\Omega_* X; \Bbbk)$ is equivalent to the category of modules over $\mathrm{C}_\bullet(\Omega_* \Omega_* X; \Bbbk)$, and is therefore compactly generated. Since compactly generated categories are stable under products ([Lur09, Proposition 5.5.7.6]), we conclude that compact generation holds in general also if $\Omega_* X$ fails to be connected. Thus $\mathrm{QCoh}(G)$ is in particular a dualizable $\Bbbk$-linear presentable categories (thanks to [Lur18, Theorem D.7.0.7]).

Moreover, Lemma 1.1.6 implies that $\mathrm{QCoh}(G)$ decomposes (non-canonically) as a product of categories of left modules

$$\mathrm{QCoh}(G) \simeq \prod_{\alpha \in \pi_0(\Omega_* X)} \mathrm{LMod}_{\Bbbk \otimes \Omega^2_* X}.$$

The convolution tensor product on the left hand side translates into a "Künneth-like" relative tensor product

$$(M_\alpha)_\alpha \otimes (N_\alpha)_\alpha \simeq \left(\bigoplus_{\beta \cdot \gamma = \alpha} M_\beta \otimes_{\Omega^2_* X} N_\gamma \right)_\alpha,$$

where $\cdot$ denotes the group law on $\pi_0(\Omega_* X) \cong \pi_1(X)$. Since we already proved that $\mathrm{QCoh}(G)$ is compactly generated, in order to prove that it is rigid it is sufficient to prove that every compact object is fully dualizable ([Gai15, Section D.1.3]). Notice that each factor $\mathrm{LMod}_{\Bbbk \otimes \Omega^2_* X}$ is a rigid monoidal category, since in a category of left modules over an $\mathbb{E}_2$-ring spectrum equipped with its relative tensor product compact objects are known to be precisely the class of fully dualizable objects. Since

$$\underline{\mathrm{Map}}_{\mathrm{QCoh}(G)}((M_\alpha)_\alpha, (N_\alpha)_\alpha) \simeq \prod_{\alpha \in \pi_1(X)} \underline{\mathrm{Map}}_{\Bbbk \otimes \Omega^2_* X}(M_\alpha, N_\alpha),$$

is not difficult to see that a collection of $\Omega^2_* X$-modules $(M_\alpha)_\alpha$ is compact if and only if each M_α is compact as a $(\Bbbk \otimes \Omega^2_* X)$-module and the collection of indices for which M_α is not trivial is finite. In particular, a compact object $(M_\alpha)_\alpha$ in $\prod_\alpha \mathrm{LMod}_{\Bbbk \otimes \Omega^2_* X} \simeq \mathrm{QCoh}(G)$ admits a both left and right dual, which is described by the collection of $(\Bbbk \otimes \Omega^2_* X)$-modules

$$\left((M_\alpha)_{\alpha\in\pi_1(X)}\right)^\vee := \left(M^\vee_{\alpha^{-1}}\right)_{\alpha\in\pi_1(X)}$$

where $M^\vee_{\alpha^{-1}}$ is the $(\Bbbk \otimes \Omega^2_* X)$-linear dual of $M_{\alpha^{-1}}$.

Finally, Proposition 4.1.12 implies that Loc_G is fully faithful, so in particular the hypotheses of Proposition 4.1.18 are satisfied by the based loop space on any space X. □

Theorem 4.1.15 guarantees that we can check the 1-affineness of the Betti stack of a pointed and connected space X by looking at the monadicity of the de-categorified global sections over its based loop space $\Omega_* X$. In virtue of Lemma 4.1.14, we know that we can always assume X to be connected. The following Lemma, which is a de-categorified analogue of the aforementioned result, allows us to reduce ourselves without loss of generality to the case when X is even *simply* connected.

Lemma 4.1.19. *Let X be a disjoint union of (possibly infinitely many) connected components*

$$X = \bigcup_{\alpha\in\pi_0(X)} X_\alpha$$

such that each $\Gamma(X_\alpha, -)\colon \mathrm{LocSys}(X_\alpha;\Bbbk) \to \mathrm{Mod}_\Bbbk$ is monadic. Then $\Gamma(X,-)\colon \mathrm{LocSys}(X) \to \mathrm{Mod}_\Bbbk$ is monadic as well.

Proof. By Barr–Beck–Lurie's theorem ([Lur17, Theorem 4.7.3.5]), the functor

$$\Gamma(X,-)\colon \mathrm{LocSys}(X;\Bbbk) \longrightarrow \mathrm{Mod}_\Bbbk$$

is monadic if and only if it is conservative and preserves colimits of $\Gamma(X,-)$-split simplicial objects. Under the equivalence

$$\mathrm{LocSys}(X;\Bbbk) \simeq \prod_{\alpha\in\pi_0(X)} \mathrm{LocSys}(X_\alpha;\Bbbk)$$

the functor $\Gamma(X,-)$ is equivalent to the functor $\prod_{\alpha\in\pi_0(X)}\Gamma(X_\alpha,-)$, and because of our assumptions each $\Gamma(X_\alpha,-)$ is monadic. So $\Gamma(X,-)$ is conservative, thanks to an analogous argument to the one used for the proof of Lemma 4.1.14.

So we only need to check that taking the global sections of a local system of $\Bbbk$-modules over X preserves colimits of $\Gamma(X,-)$-split simplicial objects. We argue as follows: let

$$\mathscr{F}_\bullet\colon \boldsymbol{\Delta}^{\mathrm{op}}_+ \longrightarrow \mathrm{LocSys}(X;\Bbbk) \simeq \prod_{\alpha\in\pi_0(X)} \mathrm{LocSys}(X_\alpha;\Bbbk)$$

be an augmented simplicial object which becomes split after taking global sections. In particular, we can interpret it as a collection of augmented simplicial objects

$$\mathscr{F}_\bullet \simeq \left(\mathscr{F}^\alpha_\bullet\right)_{\alpha\in\pi_0(X)},$$

and we have

$$\Gamma(X,\mathscr{F}_\bullet) \simeq \prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha,\mathscr{F}^\alpha_\bullet\right).$$

We claim that such product is split because each $\Gamma(X_\alpha,\mathscr{F}^\alpha_\bullet)$ is already a split simplicial object of $\mathrm{Mod}_{\Bbbk}$.

Let

$$\rho\colon \mathbf{\Delta}_+ \longrightarrow \mathbf{\Delta}_+$$

be the functor defined via the construction

$$[n] \mapsto [0]\star[n] \simeq [n+1].$$

For any category $\mathscr{C}$, pre-composition with ρ^{op} produces a functor at the level of categories of simplicial objects of $\mathscr{C}$

$$\mathrm{T} := -\circ\rho^{\mathrm{op}}\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathscr{C}) \longrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathscr{C}).$$

Applying this general machinery to our case, we obtain an augmented simplicial $\Bbbk$-module

$$\mathrm{T}\Gamma(X,\mathscr{F}_\bullet) := \Gamma(X,-)\circ\mathscr{F}_\bullet\circ\rho^{\mathrm{op}}\colon \mathbf{\Delta}_+^{\mathrm{op}} \longrightarrow \mathrm{LocSys}(X;\Bbbk) \longrightarrow \mathrm{Mod}_{\Bbbk},$$

such that for all $n \geqslant 0$ one has

$$\mathrm{T}\Gamma(X,\mathscr{F}_n) \simeq \Gamma(X,\mathrm{T}\mathscr{F}_n) \simeq \Gamma(X,\mathscr{F}_{n+1}).$$

The natural inclusion $[n]\subseteq\rho([n])$ defines a natural transformation from ρ to $\mathrm{id}_{\mathbf{\Delta}_+}$, hence a canonical map

$$\varphi_\bullet\colon \mathrm{T}\Gamma(X,\mathscr{F}_\bullet) \longrightarrow \Gamma(X,\mathscr{F}_\bullet).$$

Since the simplicial local system $\mathscr{F}_\bullet$ is a $\Gamma(X,-)$-split simplicial object, [Lur17, Corollary 4.7.2.9] tells us that one has a right homotopy inverse

$$\psi_\bullet\colon \Gamma(X,\mathscr{F}_\bullet) \longrightarrow \mathrm{T}\Gamma(X,\mathscr{F}_\bullet).$$

Unraveling all definitions, this means that we have a sequence of maps of $\Bbbk$-modules

$$\left(\varphi^\alpha_\bullet\colon \Gamma\left(X_\alpha,\mathrm{T}\mathscr{F}^\alpha_\bullet\right) \longrightarrow \Gamma\left(X_\alpha,\mathscr{F}^\alpha_\bullet\right)\right)_{\alpha\in\pi_0(X)}.$$

and after taking their product the composition

$$\prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha,\mathscr{F}^\alpha_\bullet\right) \xrightarrow{\psi_\bullet} \prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha,\mathrm{T}\mathscr{F}^\alpha_\bullet\right) \xrightarrow{\prod\varphi^\alpha_\bullet} \prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha,\mathscr{F}^\alpha_\bullet\right)$$

is homotopic to the identity.

We claim that $\psi_\bullet$ yields a right homotopy inverse to *each* map $\varphi^\alpha_\bullet$. Indeed, for each $\overline{\alpha}\in\pi_0(X)$ define

$$\psi^{\overline{\alpha}}_\bullet\colon \Gamma\left(X_\alpha,\mathscr{F}^{\overline{\alpha}}_\bullet\right) \xhookrightarrow{\iota^{\overline{\alpha}}_\bullet} \prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha,\mathscr{F}^\alpha_\bullet\right) \xrightarrow{\psi_\bullet} \prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha,\mathrm{T}\mathscr{F}^\alpha_\bullet\right) \xrightarrow{\pi^{\overline{\alpha}}_\bullet} \Gamma\left(X_\alpha,\mathrm{T}\mathscr{F}^{\overline{\alpha}}_\bullet\right).$$

Since we have by assumption homotopies making the diagram

$$\begin{tikzcd}
\prod \Gamma\left(X_\alpha, \mathscr{F}_\bullet^\alpha\right) \arrow[r, "\psi_\bullet"] \arrow[rr, bend left, "\mathrm{id}"] & \prod \Gamma\left(X_\alpha, \mathrm{T}\mathscr{F}_\bullet^\alpha\right) \arrow[r, "\varphi_\bullet"] \arrow[d, "\pi_\bullet^{\overline{\alpha}}"'] & \prod \Gamma(X_\alpha, \mathscr{F}_\bullet^\alpha) \arrow[d, "\pi_\bullet^{\overline{\alpha}}"] \\
\Gamma\left(X_\alpha, \mathscr{F}_\bullet^{\overline{\alpha}}\right) \arrow[u, "\iota_\bullet^{\overline{\alpha}}"] \arrow[r, "\psi_\bullet^\alpha"'] & \Gamma\left(X_\alpha, \mathrm{T}\mathscr{F}_\bullet^{\overline{\alpha}}\right) \arrow[r, "\varphi_\bullet^\alpha"'] & \Gamma\left(X_\alpha, \mathscr{F}_\bullet^{\overline{\alpha}}\right)
\end{tikzcd}$$

commute in every direction, and since $\pi_\bullet^{\overline{\alpha}} \circ \iota_\bullet^{\overline{\alpha}}$ is homotopic to the identity of $\mathscr{F}_\bullet^{\overline{\alpha}}$, it follows that each morphism

$$\psi_\bullet^\alpha \colon \Gamma\left(X_\alpha, \mathscr{F}_\bullet^\alpha\right) \longrightarrow \Gamma\left(X_\alpha, \mathrm{T}\mathscr{F}_\bullet^\alpha\right)$$

produces a right homotopy inverse to the natural map $\varphi_\bullet^\alpha$ of sections over X_α for each $\alpha \in \pi_0(X)$.

So applying the opposite direction of the criterion of [Lur17, Corollary 4.7.2.9] we obtain that a $\Gamma(X, -)$-split simplicial object in $\mathrm{LocSys}(X;\Bbbk)$ corresponds to a sequence of $\Gamma(X_\alpha, -)$-split simplicial objects of $\mathrm{LocSys}(X_\alpha;\Bbbk)$. This discussion implies that we have a chain of equivalences

$$\begin{aligned}
\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \Gamma\left(X, \mathscr{F}_\bullet\right) &\simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha, \mathscr{F}_\bullet^\alpha\right) \\
&\simeq \prod_{\alpha\in\pi_0(X)} \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \Gamma\left(X_\alpha, \mathscr{F}_\bullet^\alpha\right) \\
&\simeq \prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha, \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathscr{F}_\bullet^\alpha\right) \simeq \Gamma\left(X, \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathscr{F}_\bullet\right).
\end{aligned}$$

All these equivalence hold because we were assuming that all $\Gamma(X_\alpha, -)$ were monadic functors (hence, they all preserves colimit of $\Gamma(X_\alpha, -)$-split simplicial objects), together with the fact that colimits of split simplicial objects are universal ([Lur17, Remark 4.7.2.4]) and with the observation that colimits in products of categories are computed component-wise. $\square$

Lemma 4.1.19 will provide a useful tool in proving that Betti stacks of certain spaces are, or are not, 1-affine.

Corollary 4.1.20. *Let X be a 1-truncated topological space. Then the Betti stack X_{B} is 1-affine.*

Proof. We can assume that X is connected thanks to Lemma 4.1.14; in particular, we have that $X \simeq K(\pi, 1)$ is an Eilenberg-MacLane space for the discrete group $\pi := \pi_1(X)$. We apply Theorem 4.1.15: X_{B} is 1-affine precisely if

$$\Gamma(\pi, -) \colon \mathrm{LocSys}(\pi;\Bbbk) \longrightarrow \mathrm{Mod}_\Bbbk$$

is monadic. But π is a disjoint union of contractible components

$$\pi \simeq \bigcup_{\alpha \in \pi_1(X)} \{*\},$$

so Lemma 4.1.19 immediately implies our statement. □

Proposition 4.1.21. *Let X be a space, and assume $\Bbbk$ to be a semisimple commutative ring. If there exists a base point x such that $\pi_2(X,x)$ contains an element g either of infinite order, or such that the order of g is a unit in $\Bbbk$, then the Betti $\Bbbk$-stack X_{B} is not 1-affine.*

Remark 4.1.22. In particular, if $\Bbbk$ is a field of characteristic 0, the second homotopy group of a space X always provides an obstruction to the 1-affineness of its Betti stack over $\Bbbk$.

Proof. Let us write $\pi := \pi_2(X,x)$. In virtue of Lemma 4.1.14, we can assume without loss of generality that X is connected. So, taking the space of loops based at x and using Theorem 4.1.15, our statement will follow once we prove that the global sections functor on $\mathrm{LocSys}(\Omega_* X;\Bbbk)$ is not monadic over $\mathrm{Mod}_{\Bbbk}$. Lemma 4.1.19 allows us to assume that $\Omega_* X$ is itself connected, so under the equivalence of Lemma 1.1.6 we are asked to check whether the functor

$$\underline{\mathrm{Map}}_{\Bbbk\otimes\Omega^2_* X}(\Bbbk, -)\colon \mathrm{LMod}_{\Bbbk\otimes\Omega^2_* X} \longrightarrow \mathrm{Mod}_{\Bbbk}$$

is not monadic.

Notice that $\Bbbk \otimes \pi \simeq \pi_0(\Bbbk \otimes \Omega^2_* X)$ is isomorphic as a $\Bbbk$-algebra to the group ring $\Bbbk[\pi]$, and the obvious projection

$$\Omega^2_* X \longrightarrow \pi_0(\Omega^2_* X) \simeq \pi$$

turns $\Bbbk[\pi]$ into a $(\Bbbk \otimes \Omega^2_* X)$-module. Let $g \in \pi$ be as in the statement: then $(1-g)$ is a non-nilpotent element. Indeed, consider the subgroup $\langle g \rangle$ generated by g: the inclusion $\langle g \rangle \subseteq \pi$ induces an inclusion of commutative rings $\Bbbk[\langle g \rangle] \subseteq \Bbbk[\pi]$, so if $(1-g)$ is not nilpotent in $\Bbbk[\langle g \rangle]$ it will automatically be not nilpotent in $\Bbbk[\pi]$ as well. We can therefore reduce ourselves to prove the statement in the case π is cyclic and generated by g.

(1) If g has infinite order, then $\Bbbk[\pi] \cong \Bbbk[t,t^{-1}]$ is a domain, hence $(1-g)$ is not nilpotent.
(2) If g has finite order n and n is a unit in $\Bbbk$, then $\Bbbk[\pi]$ is a semisimple algebra because of Maschke's theorem (see for example [PS02, Theorem 3.4.7]). In particular, $\Bbbk[\pi]$ is reduced and $(1-g)$ is not nilpotent.

It follows that the set $S := \{(1-t)^n \mid n \geqslant 0\} \subseteq \Bbbk[\pi]$ satisfies the Ore conditions in the graded commutative ring $\pi_\bullet(\Bbbk \otimes \Omega^2_* X)$, hence there exists the localization $(\Bbbk \otimes \Omega^2_* X)[(1-g)^{-1}]$ ([Lur17, Section 7.2.3]) and since S does not contain the 0 element such localization is not trivial. Since $(1-g)$ is an element in the fiber of the map

$$\Bbbk \otimes \Omega^2_* X \longrightarrow \Bbbk \otimes \pi_0(\Omega^2_* X) \simeq \Bbbk[\pi] \longrightarrow \Bbbk,$$

it follows that $\Bbbk$ is S-nilpotent. In particular, [Lur17, Proposition 7.3.2.14] guarantees that $(\Bbbk \otimes \Omega^2_* X)[(1-g)^{-1}]$ is right orthogonal to $\Bbbk$ as a $(\Bbbk \otimes \Omega^2_* X)$-module, so the global sections cannot be conservative. □

Porism 4.1.23. Notice that, in the setting of the proof of Proposition 4.1.21, the functor

$$\underline{\mathrm{Map}}_{\Bbbk\otimes\Omega^2_* X}(\Bbbk, -)\colon \mathrm{LMod}_{\Bbbk\otimes\Omega^2_* X} \longrightarrow \mathrm{Mod}_{\Bbbk}$$

is simply the global sections functor

$$\Gamma(\Omega_* X, -)\colon \mathrm{LocSys}(\Omega_* X; \Bbbk) \longrightarrow \mathrm{Mod}_{\Bbbk}$$

under the equivalence $\mathrm{LocSys}(\Omega_* X; \Bbbk) \simeq \mathrm{LMod}_{\Bbbk\otimes\Omega^2_* X}$ of Lemma 1.1.6. So, the proof of Proposition 4.1.21 actually passes through the proof of the following de-categorified analogue.

Lemma 4.1.24. *Let X be a space, and assume $\Bbbk$ to be a semisimple commutative ring. If there exists $\Bbbk$ a base point x such that $\pi_1(X, x)$ contains an element g either of infinite order, or such that the order of g is a unit in $\Bbbk$, then the Betti stack X_{B} is not almost 0-affine.*

Remark 4.1.25. Proposition 4.1.21 is *not* a necessary condition. As we will explain in Remark 5.40, Example 5.38 shows that $\mathbf{B}\mathbb{CP}^\infty \simeq \mathbf{B}^3\mathbb{Z}$ is 1-affine.

We conclude this section by studying in detail two examples. The infinite projective space $\mathbb{CP}^\infty$ has non-trivial, free second homotopy group. Thus, by Proposition 4.1.21, it is not 1-affine. We will give a direct proof of this fact, which essentially already appeared in Teleman [Tel14]. The second example we will discuss is the circle S^1. Since S^1 is 1-truncated it is 1-affine by Corollary 4.1.20. We will give a different argument for the 1-affineness of S^1, following [Tel14] and [GHM23]. In fact, both examples can be viewed as instances of the emerging picture of 3d Homological Mirror Symmetry: we refer the reader to [Tel14], [GHM23] and references therein for additional information.

Example 4.1.26. Let $X := \mathbb{CP}^\infty$ be the infinite-dimensional projective space. It is a simply connected CW complex whose based loop space is homotopy equivalent to the circle, i.e.,

$$\Omega_* X \simeq S^1, \qquad \text{and} \qquad \Omega^2_* X \simeq \mathbb{Z}.$$

In particular, there is an equivalence of $\mathbb{E}_2$-algebras $\mathrm{C}_\bullet(\Omega^2_* X; \Bbbk) \simeq \Bbbk[t, t^{-1}]$. This yields an equivalence of monoidal categories

$$\mathrm{LMod}_{\Omega^2_* X}(\mathrm{Mod}_{\Bbbk}) \simeq \mathrm{Mod}_{\Bbbk[t,t^{-1}]}.$$

We can use this to obtain an interesting alternative description of $\mathrm{LocSysCat}(X; \Bbbk)$. Indeed, we can write a chain of equivalences

$$\psi\colon \mathrm{LocSysCat}(X; \Bbbk) \xrightarrow[2.12]{\simeq} \mathrm{Lin}_{\Bbbk[t,t^{-1}]}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \xrightarrow{\simeq} \mathrm{Lin}_{\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \xrightarrow{\simeq} \mathrm{ShvCat}(\mathbb{G}_{\mathrm{m},\Bbbk})$$

where the last equivalence follows from the fact that $\mathbb{G}_{\mathrm{m},\Bbbk}$, being affine, is obviously 1-affine. Thus categorical local systems on X can be described equivalently as quasi-coherent quasi-coherent sheaves of categories over $\mathbb{G}_{\mathrm{m},\Bbbk}$.

We remark that ψ can be seen as an instance of 3d Homological Mirror Symmetry. Let us briefly explain why this is the case. The pair of spaces

$$\mathrm{T}^*\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk} \longleftrightarrow \mathrm{T}^*\mathbb{G}_{\mathrm{m},\Bbbk}$$

is one of the basic examples of 3d mirror partners. At least if $\Bbbk = \mathbb{C}$, the category $\mathrm{LocSysCat}(X;\Bbbk)$ can be viewed as (a subcategory of) the category of 3d A-branes on $\mathrm{T}^*\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}$. The key observation here is that topologically we have a homotopy equivalence

$$\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}(\mathbb{C}) \simeq \mathbb{CP}^\infty$$

and the category of 3d A-branes of a cotangent stack is expected to contain local systems of categories over the base; we refer to [Tel14] for a fuller discussion of this point. Conversely, $\mathrm{ShvCat}(\mathbb{G}_{\mathrm{m},\Bbbk})$ is (a subcategory of) the category of 3d B-branes on $\mathrm{T}^*\mathbb{G}_{\mathrm{m},\Bbbk}$. From this perspective, equivalence ψ implements a dictionary relating A-branes on $\mathrm{T}^*\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}$ and B-branes on its mirror, thus paralleling closely the classical 2d HMS story.

Next, let us show that X is not 1-affine; see also [Tel14] for a similar discussion. Note that it is enough to show that the global sections functor

$$\Gamma(X,-)\colon \mathrm{LocSysCat}(X;\Bbbk) \longrightarrow \mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

is not conservative. This is easily done directly using the equivalences provided by Corollary 2.12

$$\mathrm{LocSysCat}(X;\Bbbk) \simeq \mathrm{LMod}_{S^1}\Big(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big) \simeq \mathrm{Lin}_{\Bbbk[t,t^{-1}]} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}. \tag{4.1.27}$$

We can easily classify the $\Bbbk[t,t^{-1}]$-linear structures on $\mathrm{Mod}_\Bbbk$: as explained in Paragraph 2.20 and proved in Proposition 2.21, these are equivalently described as characters of $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$. So, an S^1-action on $\mathrm{Mod}_\Bbbk$ corresponds to the choice of a non-trivial scalar $\lambda \in \Bbbk^\times$, which is the image of t under an $\mathbb{E}_2$-morphism $\Bbbk[t,t^{-1}] \to \Bbbk$. Let us denote by $\mathrm{Mod}_\Bbbk(\lambda)$ the corresponding categorical S^1-module.

Now, the global section functor on $\mathrm{LocSysCat}(X;\Bbbk)$ is corepresented by the trivial local system, which is the unit with respect to the ordinary monoidal structure on $\mathrm{LocSysCat}(X;\Bbbk)$. We denote this object by

$$\underline{\mathrm{LocSys}}(-) \in \mathrm{LocSysCat}(X;\Bbbk).$$

Under equivalence (4.1.27), the object $\underline{\mathrm{LocSys}}(-)$ is mapped to $\mathrm{Mod}_\Bbbk(1)$.

Notice that, for any invertible $\lambda \in \Bbbk^\times$, $\mathrm{Mod}_\Bbbk(\lambda)$ can be seen as the category of $\Bbbk(\lambda)$-modules inside $\mathrm{Mod}_{\Bbbk[t,t^{-1}]}$, where $\Bbbk(\lambda)$ is the commutative $\Bbbk[t,t^{-1}]$-algebra on the underlying $\Bbbk$-module $\Bbbk$ determined by the evaluation $\mathrm{ev}_\lambda\colon \Bbbk[t,t^{-1}] \to \Bbbk$; in other words:

$$\mathrm{Mod}_\Bbbk(\lambda) \simeq \mathrm{Mod}_{\Bbbk(\lambda)}\big(\mathrm{Mod}_{\Bbbk[t,t^{-1}]}\big) \simeq \mathrm{Mod}_\Bbbk,$$

where in the last equivalence we used [Lur17, Remark 7.1.3.7]. Invoking the categorical Eilenberg-Watts theorem, we have then

$$\begin{aligned}\underline{\mathrm{Fun}}^{\mathrm{L}}_{\Bbbk[t,t^{-1}]}(\mathrm{Mod}_{\Bbbk}(1), \mathrm{Mod}_{\Bbbk}(\lambda)) &\simeq \underline{\mathrm{Fun}}^{\mathrm{L}}_{\Bbbk[t,t^{-1}]}\big(\mathrm{Mod}_{\Bbbk(1)}\big(\mathrm{Mod}_{\Bbbk[t,t^{-1}]}\big), \mathrm{Mod}_{\Bbbk}(\lambda)\big)\\ &\simeq {}_{\Bbbk(1)}\mathrm{BMod}_{\Bbbk(\lambda)}\big(\mathrm{Mod}_{\Bbbk[t,t^{-1}]}\big)\\ &\simeq \mathrm{Mod}_{\Bbbk(1)\otimes_{\Bbbk[t,t^{-1}]}\Bbbk(\lambda)}.\end{aligned}$$

But now an easy homological computation shows that $\Bbbk(1)\otimes_{\Bbbk[t,t^{-1}]}\Bbbk(\lambda)$ is 0 whenever $\lambda \neq 1$, and so the S^1-fixed points are trivial.

It might be useful to revisit the previous calculation from a geometric standpoint using equivalence ψ. Let

$$\iota_{\lambda} : \mathrm{Spec}(\Bbbk) \longrightarrow \mathbb{G}_{\mathrm{m},\Bbbk}$$

be the $\Bbbk$-rational point $\lambda \in \mathbb{G}_{\mathrm{m},\Bbbk}(\Bbbk)$. Under ψ, the object $\mathrm{Mod}_{\Bbbk}(\lambda)$ becomes a categorified *skyscraper sheaf*. That is, it is the quasi-coherent sheaf of categories obtained by pushing-forward the unit along the functor

$$\iota_{\lambda,*} : \mathrm{ShvCat}(\mathrm{Spec}(\Bbbk)) \longrightarrow \mathrm{ShvCat}(\mathbb{G}_{\mathrm{m},\Bbbk})$$

The computation above shows that, as expected, skyscraper sheaves at different points are mutually orthogonal.

Example 4.1.28. Let us consider next the case $X = S^1 \simeq K(\mathbb{Z},1)$. We can prove directly that its Betti stack is 1-affine as follows. Note first that $\mathrm{LocSysCat}(S^1;\Bbbk)$ is the same as the category of $\Bbbk$-linear presentable categories equipped with a choice of an autoequivalence $F : \mathscr{C} \simeq \mathscr{C}$. The latter, in turn, is equivalent to the category of $\mathrm{QCoh}(\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk})$-linear presentable categories ([GHM23, Example 0.4]). Since $\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}$ is 1-affine ([Gai15, Remark 2.5.2]), it follows that

$$\mathrm{ShvCat}(S^1_{\mathrm{B}}) \simeq \mathrm{LocSysCat}(S^1;\Bbbk) \simeq \mathrm{Lin}_{\mathrm{QCoh}(\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \simeq \mathrm{ShvCat}(\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}). \tag{4.1.29}$$

The latter category is also equivalent to the category $\mathrm{Lin}_{\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ of $\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})$-linear presentable categories, where now $\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})$ is seen as a symmetric monoidal category via the convolution tensor product induced by the $\mathbb{E}_{\infty}$-group structure on $\mathbb{G}_{\mathrm{m},\Bbbk}$. This can be deduced by concatenating the equivalences (10.1) and (10.4) in [Gai15]; an earlier proof of this fact can be found in [BFN12].

Equipped with this monoidal structure, $\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})$ is monoidally equivalent to the category of representations of $\mathbb{Z}$ inside $\mathrm{Mod}_{\Bbbk}$, which is in turn equivalent to the category $\mathrm{LocSys}(S^1;\Bbbk)$ equipped with its point-wise monoidal structure. This is precisely $\mathrm{QCoh}(S^1_{\mathrm{B}})$. We deduce that there is an equivalence

$$\mathrm{ShvCat}\left(S^1_{\mathrm{B}}\right) \simeq \mathrm{Lin}_{\mathrm{QCoh}(S^1_{\mathrm{B}})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

i.e., S^1_{B} is 1-affine, as we wanted to show. We remark that equivalence (4.1.29) can be viewed as the opposite direction of 3d HMS with respect to the one considered in Example 4.1.26, see [GHM23] for more information.

4.2. **Sheaves of higher categories and n-affineness for Betti stacks.** In this Section we establish an n-analogue version of Corollary 4.1.20 for all n. Namely, we will show that if X is a n-truncated space then its Betti stack X_{B} is n-affine in the sense of [Ste21, Definition 8.7.3]. We start by recalling some preliminary notions on quasi-coherent sheaves of presentable n-categories over prestacks.

First, recall from Construction 3.2.21 that for any $n \geqslant 0$ we have the functor

$$\begin{aligned}(n+1)\mathbf{Mod}^n_{(-)}\colon \mathrm{Aff}^{\mathrm{op}}_{\Bbbk} \simeq \mathrm{CAlg}^{\geqslant 0}_{\Bbbk} &\longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n+1)}\\ \mathrm{Spec}(R) &\mapsto (n+1)\mathbf{Mod}^n_R\end{aligned}$$

sending an affine scheme $\mathrm{Spec}(R)$ to its $(n+1)$-category of n-fold R-modules (or equivalently of R-linear presentable n-categories, when $n \geqslant 1$).

Definition 4.2.1 ([Ste21, Definition 14.2.4])**.** Let $\mathscr{X}$ be a prestack defined over a commutative ring spectrum $\Bbbk$, and let $n \geqslant 1$ be an integer. The *$(n+1)$-category* $(n+1)\mathbf{ShvCat}^n(\mathscr{X})$ *of quasi-coherent sheaves of ($\Bbbk$-linear presentable) n-categories* over $\mathscr{X}$ is defined as the right Kan extension of the functor $(n+1)\mathbf{Mod}^n_{(-)}$ along the inclusion $\mathrm{Aff}^{\mathrm{op}}_{\Bbbk} \subseteq \mathrm{PSt}^{\mathrm{op}}_{\Bbbk}$.

This means that $(n+1)\mathbf{ShvCat}^n(\mathscr{X})$ is the limit computed inside $\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n+1)}$

$$(n+1)\mathbf{ShvCat}^n(\mathscr{X}) := \lim_{\substack{\mathrm{Spec}(R)\to\mathscr{X}\\ R\in\mathrm{CAlg}^{\geqslant 0}_{\Bbbk}}} (n+1)\mathbf{Mod}^n_R \simeq \lim_{\substack{\mathrm{Spec}(R)\to\mathscr{X}\\ R\in\mathrm{CAlg}^{\geqslant 0}_{\Bbbk}}} (n+1)\mathbf{Lin}_R\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}.$$

In particular, Definition 4.2.1 agrees with Construction 4.1.1 when $n = 1$.

Remark 4.2.2. For $n \geqslant 2$, the right Kan extension defining the $(n+1)$-category of quasi-coherent sheaves of n-categories in Definition 4.2.1 is also a *left* Kan extension. Indeed, for any morphism of prestacks $f\colon \mathscr{X} \to \mathscr{Y}$ and for any $n \geqslant 2$ the pullback functor $f^*\colon \mathrm{ShvCat}^n(\mathscr{Y}) \to \mathrm{ShvCat}^n(\mathscr{X})$ is part of an ambidextrous adjunction ([Ste21, Corollary 14.2.10]). In particular, [Ste20, Theorem 5.5.14] implies that the limit inside $(n+2)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n+1)}$ along the pullback $(n+1)$-functors corresponds to the colimit inside $(n+2)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n+1)}$ along the colimit-preserving pushforward $(n+1)$-functors. This also holds for $n = 1$ if the morphism f is assumed to be affine schematic.

Remark 4.2.3. Being defined as a right Kan extension, it follows that $(n+1)\mathbf{ShvCat}^n$ sends colimits of prestacks to limits of $\Bbbk$-linear presentable $(n+1)$-categories. Moreover, $(n+1)\mathbf{ShvCat}^n$ satisfies descent with respect to étale topology ([Ste21, Corollary 14.3.5]). In particular, we deduce that in the case of a Betti stack X_{B} of a CW complex X, written as a colimit of its contractible cells

$$X \simeq \operatorname*{colim}_{x\to X} \{*\},$$

we have that

$$\begin{aligned}(n+1)\mathbf{ShvCat}^n(X_{\mathrm{B}}) &\simeq \lim_{x\to X}(n+1)\mathbf{ShvCat}^n(\{*\})\\ &\simeq \lim_{x\to X}(n+1)\mathbf{Mod}_{\Bbbk}^{n+1}\\ &\simeq \lim_{x\to X}(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\\ &\simeq (n+1)\mathbf{LocSysCat}^n(X;\Bbbk).\end{aligned} \tag{4.2.4}$$

Just like for the case $n=1$, we have a naturally defined global sections $(n+1)$-functor

$$(n+1)\mathbf{\Gamma}(\mathscr{X},-)\colon (n+1)\mathbf{ShvCat}^n(\mathscr{X})\longrightarrow (n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$$

which for any quasi-coherent sheaf of n-categories $n\mathscr{F}$ over $\mathscr{X}$ computes the limit over all local sections

$$\lim_{\substack{\mathrm{Spec}(R)\to\mathscr{X}\\ R\in\mathrm{CAlg}_{\Bbbk}^{\geqslant 0}}}(n+1)\mathbf{\Gamma}(\mathrm{Spec}(R),n\mathscr{F})$$

inside $(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$.

Definition 4.2.5. Let $\mathscr{X}$ be a prestack. We say that $\mathscr{X}$ is *n-affine* if the global sections $(n+1)$-functor

$$(n+1)\mathbf{\Gamma}(\mathscr{X},-)\colon (n+1)\mathbf{ShvCat}^n(\mathscr{X})\longrightarrow (n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$$

is monadic.

Remark 4.2.6. Thanks to [Ste21, Proposition 14.3.6], we know that for $n\geqslant 2$ the monadicity requirement for n-affineness can be checked at the level of the underlying categories – i.e., the $(n+1)$-functor $(n+1)\mathbf{\Gamma}(\mathscr{X},-)$ is monadic if and only if the underlying functor of ordinary categories

$$\Gamma(\mathscr{X},-)\colon\ \mathrm{ShvCat}^n(\mathscr{X})\longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

is monadic.

However, in the case $n=1$ Definition 4.2.5 recovers Definition 4.1.5 ([Ste21, Remark 14.3.8]). As explained in Remark 4.1.6, this is is a stronger requirement than asking simply for the monadicity of the global sections functor: indeed, this is equivalent to asking for it to be a *colimit-preserving* monadic right adjoint. The fact that, for $n\geqslant 2$, this issue does not arise boils down to the fact that pullbacks and pushforwards between presentable $(n+1)$-categories of quasi-coherent sheaves of n-categories form an *ambidextrous* adjunction in $(n+2)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n+1)}$, as already mentioned in Remark 4.2.2. In particular, if $n\geqslant 2$ the monadicity requirement of Definition 4.2.5 yields a $\Bbbk$-linear equivalence of presentable $(n+1)$-categories

$$(n+1)\mathbf{ShvCat}^n(\mathscr{X})\simeq (n+1)\mathbf{Mod}_{n\mathbf{ShvCat}^{n-1}(\mathscr{X})}\Big((n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\Big).$$

Proposition 4.2.7. *Let X be any space. Then the functor*

$$\Gamma^{\mathrm{enh}}(X_{\mathrm{B}},-)\colon \operatorname{LocSysCat}^n(X;\Bbbk) \longrightarrow \operatorname{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} \operatorname{Pr}^{\mathrm{L}}_{(\infty,n)}$$

admits a fully faithful left adjoint.

Proof. The Proposition is proved in the same way as Proposition 4.1.12. The key observation is that limits of left adjointable diagrams inside $(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ exist and agree with colimits of the corresponding diagram of left adjoints ([Ste20, Theorem 5.5.14]). □

So, just like in the 2-categorical case (Porism 4.1.13), the only obstruction to n-affineness for Betti stacks is the conservativity of the global sections functor $\Gamma(X,-)\colon \operatorname{LocSysCat}^n(X;\Bbbk) \to \operatorname{Lin}_{\Bbbk}\operatorname{Pr}^{\mathrm{L}}_{(\infty,n)}$.

Theorem 4.2.8. *Let X be a space, and suppose that X is n-truncated. Then its Betti stack X_{B} is n-affine.*

In order to prove Theorem 4.2.8, we need to establish the following categorified variant of Proposition 4.1.18 – at least for Betti stacks.

Theorem 4.2.9. *Let X be a space with a choice of a base point. Then its Betti stack X_{B} is n-affine if and only if the Betti stack of the based loop space $\Omega_* X_{\mathrm{B}}$ is $(n-1)$-affine.*

Remark 4.2.10. Theorem 4.2.9 can be also interpreted as a topological analogue of the n-affineness criterion of [Ste21, Theorem 14.3.9].

Before proceeding, let us remark that Theorem 4.2.9 immediately implies Theorem 4.2.8. Indeed, for any $n \geqslant 1$, a simple induction shows that the n-affineness of X_{B} boils down to the 1-affineness of $\Omega_*^{n-1}X$. If $\Omega_*^{n-1}X$ is 1-truncated (which is equivalent to asking that X is n-truncated) then its Betti stack is 1-affine in virtue of Corollary 4.1.20. So, all is left to do is showing that Theorem 4.2.9 holds. We will devote to this task most of the remainder of this Section.

Lemma 4.2.11. *Let $n \geqslant 2$ be an integer. The assigment $X \mapsto \operatorname{Lin}_{\operatorname{LocSysCat}^{n-1}(X;\Bbbk)} \operatorname{Pr}^{\mathrm{L}}_{(\infty,n)}$ satisfies descent with respect to effective epimorphisms in $\mathcal{S}$ precisely if $\Omega_* X$ is $(n-1)$-affine.*

We shall split the proof of Lemma 4.2.11 in substeps for the convenience of the reader.

4.2.12. First, assume that X is connected: arguing as in Lemma 4.1.14, this is not a restrictive assumption. Take any homotopy effective epimorphism $U_\bullet \to X$. For every non-negative integer n the n-th space in the simplicial diagram $U_\bullet \to X$ is described as an n-fold fiber product

$$U_n \simeq \underbrace{U_0 \times_X U_0 \times_X \cdots \times_X U_0}_{n \text{ times}}.$$

Applying the functor $\operatorname{Lin}_{n\mathbf{LocSysCat}^{n-1}(-;\Bbbk)} \operatorname{Pr}^{\mathrm{L}}_{(\infty,n)}$, we obtain a cosimplicial diagram of categories

$$\operatorname{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} \operatorname{Pr}^{\mathrm{L}}_{(\infty,n)} \longrightarrow \operatorname{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_\bullet;\Bbbk)} \operatorname{Pr}^{\mathrm{L}}_{(\infty,n)},$$

hence a natural functor

$$\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \longrightarrow \lim_{[m]\in\mathbf{\Delta}} \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}, \tag{4.2.13}$$

where the limit on the right hand side is computed in the category of $(n+1)$-categories.

4.2.14. Such functor admits a right adjoint: this can be described at the level of objects by the assignment

$$\mathscr{F}_{\bullet} := (\mathscr{F}_m)_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \mapsto \lim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \iota_{m,*}\mathscr{F}_m,$$

where $\mathscr{F}_{\bullet}$ is a cosimplicial system of presentable $n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)$-linear n-categories. Notice that this limit of $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-linear n-categories *does exist*. Indeed, for $n \geqslant 2$ and for an arbitrary morphism of stacks $f:\mathscr{X}\to\mathscr{Y}$ the canonical pullback n-functor

$$f^{*}\colon (n+1)\,\mathrm{ShvCat}^{n}(\mathscr{Y}) \longrightarrow (n+1)\,\mathrm{ShvCat}^{n}(\mathscr{X})$$

is part of an *ambidextrous* adjunction: this is [Ste21, Corollary 14.2.10]. So, the existence of limits of left adjointable diagrams of presentable n-categories allows us to conclude that this formula does make sense.

To see that the functor $\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \to \lim_{[m]}\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is an equivalence, we check that both the unit and the counit of the aforementioned adjunction are equivalences of n-categories. The unit of an adjunction is an equivalence if and only if the functor (4.2.13) is fully faithful.

Lemma 4.2.15. *For any space X, for any choice of a effective epimorphism $U_{\bullet}$ and for any integer $n \geqslant 1$, the functor* (4.2.13) *is fully faithful.*

Proof. We have a commutative diagram of functors

$$\begin{tikzcd}
\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \arrow[r, "(4.2.13)"] \arrow[d, hook, "\mathrm{Loc}^{n}_{X_{\mathrm{B}}}"'] & \lim_{[m]}\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \arrow[d, hook, "\lim_{[m]}\mathrm{Loc}^{n}_{(U_m)_{\mathrm{B}}}"] \\
\mathrm{LocSysCat}^{n}(X;\Bbbk) \arrow[r, "\simeq"] & \lim_{[m]}\mathrm{LocSysCat}^{n-1}(U_m;\Bbbk).
\end{tikzcd}$$

The bottom arrow is an equivalence because for any integer $n \geqslant 2$ the assignment $X \mapsto n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$ satisfies descent in X; this is true also for $n=1$ if we interpret $1\mathbf{LocSysCat}^{0}(X;\Bbbk)$ to be $\mathrm{LocSys}(X;\Bbbk)$. Moreover, the two vertical arrows are fully faithful thanks to Proposition 4.2.7. It follows that the upper arrow has to be fully faithful as well. □

We are only left to check that the counit is an equivalence. We argue as follows: since we are assuming X to be connected, we can choose an effective epimorphism $\{*\}\to X$ given by the inclusion of any base point in the only connected component of X: the fact that this is an effective epimorphism is due to [Lur09, Proposition 7.2.1.14]. Setting $V_n := U_n \times_X \{*\}$, we obtain a commutative diagram of categories

$$\begin{tikzcd} \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \arrow[r, hook] \arrow[d] & \lim_{[m]} \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \arrow[d] \\ \mathrm{Lin}_{\Bbbk} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \arrow[r] & \lim_{[m]} \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} . \end{tikzcd} \tag{4.2.16}$$

The bottom functor is an equivalence. Indeed, effective epimorphisms of spaces are stable under pullbacks (because colimits are universal in any topos), so under base change we obtain an effective epimorphism

$$V_\bullet := U_\bullet \times_X \{*\} \longrightarrow \{*\} .$$

Choosing the inclusion of any point $\{*\} \to V_0 \simeq U_0 \times_X \{*\}$ and using [Lur17, Corollary 4.7.2.9] we obtain a splitting of the above augmented simplicial diagram: it follows that its colimit is preserved by *any* functor, and in particular by the contravariant functor $\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(-;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$.

4.2.17. On the other hand, the vertical functor on the right is conservative. Indeed, under the equivalence

$$\mathrm{Lin}_{\Bbbk} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \simeq \lim_{[m]\in\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

such functor corresponds to the composition of the fully faithful embedding

$$\lim_{[m]\in\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \hookrightarrow \lim_{[m]\in\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{LocSysCat}^n(U_m;\Bbbk) \simeq \mathrm{LocSysCat}^n(X;\Bbbk)$$

with the pullback along the chosen base point $\{*\} \to X$. Since X is assumed to be connected, this pullback corresponds to forgetting the $\Omega_* X$ action on a presentably $\Bbbk$-linear n-category under the equivalence of Theorem 3.2.24, which is a conservative operation.

So the issue is whether the diagram (4.2.16) is horizontally right adjointable: if that was the case, then one could check whether the counit is an equivalence after pulling back over the base point, and there the answer is obvious because the bottom functor is an equivalence.

4.2.18. Notice that the vertical functor on the right hand side of diagram (4.2.16) can be computed in the following way: at each step of the simplicial diagram

$$n\mathscr{C}_m \mapsto n\mathscr{C}_m \otimes_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)$$

we take the base change induced by the symmetric monoidal pullback

$$n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk) \longrightarrow n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)$$

along the natural projection $V_m := U_m \times_X \{*\} \to U_m$. On the left hand side, the vertical functor is similarly described as a base change functor

$$n\mathscr{C} \mapsto n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}.$$

So, the right adjointability of the diagram (4.2.16) amounts to asking for the natural $\Bbbk$-linear n-functor

$$\lim_{[m]\in\boldsymbol{\Delta}^{\mathrm{op}}} \iota_{m,*} n\mathscr{C}_m \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow$$
$$\longrightarrow \lim_{[m]\in\boldsymbol{\Delta}^{\mathrm{op}}} \iota_{m,*}\left(n\mathscr{C}_m \otimes_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)\right)$$

to be an equivalence.

The commutativity of the totalization of the cosimplicial diagram with the tensor product is not a problem: such limit is computed as a colimit of presentable n-categories along the simplicial diagram obtained by passing to the left adjoints. Since the tensor product of presentable n-categories commutes with colimits, we can bring the limit inside and outside of the tensor product without any harm. So we can rephrase our problem as follows: given a map of topological spaces $U \to X$ and setting $V := U \times_X \{*\}$, when is the n-functor

$$n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(V;\Bbbk) \quad (4.2.19)$$

an equivalence for an arbitrary $n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)$-linear presentable n-category $n\mathscr{C}$?

4.2.20. We will show that (4.2.19) is an equivalence by proving the following stronger statement. Let $Y \to X \leftarrow Z$ be morphisms of spaces, such that the Betti stack $(\Omega_* X)_{\mathrm{B}}$ is $(n-1)$-affine. We will show that there is an equivalence

$$n\mathbf{LocSysCat}^{n-1}(Y) \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(Z;\Bbbk) \xrightarrow{\simeq} n\mathbf{LocSysCat}^{n-1}(Y \times_X Z;\Bbbk)$$

Note that by setting $Y := U$ and $Z := \{*\}$, we can recover (4.2.19) in the case $n\mathscr{C} = n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)$. Then, using the fact that for any other $n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)$-linear presentable n-category $n\mathscr{C}$ one has a canonical equivalence

$$n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(U;\Bbbk) \simeq n\mathscr{C},$$

we can extend our result to an arbitrary $n\mathscr{C}$. Since categorical local systems satisfy hyperdescent, we can replace both Y and Z by some effective epimorphism $W_\bullet \to Y$ whose 0-th stage is described by a disjoint union of contractible spaces $\{*\}_{\alpha\in A}$, and where the m-th stage is described by the usual Čech formula

$$W_m := \coprod_{\alpha_1,\cdots,\alpha_m} \{*\}_{\alpha_1} \times_{W_{m-1}} \cdots \times_{W_{m-1}} \{*\}_{\alpha_m}.$$

In virtue of Lemma 3.2.28, such a disjoint union is sent via the functor $n\mathbf{LocSysCat}^{n-1}(-;\Bbbk)$ to a coproduct of presentably $\Bbbk$-linear n-categories. The latter distributes over tensor products:

we can therefore reduce ourselves to the case in which W_0 is just a point. Using the fact that

$$\Omega^p_* X \times_X \Omega^q_* X \simeq \Omega^{p+q}_* X,$$

it is easy to see that W_m is described by an m-fold based loop space $\Omega^m_* X$. So, we are left to prove that there is an equivalence

$$n\mathbf{LocSysCat}^{n-1}(\Omega^m_* X;\Bbbk) \simeq \underbrace{n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} \cdots \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}}_{m \text{ times}}.$$

Arguing by induction, and using the fact that

$$n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

can be written as

$$\left(n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\right) \otimes \left(n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\right),$$

we are reduced to prove that there is an equivalence

$$n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk) \simeq n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}.$$

In the formulas above, we are writing $\otimes$ for $\otimes_{n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}}$.

The symmetric monoidal pullback n-functor $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk) \to n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ induced by the inclusion of the base point in X turns $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ into an $\mathbb{E}_\infty$-$n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-algebra inside $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$. In particular, we have a $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-linear n-functor

$$n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \tag{4.2.21}$$

corresponding to such symmetric monoidal operation.

Lemma 4.2.22. *The underlying functor of the action $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-linear functor*

$$n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

is a monadic functor of categories.

Proof. The tensor product $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ is computed as a geometric realization of a simplicial diagram of n-categories $n\boldsymbol{\mathscr{C}}_\bullet$, whose i-th term is described as

$$n\mathscr{C}_i \simeq n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)^{\otimes i} \otimes n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \simeq n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)^{\otimes i},$$

where the faces and degeneracies are induced by pullback n-functors. Here, the tensor product is understood as the tensor product of $\Bbbk$-linear presentable n-categories, whose monoidal unit is $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$. Under the equivalences

$$n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)^{\otimes i} \otimes n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \simeq n\mathbf{LocSysCat}^{n-1}\left(X^{\times i};\Bbbk\right),$$

we can describe such simplicial object in more detail.

(1) The degeneracy morphisms of such simplicial diagram correspond to pulling back categorical local systems along projections $X^{\times i} \to X^{\times i-1}$.

(2) The face morphisms correspond either to pulling back categorical local systems along the extremal inclusions $X^{i-1} \simeq \{*\} \times X^{i-1} \subseteq X^i$ and $X^{i-1} \simeq X^{i-1} \times \{*\} \subseteq X^i$ (these are the face morphisms ∂_0 and ∂_i), or to pulling back categorical local systems along the morphism $\Delta_p \colon X^{i-1} \to X^i$ described informally by $(x_1, \dots, x_{p-1}, x_p, x_{p+1}, \dots, x_{i-1}) \mapsto (x_1, \dots, x_{p-1}, x_p, x_p, x_{p+1}, \dots, x_{i-1})$ (these are the face morphisms ∂_p, for $p \in \{1, \dots, i-1\}$).

Thanks to this description of the faces n-functors in this simplicial n-category, we see immediately that for any morphism $\alpha\colon [i] \to [j]$ the diagram

$$\begin{tikzcd}
n\mathbf{LocSysCat}^{n-1}\left(X^{\times(j+1)};\Bbbk\right) \arrow[r, "\partial_{i+1,*}"] \arrow[d, "(\alpha\star\mathrm{id}_{[0]})^*"'] & n\mathbf{LocSysCat}^{n-1}\left(X^{\times j};\Bbbk\right) \arrow[d, "\alpha^*"] \\
n\mathbf{LocSysCat}^{n-1}\left(X^{\times(i+1)};\Bbbk\right) \arrow[r, "\partial_{i,*}"'] & n\mathbf{LocSysCat}^{n-1}\left(X^{\times i};\Bbbk\right)
\end{tikzcd}$$

is commutative. This means that such simplicial diagram of n-categories (or better, the underlying simplicial diagram of categories) satisfies the monadic Beck-Chevalley condition ([Gai15, Definition C.1.5]), after applying suitably [Gai15, Lemma C.1.6]. Hence, [Gai15, Lemma C.1.8] implies that such action functor is indeed monadic. □

Lemma 4.2.22 is what we need in order to apply [Gai15, Corollary C.2.3], which guarantees that we can compute the monad described by the action functor

$$\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{\mathrm{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

as the composition

$$\eta^* \circ \eta_* \colon \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

where $\eta\colon \{*\} \hookrightarrow X$ is the inclusion of the base point. This implies that the naturally defined functor

$$\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{\mathrm{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow \mathrm{LocSysCat}^{n-1}(\Omega_* X;\Bbbk),$$

obtained by taking the right adjoint to

$$\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \simeq \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{\mathrm{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

and then composing with the pullback n-functors induced by the two projections $\pi_1, \pi_2\colon \Omega_* X \rightrightarrows \{*\}$, makes the diagram

$$\begin{tikzcd}
\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{\mathrm{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \arrow[rr] \arrow[dr, bend right] & & \mathrm{LocSysCat}^{n-1}(\Omega_* X;\Bbbk) \arrow[dl, bend left] \\
& \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} &
\end{tikzcd}$$

commute. In the above picture, the horizontal arrow is the one described in this paragraph; the right-hand side arrow is the push-forward along the natural terminal morphism $\Omega_* X \to \{*\}$ (i.e., it is the global sections functor), and the left-hand side arrow is the functor (4.2.21) corresponding $\mathrm{LocSysCat}^{n-1}(X;\Bbbk)$-linear monoidal structure of $\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$.

So, by Barr–Beck–Lurie, the horizontal arrow is an equivalence precisely if

$$\Gamma(\Omega_* X_{\mathrm{B}}, -)\colon \mathrm{LocSysCat}^{n-1}(\Omega_* X;\Bbbk) \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

is a monadic functor. But this is equivalent to $n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)$ being monadic over $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ as n-categories, and this is precisely the definition of $(n-1)$-affineness for the Betti stack $(\Omega_* X)_{\mathrm{B}}$. Combining all the arguments of the last paragraphs, we obtain the proof of Lemma 4.2.11.

Proof of Theorem 4.2.9. First, using again [Ste21, Proposition 14.3.6], we reduce ourselves to check the monadicity at the level of the underlying category. Notice that the Seifert-Van Kampen Theorem implies that the sheafy side $\mathrm{LocSysCat}^{n}(X;\Bbbk)$ satisfies hyperdescent in X for any topological space X. When the side of presentable $\mathrm{LocSysCat}^{n-1}(X;\Bbbk)$-modules satisfies hyperdescent in X as well, we ccan conclude that the left adjoint to $\Gamma^{\mathrm{enh}}(X_{\mathrm{B}}, -)$ is an equivalence: indeed, it is be sufficient to choose a hypercover $U_\bullet \to X$ described in each degree by a disjoint union of contractible spaces, and then obtain that

$$\begin{aligned}
\mathrm{LocSysCat}^{n}(X;\Bbbk) &\simeq \lim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{LocSysCat}^{n}(U_m;\Bbbk) \\
&\simeq \lim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Mod}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \\
&\simeq \mathrm{Mod}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)},
\end{aligned}$$

using the fact that disjoint unions of contractible spaces are obviously n-affine.

We can reformulate questions concerning hyperdescent of local systems on X as questions concerning descent, thanks to the hypercompleteness of the topos $\mathrm{Fun}(X, \mathcal{S}) =: \mathrm{LocSys}(X)$: this is always hypercomplete without any assumptions on X ([Lur09, Example 7.2.1.9 and Corollary 7.2.1.12]). In particular, if

$$X \longmapsto \mathrm{Mod}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

satisfies descent with respect to any effective epimorphism $U_\bullet \to X$ inside $\mathcal{S}$, it automatically satisfies hyperdescent as well, and hence we can conclude that the global sections functor

$\Gamma(X_{\mathrm{B}}, -)\colon \mathrm{LocSysCat}^n(X;\Bbbk) \to \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is monadic. So we can conclude thanks to the n-affineness criterion for Betti stacks provided in Lemma 4.2.11. □

Remark 4.2.23. Theorem 4.2.8 yields, for any $n \geqslant 1$ and for any n-truncated space X, an equivalence of symmetric monoidal $(n+1)$-categories

$$(n+1)\mathbf{LocSysCat}^n(X;\Bbbk) \simeq (n+1)\mathbf{Mod}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)},$$

where $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$ is seen as a symmetric monoidal n-category via the natural (point-wise) symmetric monoidal structure. On the other hand, if X is connected we have a symmetric monoidal equivalence

$$(n+1)\mathbf{LocSysCat}^n(X;\Bbbk) \simeq (n+1)\mathbf{Mod}_{n\mathbf{LocSysCat}^n(\Omega_* X;\Bbbk)}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$$

in virtue of Theorem 3.2.24. Here, however, we consider the monoidal structure on $n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)$ provided by the Day convolution tensor product, which takes into account the $\mathbb{E}_1$-algebra structure of $\Omega_* X$. Combining these two equivalences, we obtain that for any connected and n-truncated space X there is an equivalence between presentable n-categorical modules for the standard monoidal structure on $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$ and n-categorical modules for the convolution monoidal structure on $n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)$. Unraveling all the constructions, we can see that the explicit equivalence is provided by sending a $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-module $n\mathscr{C}$ to the presentable n-category

$$n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)},$$

which inherits an $n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)$-action from the one on $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$. This can be seen as a topological analogue of the Morita equivalence for convolution categories of [BFN12, Theorem 1.3].

We end this section with another immediate consequence of Theorem 4.2.9.

Corollary 4.2.24. *Let X be a space, and assume $\Bbbk$ to be a semisimple commutative ring. If there exists a base point x such that $\pi_{n+1}(X,x)$ contains an element g either of infinite order, or such that the order of g is a unit in $\Bbbk$, then the Betti $\Bbbk$-stack X_{B} is not n-affine. In particular, if k is a field of characteristic* 0*, then if $\pi_{n+1}(X)$ does not vanish then X is not n-affine.*

Proof. In virtue of Theorem 4.2.9, we only need to check whether the based loop space $\Omega^{n-1}_* X$ is 1-affine. But its second homotopy group is isomorphic to $\pi_{n+1}(X,x)$, so the conclusion follows from Proposition 4.1.21. □

5. Categorified Koszul duality via coaffine stacks

This Section contains our main contribution to $\mathbb{E}_n$-Koszul duality, at least in the topological setting. Consider first the case $n=1$. If X is a pointed simply connected space with the same homotopy type of a CW complex of finite type, the algebras $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$

are $\mathbb{E}_1$-Koszul dual ([DGI06, §4.22]). When $\Bbbk$ is a field of characteristic 0, there is a kind of Morita equivalence relating modules over $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$ but the right statement is subtle: it requires to either restrict to appropriately bounded modules; or to change the notion of module we work with. In particular, if we work with *ind-coherent* modules, i.e. if we replace $\mathrm{LMod}_{\mathrm{C}^\bullet(X;\Bbbk)}$ with $\mathrm{IndCoh}_{\mathrm{C}^\bullet(X;\Bbbk)}$, we do obtain an equivalence

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{IndCoh}_{\mathrm{C}^\bullet(X;\Bbbk)}\,. \tag{5.1}$$

In this Section, we will explain how to define a category that should be viewed as the category of iterated "ind-coherent" modules over $\mathrm{C}^\bullet(X;\Bbbk)$. This will allow us to prove an n-categorical Morita equivalence statement relating $\mathrm{C}_\bullet(\Omega^n_* X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$, which in particular recovers (5.1) when $n = 1$. In fact we will not attempt to define directly a categorification of the notion of ind-coherent module. Rather, the key idea in our approach is using the theory *coaffine stacks* introduced in [Toë06] and further studied in [Lur11a]. We stress that our approach is new even in the classical case of $\mathbb{E}_1$-Koszul duality, although in that setting it is ultimately equivalent to (5.1).

We start by recalling some fundamental results in the theory of coaffine stacks defined over a field $\Bbbk$ of characteristic 0. We will mostly adopt the conventions from [Lur11a]. In particular, as in [Lur11a], we will call these objects *coaffine* rather than *affine* stacks, to stress the difference with affine schemes (which in turn are the spectra of *connective* $\Bbbk$-algebras). We will then use this theory to revisit the classical $\mathbb{E}_1$-Koszul duality between $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$, and in particular equivalence (5.1). We will conclude this Section proving our main result (Theorem 5.22), which provides an n-categorification of equivalence (5.1).

Definition 5.2. Let $n \geqslant 1$ be an integer.

1) We say that a $\Bbbk$-algebra A is *n-coconnective* if the structure morphism $\Bbbk \to A$ induces an isomorphism of abelian groups

$$\Bbbk \xrightarrow{\cong} \pi_0 A$$

and the homotopy groups $\pi_i A$ vanish for both $i \geqslant 1$ and $-n < i < 0$. If A is 1-coconnective, we shall simply say that A is *coconnective*.

2) A *coaffine stack* is a stack X which is equivalent to the stack

$$\mathrm{Map}_{\mathrm{CAlg}_\Bbbk}(A,-)\colon \mathrm{CAlg}_\Bbbk^{\geqslant 0} \longrightarrow \mathcal{S}$$

for some coconnective $\Bbbk$-algebra A. In this case, we shall say that X is the *cospectrum of A*, and we shall denote it as $\mathrm{cSpec}(A)$.

5.3. Coaffine stacks behave in a very similar way to affine schemes: for any stack $\mathcal{Y}$ defined over $\Bbbk$, giving a morphism $\mathcal{Y} \to \mathrm{cSpec}(A)$ is equivalent to giving a morphism of commutative $\Bbbk$-algebras $A \to \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ ([Lur11a, Theorem 4.4.1]). Moreover, any coaffine stack $\mathcal{X} \simeq \mathrm{cSpec}(A)$ can be realized as the left Kan extension of its restriction to *discrete*

$\Bbbk$-algebras (i.e., coaffine stacks are 0-*coconnective* in the sense of [GR17, Chapter 2, §1.3.4]). In particular, defining the category of classical affine schemes

$$\mathrm{Aff}_{\Bbbk}^{\mathrm{cl}} := \left(\mathrm{CAlg}_{\Bbbk}^{\mathrm{disc}}\right)^{\mathrm{op}}$$

as the opposite of the category of discrete $\Bbbk$-algebras, we have that for any coaffine stack $\mathcal{X} \simeq \mathrm{cSpec}(A)$ the inclusion

$$\left(\mathrm{Aff}_{\Bbbk}^{\mathrm{cl}}\right)_{/\mathcal{X}} \simeq \left(\left(\mathrm{CAlg}_{\Bbbk}^{\mathrm{disc}}\right)_{A/}\right)^{\mathrm{op}} \subseteq \left(\left(\mathrm{CAlg}_{\Bbbk}\right)_{A/}\right)^{\mathrm{op}} \simeq \left(\mathrm{Aff}_{\Bbbk}\right)_{/\mathcal{X}}$$

is cofinal, as stated in the proof of [GR17, Chapter 3, Lemma 1.2.2].

However, contrarily to the case of affine schemes, the category

$$\mathrm{QCoh}(\mathcal{X}) := \lim_{\substack{\mathrm{Spec}(R)\to\mathcal{X} \\ R\in\mathrm{CAlg}_{\Bbbk}^{\geqslant 0}}} \mathrm{Mod}_R$$

of quasi-coherent sheaves over a coaffine stack $\mathcal{X} \simeq \mathrm{cSpec}(A)$ does not recover the category of A-modules. Rather, we have the following result.

Proposition 5.4 ([Lur11a, Propositions 3.5.2 and 3.5.4, Remark 3.5.6])**.** *Let A be a coconnective $\Bbbk$-algebra, let $\mathcal{X} := \mathrm{cSpec}(A)$ be its corresponding coaffine stack. Let $\eta \in \mathcal{X}(\Bbbk)$ be any $\Bbbk$-point of $\mathcal{X}$.*

(1) There exists a right complete t-structure on Mod_A defined as follows.
- *The coconnective objects are detected via the forgetful functor*

$$\mathrm{oblv}_A \colon \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\Bbbk}.$$

- *The connective objects are those A-modules such that, for any morphism of commutative $\Bbbk$-algebras $A \to R$ with R connective, the R-module $M \otimes_A R$ is connective.*

(2) There exists a both left and right complete t-structure on $\mathrm{QCoh}(\mathcal{X})$, whose heart is equivalent to the ordinary abelian category of algebraic representations of the prounipotent group scheme $\pi_1(\mathcal{X}, \eta)$. Both connective and coconnective objects are detected via the pullback functor

$$\eta^* \colon \mathrm{QCoh}(\mathcal{X}) \longrightarrow \mathrm{QCoh}(\mathrm{Spec}(\Bbbk)) \simeq \mathrm{Mod}_{\Bbbk}.$$

(3) Let $F \colon \mathrm{Mod}_A \to \mathrm{QCoh}(\mathcal{X})$ be the natural symmetric monoidal pullback functor. Then F exhibits the t-structure of $\mathrm{QCoh}(\mathcal{X})$ as the left completion of the t-structure on Mod_A.

Definition 5.5 ([Lur11b, Definitions 3.0.1, 3.1.13 and 3.4.1])**.** Let A be an associative $\Bbbk$-algebra, let M be a left A-module.

1) We say that M is *locally small* if $\pi_k M$ is a finite dimensional $\Bbbk$-vector space for any integer k. We say that A is *locally small* if its underlying A-module is locally small.
2) We say that M is *small* if

$$\pi_\bullet M := \bigoplus_{k\in\mathbb{Z}} \pi_k M$$

is a finite dimensional $\Bbbk$-vector space.

3) We say that A is *small* if its underlying A-module is connective and small, and the structure morphism $\Bbbk \to A$ induces an isomorphism of discrete $\Bbbk$-algebras $\Bbbk \cong \pi_0 A/\mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\pi_0 A$.

Remark 5.6. Let A be an associative $\Bbbk$-algebra, and let $\mathrm{LMod}_A^{\mathrm{sm}}$ be the full sub-category of LMod_A spanned by small objects. We have a Cartesian diagram of categories

$$\begin{tikzcd} \mathrm{LMod}_A^{\mathrm{sm}} \arrow[r, "\mathrm{oblv}_A"] \arrow[d, hook] & \mathrm{Perf}_\Bbbk \arrow[d, hook] \\ \mathrm{LMod}_A \arrow[r, "\mathrm{oblv}_A"'] & \mathrm{Mod}_\Bbbk . \end{tikzcd}$$

Forgetting the A-module structure commutes with all limits and colimits, and $\mathrm{Perf}_\Bbbk$ is stable under finite limits and colimits inside $\mathrm{Mod}_\Bbbk$. It follows that $\mathrm{LMod}_A^{\mathrm{sm}}$ is a stable (but of course not complete or cocomplete) sub-category of LMod_A. In particular, its ind-completion

$$\mathrm{IndCoh}_A^{\mathrm{L}} := \mathrm{Ind}\left(\mathrm{LMod}_A^{\mathrm{sm}}\right)$$

is stable ([Lur17, Proposition 1.1.3.6]). Moreover, since $\mathrm{LMod}_A^{\mathrm{sm}}$ admits all finite coproducts, it follows that $\mathrm{IndCoh}_A^{\mathrm{L}}$ admits *all* coproducts (since they are realized as filtered colimits of finite coproducts).

Definition 5.7. Let A be an associative $\Bbbk$-algebra. The category $\mathrm{IndCoh}_A^{\mathrm{L}}$ is the *category of left ind-coherent modules on A.*

Warning 5.8. The notation can be misleading: in [GR17], *ind-coherent sheaves* over an affine scheme are interpreted as bounded modules with coherent homology. Rather, our definition of ind-coherent modules matches the one in [Lur11b, Definition 3.4.4]. Yet, if A is a discrete local Artinian ring, or a small $\Bbbk$-algebra in the sense of Definition 5.5.(3), then the two notions coincide.

In the following, we revisit $\mathbb{E}_1$-Koszul duality for associative algebras and its formulation in terms of correspondences between categories of ind-coherent and quasi-coherent modules. Recall that, over any field $\Bbbk$, an augmented associative $\Bbbk$-algebra A admits a $\mathbb{E}_1$-Koszul dual $A^!$ if and only if there exists a morphism

$$\mu\colon A \otimes A^! \longrightarrow \Bbbk$$

which exhibits $A^!$ as the classifying object $\underline{\mathrm{Map}}_A(\Bbbk, \Bbbk)$ of morphisms of left A-modules from $\Bbbk$ to itself, see [Lur11b, Remark 3.1.12].

Proposition 5.9. *Let A be an augmented associative $\Bbbk$-algebra such that the augmentation of A induces an isomorphism*

$$\pi_0 A \xrightarrow{\cong} \Bbbk.$$

Suppose that the Koszul dual $A^!$ is locally small as a $\Bbbk$-module. Then there is an equivalence of categories

$$\mathrm{IndCoh}^{\mathrm{L}}_A \simeq \mathrm{RMod}_{A^!}. \tag{5.10}$$

Proof. Using [Lur11b, Remark 3.4.2], we know that $\mathrm{LMod}^{\mathrm{sm}}_A$ is the thick sub-category spanned by $\Bbbk$ inside LMod_A – i.e., it is the smallest stable category sitting inside LMod_A containing $\Bbbk$ and closed under retracts. In particular, the Koszul duality functor for modules

$$\mathrm{LMod}^{\mathrm{op}}_A \longrightarrow \mathrm{LMod}_{A^!}$$

restricts to an equivalence between $\mathrm{LMod}^{\mathrm{sm}}_A$ and $\mathrm{Perf}^{\mathrm{op}}_{A^!}$ ([Lur11b, Proposition 3.5.6]), so applying the $A^!$-linear duality self-functor we have

$$\mathrm{LMod}^{\mathrm{sm}}_A \xrightarrow{\simeq} \mathrm{Perf}_{A^!}$$

hence an equivalence on their ind-completions. □

Remark 5.11. At first sight, [Lur11b, Proposition 3.5.2] would seem to imply the need for some smallness assumption on $A^!$ in Proposition 5.9. Actually, this is not the case: the smallness is only needed in order to have an equivalence of *functors* from $\mathrm{Alg}^{\mathrm{sm}}_{\Bbbk}$ to $\widehat{\mathrm{Cat}}_{(\infty,1)}$ between $\mathrm{IndCoh}^{\mathrm{L}}_{(-)}$ and $\mathrm{RMod}_{(-)^!}$. Of course, if we do not assume our algebras to be small the tensor product does not preserve small modules, so the functor $\mathrm{IndCoh}^{\mathrm{L}}_{(-)}$ is not even well-defined; yet, the *point-wise* equivalence (5.10) still applies under our, milder, assumptions on A.

We shall now equip $\mathrm{IndCoh}^{\mathrm{L}}_{A^!}$ with a t-structure using the following general recipe.

Lemma 5.12 ([GR17, Chapter IV, Lemma 1.2.4])**.** *Let $\mathscr{C}$ be a (non cocomplete) stable category, endowed with a t-structure. Then $\mathrm{Ind}(\mathscr{C})$ carries a unique t-structure which is compatible with filtered colimits (i.e., such that truncation functors commute with filtered colimits), and for which the tautological inclusion $\mathscr{C} \subseteq \mathrm{Ind}(\mathscr{C})$ is t-exact. Moreover:*

1) *The sub-categories $\mathrm{Ind}(\mathscr{C})_{\geqslant 0}$ and $\mathrm{Ind}(\mathscr{C})_{\leqslant 0}$ are compactly generated by $\mathscr{C}_{\geqslant 0}$ and $\mathscr{C}_{\leqslant 0}$, respectively.*
2) *Given any other stable category $\mathscr{D}$ equipped with a t-structure compatible with filtered colimits, any functor $F\colon \mathrm{Ind}(\mathscr{C}) \to \mathscr{D}$ is t-exact if and only if $F|_{\mathscr{C}}$ is t-exact.*

Proposition 5.13. *Let A be a connective associative $\Bbbk$-algebra. Then $\mathrm{IndCoh}^{\mathrm{L}}_A$ admits a right complete t-structure. Moreover, if A is locally small the t-exact functor*

$$\Phi_A\colon \mathrm{IndCoh}^{\mathrm{L}}_A \longrightarrow \mathrm{LMod}_A$$

induced by the natural inclusion $\mathrm{LMod}^{\mathrm{sm}}_A \subseteq \mathrm{LMod}_A$ exhibits LMod_A as the left completion of the t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_A$.

Warning 5.14. If one assumes A to be *small* rather than only locally small, Proposition 5.13 boils down to [Lur11b, Proposition 3.4.18]. One could be confused by the fact that there the

t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_A$ fails to be *right* complete, but this is easily explained: in [Lur11b] the functor Φ_A is replaced by another functor Ψ_A, which is more compatible with base change. The functor Ψ_A is closely related to Φ_A but involves A-linear duality as well; in particular, it swaps connective and coconnective objects, and this explains why the t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_A$ described in [Lur11b, Definition 3.4.16 and Remark 3.4.17] is left but not right complete. Rather, our definition of the t-structure in Proposition 5.13 resembles the t-structure on ind-coherent sheaves over Noetherian schemes defined in [GR17, Proposition 1.2.2].

Proof of Proposition 5.13. For any connective associative $\Bbbk$-algebra the restriction of the ordinary t-structure on the category of left A-modules yields a t-structure on $\mathrm{LMod}^{\mathrm{sm}}_A$: this boils down to the fact that this is true for $\mathrm{Perf}_{\Bbbk}$, and that forgetting the A-module structure is a conservative operation which preserves all limits and colimits. Thus, the existence of the t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_A$ follows from Lemma 5.12.

We can easily see that such t-structure is right complete as follows. Since $\mathrm{IndCoh}^{\mathrm{L}}_A$ is stable and admits uncountable coproducts, and coconnective objects are stable under uncountable coproducts (because this is true in LMod_A), we can use the (dual of the) criterion [Lur11b, Proposition 1.2.1.19] for the right completeness of t-structures on stable categories. Indeed, we have that

$$\left(\mathrm{IndCoh}^{\mathrm{L}}_A\right)_{\leqslant -\infty} := \bigcap_{n\geqslant 0}\left(\mathrm{IndCoh}^{\mathrm{L}}_A\right)_{\leqslant -n} \simeq \bigcap_{n\geqslant 0}(\mathrm{LMod}_A)_{\leqslant -n} \simeq 0.$$

Moreover, the functor $\Phi_A\colon \mathrm{IndCoh}^{\mathrm{L}}_A \to \mathrm{LMod}_A$ is t-exact: this is an obvious consequence of Lemma 5.12.(2) because the inclusion $\mathrm{LMod}^{\mathrm{sm}}_A \subseteq \mathrm{LMod}_A$ is t-exact.

To prove the claim about the left completion, we simply need to check that for any integer n the functor Φ_A induces an equivalence of categories

$$\left(\mathrm{IndCoh}^{\mathrm{L}}_A\right)_{\leqslant n} \xrightarrow{\simeq} (\mathrm{LMod}_A)_{\leqslant n}\,. \tag{5.15}$$

Indeed, the equivalence (5.15) would yield an equivalence between the categories of eventually coconnective objects

$$\mathrm{IndCoh}^{\mathrm{L},+}_A := \bigcup_{n\in\mathbb{Z}}\left(\mathrm{IndCoh}^{\mathrm{L}}_A\right)_{\leqslant n} \simeq \bigcup_{n\in\mathbb{Z}}(\mathrm{LMod}_A)_{\leqslant n} =: \mathrm{LMod}^+_A.$$

Restriction to eventually coconnective objects does not alter the left completion of a t-structure ([Lur17, Remark 1.2.1.18]), so this implies that the left completion of $\mathrm{IndCoh}^{\mathrm{L}}_A$ and LMod_A are equivalent. But the canonical t-structure on LMod_A is left complete ([Lur17, Proposition 7.1.1.13]), so we conclude that it has to be the left completion of the t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_A$ as well.

Since the functor Φ_A is exact, we can reduce ourselves to consider the case $n = 0$. We first prove that any perfect and coconnective left A-module M is obtained as a colimit of small coconnective left A-modules. Write any such M as a colimit

$$M \simeq \operatorname*{colim}_{i\in I} A^{\oplus r_i}[n_i]$$

over some diagram I. Notice that, even if M is perfect, the diagram cannot be assumed to be finite because M could be obtained from A via shifts, finite direct sums or *retracts*, and the latter are only realized as *countably infinite* colimits ([Lur09, Section 4.4.5]). Since M is coconnective, we have

$$M \simeq \tau_{\leqslant 0} M \simeq \tau_{\leqslant 0}\left(\operatorname*{colim}_{i\in I} A^{\oplus r_i}[n_i]\right) \simeq \operatorname*{colim}_{i\in I} \tau_{\leqslant 0}\left(A^{\oplus r_i}[n_i]\right),$$

where in the last equivalence we used the fact the truncation functor $\tau_{\leqslant 0}$ is a left adjoint. Since A is locally finite and connective, each $\tau_{\leqslant 0}(A^{\oplus r_i}[n_i])$ is a small A-module. Moreover, by the very same definition of the t-structure on $\mathrm{LMod}_A^{\mathrm{sm}}$, we conclude that $\tau_{\leqslant 0}(A^{\oplus r_i}[n_i])$ is coconnective inside $\mathrm{LMod}_A^{\mathrm{sm}}$. Next, we prove that the functor Φ_A is fully faithful when restricted to $\left(\mathrm{IndCoh}_A^{\mathrm{L}}\right)_{\leqslant 0}$. We will actually prove that for *any* small left A-module M (seen trivially as a left ind-coherent A-module) and for any coconnective left ind-coherent A-module N the map of spaces

$$\mathrm{Map}_{\mathrm{IndCoh}_A^{\mathrm{L}}}(M, N) \longrightarrow \mathrm{Map}_{\mathrm{LMod}_A}(\Phi_A(M), \Phi_A(N))$$

is an equivalence. Writing N as a filtered colimit $\operatorname{colim}_i N_i$, with each N_i small and coconnective, we have that

$$\begin{aligned}\mathrm{Map}_{\mathrm{IndCoh}_A^{\mathrm{L}}}(M, N) &\simeq \mathrm{Map}_{\mathrm{IndCoh}_A^{\mathrm{L}}}\left(M, \operatorname*{colim}_i N_i\right)\\ &\simeq \operatorname*{colim}_i \mathrm{Map}_{\mathrm{IndCoh}_A^{\mathrm{L}}}(M, N_i) \simeq \operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A^{\mathrm{sm}}}(M, N_i),\end{aligned}$$

because each small A-module is compact in $\mathrm{IndCoh}_A^{\mathrm{L}}$. The functor Φ_A sends M and N to their actual colimits in LMod_A, so Φ_A is fully faithful on coconnective objects if

$$\operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A}(M, N_i) \longrightarrow \mathrm{Map}_{\mathrm{LMod}_A}\left(M, \operatorname*{colim}_i N_i\right)$$

is an equivalence. As we already observed, $\mathrm{LMod}_A^{\mathrm{sm}}$ is the thick stable sub-category of LMod_A spanned by $\Bbbk$: so it sufficient to write M as a retract of shifts and direct sums of $\Bbbk$

$$M \simeq \operatorname*{colim}_{j\in J} \Bbbk^{\oplus r_j}[n_j],$$

and observe that the augmentation $A \to \Bbbk$ produces a map

$$f\colon \operatorname*{colim}_j A^{\oplus r_j}[n_j] \longrightarrow M$$

whose fiber is at least 1-connective. In particular, recalling that each N_i is coconnective, we obtain

$$\begin{aligned}
\operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A}(M, N_i) &\simeq \operatorname*{colim}_i \mathrm{fib}\left(\mathrm{Map}_{\mathrm{LMod}_A}\left(\operatorname*{colim}_j A^{\oplus r_j}[n_j], N_i\right) \longrightarrow \mathrm{Map}_{\mathrm{LMod}_A}(\mathrm{fib}(f), N_i)\right) \\
&\simeq \operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A}\left(\operatorname*{colim}_j A^{\oplus r_j}[n_j], N_i\right) \\
&\simeq \operatorname*{colim}_i \lim_j \mathrm{Map}_{\mathrm{LMod}_A}\left(\operatorname*{colim}_j A^{\oplus r_j}[n_j], N_i\right).
\end{aligned}$$

Using the fact that the diagram J is a limit over the category Idem^+, and such limits are universal because they are colimits as well, we obtain that

$$\begin{aligned}
\operatorname*{colim}_i \lim_j \mathrm{Map}_{\mathrm{LMod}_A}\left(\operatorname*{colim}_j A^{\oplus r_j}[n_j], N_i\right) &\simeq \lim_j \operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A}\left(\operatorname*{colim}_j A^{\oplus r_j}[n_j], N_i\right) \\
&\simeq \lim_j \mathrm{Map}_{\mathrm{LMod}_A}\left(A^{\oplus r_j}[n_j], N\right) \simeq \mathrm{Map}_{\mathrm{LMod}_A}(M, N),
\end{aligned}$$

and this concludes the proof. □

Remark 5.16. The fact that LMod_A is the left completion of $\mathrm{IndCoh}^{\mathrm{L}}_A$ implies that such t-structure is left complete if and only if $\mathrm{IndCoh}^{\mathrm{L}}_A$ is equivalent to LMod_A. This cannot be true if small and compact objects are not the same – which is *never* the case, unless A is discrete and finite as a $\Bbbk$-module. Indeed, the equality between the smallness and compactness conditions implies that the perfect left A-module A has to be small, hence eventually coconnective; but if A is not discrete, it is easy to see via a homological computation that the small left A-module $\pi_0 A$ does not admit a finite resolution of semi-free A-modules, hence it cannot be perfect.

The previous discussion allows us to reformulate (5.10) in terms of algebraic geometry. The key ingredient is the concept of cospectrum of a coconnective algebra.

Proposition 5.17. *Let A be a coconnective and locally finite commutative $\Bbbk$-algebra, which as a mere associative algebra admits a $\mathbb{E}_1$-Koszul dual $A^!$. Then we have an equivalence of categories*

$$\mathrm{LMod}_{A^!} \simeq \mathrm{QCoh}(\mathrm{cSpec}(A)).$$

Proof. The augmented associative algebra $A^!$ is connective ([Lur11b, Theorem 3.1.14]); if A is locally finite, one can see that $A^! \simeq \underline{\mathrm{Map}}_A(\Bbbk, \Bbbk)$ is locally finite as well. In particular, Propositions 5.4 and 5.13 provides us with the following characters.

(1) The t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_{A^!}$.
(2) The t-structure on $\mathrm{LMod}_{A^!}$, which is the left completion of the one on $\mathrm{IndCoh}^{\mathrm{L}}_{A^!}$.
(3) The t-structure on Mod_A.
(4) The t-structure on $\mathrm{QCoh}(\mathrm{cSpec}(A))$, which is the left completion of the one on Mod_A.

Since A is coconnective and locally finite, we are in the setting of Proposition 5.9 and we obtain the equivalence (5.10). If such equivalence is t-exact then we can deduce our statement from the universal property of the left completion. Again, Lemma 5.12 implies that we just need to check whether the restriction of this equivalence to the full sub-category $\mathrm{LMod}^{\mathrm{sm}}$ is t-exact.

(1) First, notice that the duality functor $\mathrm{IndCoh}^{\mathrm{L}}_{A^!} \to \mathrm{Mod}_A$ preserves coconnective objects. Indeed, let $M^!$ be a coconnective small left $A^!$-module. Its image inside Mod_A is the module
$$M := \underline{\mathrm{Map}}_{A^!}\left(\Bbbk, M^!\right),$$
and this mapping $\Bbbk$-module is immediately seen to be coconnective since it is a mapping spectrum from a connective object to a coconnective one.

(2) The duality functor also preserves connective objects. We can see it as follows: notice that, inside Mod_A, a module M is connective precisely if for any (or, equivalently, *one*) map of $\Bbbk$-algebras $A \to R$ where R is connective, the R-module $R \otimes_A M$ is connective ([Lur11a, Proposition 4.5.4]). So we can test whether for a connective small left $A^!$-module $M^!$ the $\Bbbk$-module
$$M \otimes_A \Bbbk := \underline{\mathrm{Map}}_{A^!}\left(\Bbbk, M^!\right) \otimes_A \Bbbk$$
is connective. But this is just the underlying $\Bbbk$-module of the $A^!$-module $M^!$, since the inverse to
$$\underline{\mathrm{Map}}_{A^!}(\Bbbk, -)\big|_{\mathrm{LMod}^{\mathrm{sm}}_{A^!}} : \mathrm{LMod}^{\mathrm{sm}}_{A^!} \longrightarrow \mathrm{Perf}_A$$
is realized precisely by its left adjoint $- \otimes_A \Bbbk$. So our claim follows from the fact that $M^!$ was assumed to be connective in the first place.

□

We shall now apply these results and construction to a certain class of spaces to which Koszul duality applies. We fix the following definition.

Definition 5.18. Let $n \geqslant 0$ be an integer. A space X is *n-Koszul (over a field $\Bbbk$ of characteristic* 0 if the following conditions hold.

1) The space X is cohomologically of finite type over $\Bbbk$: the commutative algebra $\mathrm{C}^\bullet(X;\Bbbk)$ is locally small in the sense of Definition 5.5.(1).
2) The space X is $(n-1)$-connected.
3) The n-th homotopy group $\pi_n(X)$ is finite.
4) The space X is nilpotent: for any choice of a base point x, the fundamental group $\pi_1(X,x)$ is a nilpotent group which acts nilpotently on every higher homotopy group $\pi_k(X)$ for $k \geqslant 2$.

Remark 5.19.

(1) The property of being n-Koszul is obviously closed under finite products, essentially because of the Künneth formula.

(2) When $n \geqslant 1$, then an n-Koszul space is in particular k-Koszul for all $0 \leqslant k \leqslant n$.

(3) When $n \geqslant 1$, n-connectedness implies simply connectedness. Therefore, in order to check whether an n-connected space X is n-Koszul over $\Bbbk$ for some $n \geqslant 1$ it is sufficient to check whether it is cohomologically of finite type over $\Bbbk$.

(4) When X is simply connected, then the homotopy groups $\pi_k(X)$ are finitely generated for all $k \in \mathbb{N}$ if and only if the homology groups $\mathrm{H}_k(X;\mathbb{Z})$ are finitely generated for all $k \in \mathbb{N}$ as well ([MP12, Theorem 4.5.4]). Flatness of fields of characteristic 0 over $\mathbb{Z}$ and the Künneth formula hence imply that $\mathrm{H}_k(X;\Bbbk)$ is finitely generated over $\Bbbk$ for all $k \in \mathbb{N}$. Finally, the universal coefficients theorem implies that $\mathrm{H}^k(X;\Bbbk)$ is finitely generated for all $k \in \mathbb{N}$. It follows that for all integers $n \geqslant 1$, any n-connected space with the same homotopy type as a CW complex of finite type is n-Koszul.

(5) Since the multiplication $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is an isomorphism, the discussion above carries on *verbatim* also to the case of simply connected spaces which are only of *rational* finite type.

In other words: whenever $n \geqslant 1$, if X is a pointed n-connected space with the same homotopy type as a CW complex which is either of finite type, or of rational finite type, then X is k-Koszul for all $0 \leqslant k \leqslant n$.

Whenever X is a pointed and $(n-1)$-connected space, the $\mathbb{E}_n$-Kozul dual of the $\mathbb{E}_n$-$\Bbbk$-algebra $\mathrm{C}_\bullet(\Omega^n_* X;\Bbbk)$ is computed by the underlying $\mathbb{E}_n$-algebra of the commutative algebra of $\Bbbk$-cochains on X, i.e.,

$$\mathrm{C}^\bullet(X;\Bbbk) \simeq \mathrm{C}_\bullet(\Omega^n_* X;\Bbbk)^{!_{\mathbb{E}_n}}.$$

The n-Koszul hypothesis is a sufficient condition for the reciprocal duality. That is, when X is pointed and n-Koszul we also have the equivalence

$$\mathrm{C}_\bullet(\Omega^n_* X;\Bbbk) \simeq \mathrm{C}^\bullet(X;\Bbbk)^{!_{\mathbb{E}_n}},$$

compare with [AF15, Proposition 5.3, ArXiv v6].

We can apply the machinery of Proposition 5.17 to deduce the following.

Corollary 5.20. *Let X be a pointed* 1*-Koszul space. Then we have an equivalence*

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))).$$

Remark 5.21. If X is a pointed 1-Koszul space then the equivalence of Corollary 5.20 arises geometrically as follows. Corollary 2.12 implies that $\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{LocSys}(X;\Bbbk)$, which is the category of quasi-coherent sheaves over the Betti stack X_{B} (as already observed in Paragraph 4.1.9). Since $\Gamma(X_{\mathrm{B}}, \mathcal{O}_{X_{\mathrm{B}}}) \simeq \mathrm{C}^\bullet(X;\Bbbk)$, the identity map of $\mathrm{C}^\bullet(X;\Bbbk)$ induces an affinization map

$$\mathrm{aff}_X \colon X_{\mathrm{B}} \longrightarrow \mathrm{C}^\bullet(X;\Bbbk).$$

The equivalence of Corollary 5.20 is then realized by pulling back and pushing forward along aff_X. Indeed, the pullback functor aff_X^* is a functor between stable categories equipped with both left and right complete t-structures which is strongly monoidal and right t-exact (i.e., it preserves connective objects). Therefore, using [Lur11b, Corollary 4.6.18], we can deduce that it is uniquely determined by the symmetric monoidal and right t-exact functor

$$\widetilde{\mathrm{aff}}_X^* \colon \mathrm{Mod}_{\mathrm{C}^\bullet(X;\Bbbk)} \longrightarrow \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}$$

which is obtained by pre-composing aff_X^* with the natural left completion functor $\mathrm{Mod}_{\mathrm{C}^\bullet(X;\Bbbk)} \to \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$.

So, it will suffice to understand the behaviour of $\widetilde{\mathrm{aff}}^*$. Let $\eta\colon \{*\} \to X$ be the chosen base point in X, and let $\eta_{\mathrm{B}}\colon \mathrm{Spec}(\Bbbk) \to X_{\mathrm{B}}$ be its image under the Betti stack functor. Pullback along η_{B} yields a forgetful functor $\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \to \mathrm{Mod}_{\Bbbk}$, and using again [Lur11b, Corollary 4.6.18] we can see that the symmetric monoidal and right t-exact functor

$$\eta_{\mathrm{B}}^* \circ \mathrm{aff}_X^* \colon \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \longrightarrow \mathrm{Mod}_{\Bbbk}$$

uniquely corresponds to the natural base change functor $\mathrm{Mod}_{\mathrm{C}^\bullet(X;\Bbbk)} \to \mathrm{Mod}_{\Bbbk}$ along the coaugmentation $\mathrm{C}^\bullet(X;\Bbbk) \to \Bbbk$ induced at the level of $\Bbbk$-cochains by η. In particular, for any $\mathrm{C}^\bullet(X;\Bbbk)$-module M the underlying $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$-module of $\widetilde{\mathrm{aff}}_X^*(M)$ is equivalent to $M \otimes_{\mathrm{C}^\bullet(X;\Bbbk)} \Bbbk$. This is just the left adjoint of the Koszul duality functor which induces the equivalence of Proposition 5.17.

Corollary 5.20 allows us to revisit the classical Koszul duality for modules (5.10), in a substantially equivalent formulation. However, this point of view has a considerable advantage. Namely, while the concept of n-categorical ind-coherent modules is somewhat mysterious and it is far from clear how to define it directly, quasi-coherent sheaves on coaffine stacks can be categorified in a natural way: that is, we can consider quasi-coherent sheaves of n-categories over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ (Definition 4.2.1). Hence, $\mathbb{E}_n$-Koszul duality for categorified modules over $\mathbb{E}_n$-Koszul dual algebras in the topological setting can be straightforwardly generalized as follows.

Theorem 5.22. *Let $n \geqslant 1$ be an integer, and let X be a pointed $(n+1)$-Koszul space over a field $\Bbbk$ of characteristic 0 whose homotopy groups $\pi_q(X)$ are finitely generated for each $q \geqslant 0$. Then the natural $(n+1)$-functor*

$$\mathrm{aff}_X^* \colon (n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \longrightarrow (n+1)\mathbf{LocSysCat}^n(X;\Bbbk)$$

is an equivalence of $(n+1)$-categories.

Remark 5.23. If we set

$$\mathrm{ShvCat}^0(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \coloneqq \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$$

and

$$\mathrm{LocSysCat}^0(X;\Bbbk) \coloneqq \mathrm{LocSys}(X;\Bbbk),$$

then we can extend Theorem 5.22 also to $n=0$: indeed, this reduces to the combination of Corollary 5.20 with Remark 5.21.

Remark 5.24. We stress that Theorem 5.22 does provide a generalization of the usual $\mathbb{E}_1$-Koszul duality equivalence between categories of modules. Let us briefly comment on the two characters appearing in the statement: the $(n+1)$-category $(n+1)\mathbf{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ provides the natural higher categorification of the concept of the category of quasi-coherent sheaves over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. On the other hand, if X is $(n+1)$-Koszul (hence, n-connected), Theorem 3.2.24 provides an equivalence

$$(n+1)\mathbf{LocSysCat}^n(X;\Bbbk) \simeq (n+1)\mathbf{Lin}_{\mathrm{C}_\bullet(\Omega_*^{n+1}X;\Bbbk)}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}.$$

Thus, for X an $(n+1)$-Koszul space, Theorem 5.22 does relate (the categorification of) quasi-coherent sheaves over the coaffine stack $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$, and (the categorification of) left modules over the $\mathbb{E}_n$-algebra $\mathrm{C}_\bullet(\Omega_*X;\Bbbk)$.

5.25. Even if the proof of Theorem 5.22 is essentially carried out via an inductive argument, proving the $n=1$ case is strikingly more technically-demanding than the $n \geqslant 2$ case. Indeed, for $n \geqslant 2$ the proof is somewhat formal and essentially depends on the general behaviour of pullbacks and pushforwards of sheaves of $(n+1)$-categories along maps of prestacks (Remark 4.2.2); however, the case $n=1$ requires some more *ad hoc* arguments. Therefore, we first study this latter case in full detail – which provides the base case for the induction, and then prove the theorem for an arbitrary $n \geqslant 2$.

We start by proving some helpful results concerning Koszul spaces and their cochain $\Bbbk$-algebras, which will be used extensively in the proof of Theorem 5.22. Lemma 5.26 allows us to set up the inductive argument, while Lemma 5.27 and Proposition 5.28 are pivotal in relating the based loop stack of a coaffine stack $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ and the coaffine stack $\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$.

Lemma 5.26. *Let $n \geqslant 1$ be an integer, and let X be a pointed $(n+1)$-Koszul space over a field $\Bbbk$ of characteristic 0 such that the homotopy groups $\pi_q(X)$ are finitely generated for all integers $q \geqslant 0$. Then for any $1 \leqslant k < n$ the iterated based loop space Ω_*^kX is $(n-k)$-Koszul over $\Bbbk$.*

Proof. Fix $1 \leqslant k < n$. Obviously, if X is $(n-1)$-connected with finite $\pi_n(X)$ then Ω_*^kX is $(n-k-1)$-connected with finite $\pi_{n-k}(\Omega_*^kX) \cong \pi_n(X)$. Moreover, all connected based loop spaces are connected H-spaces, hence they are nilpotent ([MP12, Pag. 49]). The only non-trivial part of the statement is proving that Ω_*^kX inherits the condition of being of cohomological $\Bbbk$-finite type (Definition 5.18.(1)). Recall that for any field $\Bbbk$ of characteristic 0, and for any simply connected space X whose $\Bbbk$-algebra of $\Bbbk$-cochains is locally small (in the sense of Definition 5.5.(1)), we have an isomorphism of $\Bbbk$-algebras

$$\mathrm{H}_\bullet(\Omega_*X;\Bbbk) \cong \mathrm{U}(\pi_\bullet(\Omega_*X) \otimes_{\mathbb{Z}} \Bbbk)$$

between the graded $\Bbbk$-algebra of $\Bbbk$-chains on Ω_*X and the graded universal enveloping $\Bbbk$-algebra of the graded Lie algebra $\pi_\bullet(\Omega_*X) \otimes_{\mathbb{Z}} \Bbbk$ endowed with the Whitehead bracket (see for example [FHT01, Theorem 16.13]). Since X is n-connected with finite $\pi_{n+1}(X)$, the underlying graded $\Bbbk$-vector space of

$$\pi_\bullet(\Omega_* X)\otimes_{\mathbb{Z}}\Bbbk \cong \pi_{\bullet+1}(X)\otimes_{\mathbb{Z}}\Bbbk$$

has non-trivial generators lying only in degrees $q \geqslant n$. It follows that the homology $\mathrm{H}_0(\Omega_* X;\Bbbk)$ is isomorphic to $\Bbbk$, the homology $\mathrm{H}_q(\Omega_* X;\Bbbk)$ is trivial for $1 \leqslant q \leqslant n-1$, and for $q \geqslant n$ we have that

$$\dim_{\Bbbk} \mathrm{H}_q(\Omega_* X;\Bbbk) \leqslant \dim_{\Bbbk}\left(\bigoplus_{p\geqslant 0}\left(\bigoplus_{i_1+\cdots+i_p=q}\pi_{i_1}(\Omega_* X)\otimes_{\mathbb{Z}}\cdots\otimes_{\mathbb{Z}}\pi_{i_p}(\Omega_* X)\otimes_{\mathbb{Z}}\Bbbk\right)\right).$$

The right hand side is obviously finite, because $\pi_\bullet(\Omega_* X)\otimes_{\mathbb{Z}}\Bbbk$ is bounded below and finitely generated in each degree in the first place. It follows that the algebra $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ is locally small, hence the commutative algebra $\mathrm{C}^\bullet(\Omega_* X;\Bbbk)$ is locally small as well because of the universal coefficients theorem. Since $\Omega_* X$ is now an n-Koszul space whose homotopy groups are once again finitely generated for all integers $q \geqslant 0$, the claim for the iterated based loop space follows by induction. □

Lemma 5.27. *Let A be a coconnective $\Bbbk$-algebra, and let $A \to R$ and $A \to S$ be two A-algebras. Assume that R and S are coconnective as $\Bbbk$-algebras. Then we have a natural equivalences of stacks*

$$\mathrm{cSpec}(R\otimes_A S) \simeq \mathrm{cSpec}(R)\times_{\mathrm{cSpec}(A)}\mathrm{cSpec}(S).$$

Proof. First, notice that, if $\Bbbk$ is a field, then coconnective $\Bbbk$-algebras are stable under tensor products ([Lur11a, Proposition 4.5.4.(6)]). Moreover, [Lur17, Proposition 7.2.1.19] yields that $\pi_0(R\otimes_A S)\cong \Bbbk$, so $R\otimes_A S$ is itself a coconnective $\Bbbk$-algebra and it does make sense to consider the associated coaffine stack over $\Bbbk$.

We first observe that the functor cSpec sends tensor products of coconnective $\Bbbk$-algebras to products of stacks. Indeed, for any stack $\mathcal{X}$ and for any couple of coconnective $\Bbbk$-algebras A_1 and A_2, we have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{St}_{\Bbbk}}(\mathcal{X}, \mathrm{cSpec}(A_1)\times \mathrm{cSpec}(A_2)) &\simeq \mathrm{Map}_{\mathrm{St}_{\Bbbk}}(\mathcal{X}, \mathrm{cSpec}(A_1))\times \mathrm{Map}_{\mathrm{St}_{\Bbbk}}(\mathcal{X}, \mathrm{cSpec}(A_2))\\
&\simeq \mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(A_1, \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))\times \mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(A_2, \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))\\
&\simeq \mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(A_1\otimes_{\Bbbk} A_2, \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))\\
&\simeq \mathrm{Map}_{\mathrm{St}_{\Bbbk}}(\mathcal{X}, \mathrm{cSpec}(A_1\otimes_{\Bbbk} A_2)),
\end{aligned}$$

where we used that the tensor product is the coproduct in the category of $\Bbbk$-commutative algebras. So, let A be a coconnective $\Bbbk$-algebra, and let R and S be A-algebras which are coconnective as $\Bbbk$-algebras. Again, for any stack $\mathcal{X}$ we have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{St}_\Bbbk}(\mathcal{X}, \mathrm{cSpec}(R\otimes_A S)) &\simeq \mathrm{Map}_{\mathrm{CAlg}_\Bbbk}(R\otimes_A S, \Gamma(\mathcal{X},\mathcal{O}_\mathcal{X}))\\
&\simeq \mathrm{Map}_{\mathrm{CAlg}_\Bbbk}\Big(\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} R\otimes_\Bbbk A^{\otimes n}\otimes_\Bbbk S, \Gamma(\mathcal{X},\mathcal{O}_\mathcal{X})\Big)\\
&\simeq \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Map}_{\mathrm{CAlg}_\Bbbk}\big(R\otimes_\Bbbk A^{\otimes n}\otimes_\Bbbk S, \Gamma(\mathcal{X},\mathcal{O}_\mathcal{X})\big)\\
&\simeq \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Map}_{\mathrm{St}_\Bbbk}\big(\mathcal{X}, \mathrm{Spec}(R)\times\mathrm{cSpec}(A)^{\times n}\times\mathrm{Spec}(S)\big)\\
&\simeq \mathrm{Map}_{\mathrm{St}_\Bbbk}\Big(\mathcal{X}, \lim_{[n]\in\Delta^{\mathrm{op}}}\mathrm{Spec}(R)\times\mathrm{cSpec}(A)^{\times n}\times\mathrm{Spec}(S)\Big)\\
&\simeq \mathrm{Map}_{\mathrm{St}_\Bbbk}\big(\mathcal{X}, \mathrm{Spec}(R)\times_{\mathrm{cSpec}(A)}\mathrm{Spec}(S)\big).
\end{aligned}$$

□

Proposition 5.28. *Let X be a pointed 2-Koszul space over a field $\Bbbk$ of characteristic 0. Then there is an equivalence of stacks*

$$\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk)) \simeq \mathrm{Spec}(\Bbbk)\times_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}\mathrm{Spec}(\Bbbk).$$

Proof. Since X is 2-Koszul, it is simply connected and Lemma 5.26 implies that the algebra of $\Bbbk$-cochains of $\Omega_* X$ is of finite type over $\Bbbk$. So, we can apply the Eilenberg-Moore theorem (see for example [Lur11c, Corollary 1.1.10]) and deduce the existence of a canonical equivalence

$$\Bbbk\otimes_{\mathrm{C}^\bullet(X;\Bbbk)}\Bbbk \simeq \mathrm{C}^\bullet(\Omega_* X;\Bbbk).$$

Applying the cospectrum functor and using Lemma 5.27 we deduce our claim. □

The following is the key lemma for the proof of Theorem 5.22 when $n=1$.

Lemma 5.29. *Let X be a pointed 1-Koszul space over a field $\Bbbk$ of characteristic 0. Then the functor*

$$\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}\colon \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\Big(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big) \longrightarrow \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$$

is fully faithful.

Proof. Since X is 1-Koszul, Corollary 5.20 applies: so we obtain a commutative diagram of categories

$$\begin{tikzcd}
\mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\Big(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}"'] & \mathrm{Lin}_{\mathrm{LocSys}(X;\Bbbk)}\Big(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big). \arrow[d, "\mathrm{Loc}_{X_{\mathrm{B}}}"] \\
\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \arrow[r, "\mathrm{aff}_X^*"] & \mathrm{LocSysCat}(X;\Bbbk)
\end{tikzcd} \tag{5.30}$$

and Proposition 4.1.12 implies that the composition

$$\mathrm{aff}_X^* \circ \mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}\colon \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \longrightarrow \mathrm{LocSysCat}(X;\Bbbk)$$

is fully faithful. This means that for every categorical $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$-modules $\mathscr{C}$ and $\mathscr{D}$ and for all $k \geqslant 0$, any k-simplex

$$[\sigma] \in \pi_k\big(\mathrm{Map}_{\mathrm{LocSysCat}(X;\Bbbk)}\big(\mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{C}), \mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{D})\big)\big)$$

is homotopic to the image of an essentially unique k-simplex

$$[\widetilde{\sigma}] \in \pi_k\left(\mathrm{Map}_{\mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)}(\mathscr{C}, \mathscr{D})\right)$$

under the functor $\mathrm{Loc}_{X_{\mathrm{B}}}$. We will prove that this forces every k-simplex

$$[\tau] \in \pi_k\big(\mathrm{Map}_{\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\big(\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{C}), \mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{D})\big)\big)$$

to arise in the same way. In virtue of the commutativity of the diagram (5.30), if a k-simplex $[\tau]$ as above is the image under $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}$ of some k-simplex $[\widetilde{\tau}]$ then such $[\widetilde{\tau}]$ is unique up to homotopy. Indeed, $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}$ is the first map in the composition $\mathrm{aff}_X^* \circ \mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))} \simeq \mathrm{Loc}_{X_{\mathrm{B}}}$. Since the latter is a fully faithful functor, it induces a morphism between mapping spaces which is an isomorphism on all homotopy groups; therefore, composition with $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}$ produces a morphism between mapping spaces which is forced to be injective on all homotopy groups. So we only need to prove that for all categorical $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$-modules $\mathscr{C}$ we can lift every k-simplex

$$[\tau] \in \pi_k\big(\mathrm{Map}_{\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\big(\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{C}), \mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{D})\big)\big)$$

to a k-simplex

$$[\widetilde{\tau}] \in \pi_k\left(\mathrm{Map}_{\mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)}(\mathscr{C}, \mathscr{D})\right).$$

Since X is assumed to be 1-Koszul, the commutative $\Bbbk$-algebra $\mathrm{C}^\bullet(X;\Bbbk)$ is in particular a coconnective $\Bbbk$-algebra in the sense of Definition 5.2.(1). The chosen base point $\eta\colon \{*\} \to X$ induces an augmentation $\mathrm{C}^\bullet(X;\Bbbk) \to \Bbbk$: this augmentation is essentially unique up to – non-unique – homotopy because of [Lur11a, Corollary 4.1.7]. In turn, this augmentation yields an essentially unique pointing $\mathrm{cSpec}(\eta)\colon \mathrm{Spec}(\Bbbk) \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$, which therefore can be assumed to factor as a composition

$$\mathrm{cSpec}(\eta)\colon \mathrm{Spec}(\Bbbk) \xrightarrow{\eta_{\mathrm{B}}} X_{\mathrm{B}} \xrightarrow{\mathrm{aff}_X} \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)).$$

So, let $[\tau]$ be a k-simplex in the space of maps between $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{C})$ and $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{D})$, as before. Composing with aff_X^*, we obtain a k-simplex

$$[\mathrm{aff}_X^*(\tau)] \in \pi_k\big(\mathrm{Map}_{\mathrm{LocSysCat}(X;\Bbbk)}\big(\mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{C}), \mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{D})\big)\big).$$

Since $\mathrm{Loc}_{X_{\mathrm{B}}}$ is fully faithful, the k-simplex $[\tau]$ comes from a k-simplex $[\widetilde{\tau}]$ in the space of morphisms between $\mathscr{C}$ and $\mathscr{D}$ as presentably $\mathrm{LocSys}(X;\Bbbk)$-linear categories (or, equivalently, as presentably $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$-linear categories). Since the space of morphisms in a limit of categories is the limit of the spaces of morphisms in each category, the fact that $[\mathrm{aff}_X^*(\tau)] \simeq [\mathrm{Loc}_{X_{\mathrm{B}}}(\widetilde{\tau})]$ means that for every R-point $\mathrm{Spec}(R) \to X_{\mathrm{B}}$ of the Betti stack X_{B} there exists a homotopy

$$[\Gamma(\mathrm{Spec}(R), \mathrm{aff}_X^*(\tau))] \simeq [\Gamma(\mathrm{Spec}(R), \mathrm{Loc}_{X_{\mathrm{B}}}(\widetilde{\tau}))]$$

and all such homotopies come equipped with a system of higher homotopies which are compatible with base change. In particular, for $R = \Bbbk$, we have a homotopy

$$[\Gamma(\mathrm{Spec}(\Bbbk), \mathrm{aff}_X^*(\tau))] \simeq [\widetilde{\tau} \otimes_{\mathrm{LocSys}(X;\Bbbk)} \mathrm{id}_{\mathrm{Mod}_\Bbbk}]. \tag{5.31}$$

Since X is 1-Koszul and the pointing $\mathrm{cSpec}(\eta)\colon \mathrm{Spec}(\Bbbk) \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ can be assumed to factor through X_{B}, given any categorical $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$-modules $\mathscr{C}$ the local sections over $\mathrm{Spec}(\Bbbk)$ of $\mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{C})$ and $\mathrm{aff}_X^*(\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{C}))$ are the same. Indeed, they both are equivalent to $\mathscr{C} \otimes_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))} \mathrm{Mod}_\Bbbk \simeq \mathscr{C} \otimes_{\mathrm{LocSys}(X;\Bbbk)} \mathrm{Mod}_\Bbbk$. Therefore, (5.31) yields also a homotopy

$$[\Gamma(\mathrm{Spec}(\Bbbk), \tau)] \simeq [\widetilde{\tau} \otimes_{\mathrm{LocSys}(X;\Bbbk)} \mathrm{id}_{\mathrm{Mod}_\Bbbk}]. \tag{5.32}$$

Now, using [Lur11a, Proposition 4.4.4], write $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ as a colimit of a simplicial diagram

$$\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)) \simeq \underset{[n]\in\mathbf{\Delta}^{\mathrm{op}}}{\mathrm{colim}}\, \mathrm{Spec}(A^n)$$

where $A^0 \simeq \Bbbk$ and each A^n is discrete. This allows us to write

$$\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{ShvCat}(\mathrm{Spec}(A^n)) \simeq \lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Lin}_{A^n} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

Thus, we can interpret a sheaf of categories $\mathscr{F}$ over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ as the datum of a presentably $\Bbbk$-linear category $\Gamma(\mathrm{Spec}(\Bbbk), \mathscr{F})$ together with a system of equivalences

$$\Gamma(\mathrm{Spec}(\Bbbk), \mathscr{F}) \otimes_{\mathrm{Mod}_\Bbbk} \mathrm{Mod}_{A^n} \simeq \Gamma(\mathrm{Spec}(A^n), \mathscr{F})$$

which has to be compatible with pullback along the maps forming the simplicial diagram $\mathrm{Spec}(A^\bullet) \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. In particular, taking the base change of the homotopy (5.32) along the maps $\mathrm{Spec}(A^n) \to \mathrm{Spec}(\Bbbk)$ lifts the homotopy $[\mathrm{aff}_X^*(\tau)] \simeq [\mathrm{Loc}_{X_{\mathrm{B}}}(\widetilde{\tau})]$ to a homotopy $[\tau] \simeq [\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\widetilde{\tau})]$. ☐

We are ready to prove Theorem 5.22 when $n = 1$.

Proposition 5.33. *Let X be a pointed 2-Koszul space over a field $\Bbbk$ of characteristic 0 whose homotopy groups $\pi_q(X)$ are finitely generated for each $q \geqslant 0$. Then the affinization map $\mathrm{aff}_X : X_{\mathrm{B}} \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ induces an equivalence of 2-categories*

$$\mathrm{aff}_X^* : 2\mathbf{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \xrightarrow{\simeq} 2\mathbf{LocSysCat}(X;\Bbbk).$$

Proof. The $\Bbbk$-cochains $\mathrm{C}^\bullet(\Omega_* X;\Bbbk)$ on the based loop space $\Omega_* X$ are equipped with the structure of a Hopf algebra because $\Omega_* X$ is a grouplike $\mathbb{E}_1$-monoid. Therefore, $\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk))$

is a group stack in virtue of Lemma 5.27, and Proposition 5.28 allows us to interpret $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ as the delooping of $\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$. Since Ω_*X is 1-Koszul in virtue of Lemma 5.26, we know that pulling back along affinization map $\mathrm{aff}_{\Omega_*X}\colon (\Omega_*X)_{\mathrm{B}} \to \mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$ induces a strongly monoidal equivalence

$$\mathrm{aff}_X^*\colon \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))) \xrightarrow{\simeq} \mathrm{LocSys}(\Omega_*X;\Bbbk).$$

In particular, $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk)))$ is fully dualizable and self-dual as an object of $\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ (because $\mathrm{LocSys}(\Omega_*X;\Bbbk)$ is self-dual). Moreover, the functor

$$\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))}\colon \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \longrightarrow \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk)))$$

is fully faithful (Lemma 5.29), so we can apply the discussion in [Gai15, Section 10.2] and write

$$\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right),$$

where now $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk)))$ is seen as a monoidal category via the convolution tensor product induced by the group structure on $\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$. Under the equivalence of Corollary 5.20, this monoidal structure corresponds to the Day convolution monoidal structure on $\mathrm{LocSys}(\Omega_*X;\Bbbk)$, hence we obtain a chain of equivalences

$$\begin{aligned}\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) &\xrightarrow{\simeq} \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)\\ &\xrightarrow{\simeq} \mathrm{Lin}_{\mathrm{LocSys}(\Omega_*X;\Bbbk)}\left(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)\\ &\xrightarrow{\simeq} \mathrm{LocSysCat}(X;\Bbbk),\end{aligned}$$

where the second equivalence is obtained by base change along $\mathrm{aff}_{\Omega_*X}^*$ and the third equivalence is due to Lemma 2.2 combined with Proposition 2.8. To check that this functor agrees with aff_X^*, we simply notice that the first equivalence

$$\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$$

sends a sheaf of categories $\mathscr{F}$ to the presentably $\Bbbk$-linear category $\Gamma(\mathrm{Spec}(\Bbbk),\mathscr{F})$ equipped with a $\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$-module structure. Indeed, the inverse of the above equivalence factors as a chain of equivalences

$$\begin{aligned}\mathrm{Lin}_{\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))}\left(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) &\simeq \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk)^{\times n})}\left(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)\\ &\simeq \lim_{[n]\in\Delta^{\mathrm{op}}} \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))^{\times n})\\ &\simeq \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))).\end{aligned}$$

The first equivalence is given by taking the dual $\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$-comodule structure on a categorical $\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$-module $\mathscr{C}$ and producing the associated co-bar cosimplicial category $\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))^{\otimes n} \otimes_{\mathrm{Mod}_\Bbbk} \mathscr{C}$ ([Gai15, Corollary 10.1.5]). The second equivalence is given by taking the term-wise $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))^{\times n}}$ functor ([Gai15, Proposition 10.1.3]). In both cosimplicial categories, the 0-th term is $\mathscr{C}$ itself. Under the third and last equivalence, this is precisely the category of local sections on $\operatorname{Spec}(\Bbbk)$ on the corresponding sheaf of categories over $\operatorname{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))$. Since the equivalence

$$\operatorname{Lin}_{\operatorname{QCoh}(\operatorname{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk)))}\left(\operatorname{Lin}_{\Bbbk}\operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \simeq \operatorname{LocSysCat}(X;\Bbbk)$$

sends $\mathscr{C}$ to the categorical local system over X with stalk at the base point $\eta\colon \{*\} \to X$ equivalent to the underlying presentably $\Bbbk$-linear category of $\mathscr{C}$, it follows that the equivalence

$$\operatorname{ShvCat}(\operatorname{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))) \simeq \operatorname{LocSysCat}(X;\Bbbk)$$

does not alter the local sections of a sheaf of categories over $\operatorname{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))$, so it is realized by the pullback along the affinization map.

We are left to promote such equivalence to a 2-categorical equivalence. In order to do this, we just need to check that the equivalence aff_X^* intertwines the coaugmentations from $\operatorname{Lin}_{\Bbbk}\operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}$ on both sides. This is clear since such coaugmentations are induced by pulling back along the terminal morphisms $X_{\mathrm{B}} \to \operatorname{Spec}(\Bbbk)$ and $\operatorname{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk)) \to \operatorname{Spec}(\Bbbk)$, and aff_X obviously commutes with them. □

Proposition 5.33 is the stepping stone for the inductive proof of Theorem 5.22. Before completing the proof, we observe the following easy fact concerning pushforward $(n+1)$-functors of quasi-coherent sheaves of n-categories for $n \geqslant 2$, which will be used in order to apply the inductive argument.

Remark 5.34. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of prestacks over a commutative ring spectrum $\Bbbk$, and let $n \geqslant 2$ be an integer. For a quasi-coherent sheaf of n-categories $n\mathscr{F}$ over $\mathscr{X}$, the $(n+1)$-functor f_* sends $n\mathscr{F}$ to a quasi-coherent sheaf of n-categories over $\mathscr{Y}$ whose local sections on an affine scheme $\operatorname{Spec}(R)$ over $\mathscr{Y}$ are described as

$$n\boldsymbol{\Gamma}(\operatorname{Spec}(R), f_*(n\mathscr{F})) \simeq \lim_{\operatorname{Spec}(S)\to\operatorname{Spec}(R)\times_{\mathscr{Y}}\mathscr{X}} n\boldsymbol{\Gamma}(\operatorname{Spec}(S), n\mathscr{F}).$$

Such limit is computed along the pullback $(n+1)$-functors. Since the pushforward functor f_* is both right and left adjoint to the pullback functor f^*, the above limit can be equivalently computed as the colimit along the pushforward $(n+1)$-functors, i.e.,

$$n\boldsymbol{\Gamma}(\operatorname{Spec}(R), f_*(n\mathscr{F})) \simeq \operatorname*{colim}_{\operatorname{Spec}(S)\to\operatorname{Spec}(R)\times_{\mathscr{Y}}\mathscr{X}} n\boldsymbol{\Gamma}(\operatorname{Spec}(S), n\mathscr{F}).$$

Both the above limit and colimit are computed inside $(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$.

Proof of Theorem 5.22. Proposition 5.33 proves the case for $n = 1$. For a general $n \geqslant 2$: assume that we have proved Theorem 5.22 for all integers $1 \leqslant k \leqslant n-1$. Let

$$\operatorname{cSpec}(\eta)\colon \operatorname{Spec}(\Bbbk) \xrightarrow{\eta_{\mathrm{B}}} X_{\mathrm{B}} \xrightarrow{\operatorname{aff}_X^*} \operatorname{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))$$

be the pointing of $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ induced by the chosen base point $\eta\colon \{*\} \to X$. This produces a commutative diagram of categories

$$\begin{tikzcd} \mathrm{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)) \arrow[rr, "\mathrm{aff}_X^*"] \arrow[dr, bend right, "\mathrm{cSpec}(\eta)^*"'] & & \mathrm{LocSysCat}^n(X;\Bbbk) \arrow[dl, bend left, "\eta_{\mathrm{B}}^*"] \\ & \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}. & \end{tikzcd} \tag{5.35}$$

We will prove that the diagram (5.35) satisfies the hypotheses of [Lur17, Corollary 4.7.3.16]. This will allow us to apply the Barr–Beck–Lurie's monadicity theorem, and then conclude that the n-categorical equivalence holds as well thanks to Remark 4.2.6.

(1) The functor η_{B}^* is both monadic and comonadic: it is conservative, it commutes with all colimits, and is part of an ambidextrous adjunction. Its adjoint is computed as a left Kan extension along the pointing $\eta\colon \{*\} \to X$, which is the same as a right Kan extension in virtue of Lemma 2.6. With our connectedness assumptions on X, this adjoint is extremely simple to describe: under the equivalence

$$\mathrm{LocSysCat}^n(X;\Bbbk) \simeq \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

the functor η_{B}^* corresponds to forgetting the $\mathrm{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)$-module structure, and the adjoint is given by

$$\mathscr{C} \mapsto \mathscr{C} \otimes_{n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk).$$

(2) The functor $\mathrm{cSpec}(\eta)^*$ is conservative. Indeed, suppose that a morphism of two quasi-coherent sheaves of n-categories $F\colon n\boldsymbol{\mathcal{F}} \to n\boldsymbol{\mathcal{G}}$ over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ is an equivalence when considering local sections over $\mathrm{Spec}(\Bbbk)$: we want to prove that it is actually an equivalence on *all* local sections. Since $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ is a coaffine stack, we argue as in the proof of Lemma 5.29 and write

$$\mathrm{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \lim_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Lin}_{A^n}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

for some colimit simplicial diagram $\mathrm{Spec}(A^\bullet) \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$, where $A^0 \simeq \Bbbk$ and each A^n is discrete. Then the claim is clear because for any stack $\mathscr{X}$ and any quasi-coherent sheaf of n-categories $n\boldsymbol{\mathcal{F}}$, for a morphism of affine schemes $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ living over $\mathscr{X}$ one has an equivalence of n-categories

$$n\boldsymbol{\Gamma}(\mathrm{Spec}(R), n\boldsymbol{\mathcal{F}}) \simeq n\boldsymbol{\Gamma}(\mathrm{Spec}(S), n\boldsymbol{\mathcal{F}}) \otimes_{n\mathbf{Lin}_S\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{Lin}_R\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}.$$

(3) The functor $\mathrm{cSpec}(\eta)^*$ commutes with all limits and colimits. Indeed, as observed in Remark 4.2.2, it admits a both left and right adjoint $\mathrm{cSpec}(\eta)_*$.

(4) For any $\Bbbk$-linear presentable n-category $n\boldsymbol{\mathscr{C}}$, the natural n-functor

$$\mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\boldsymbol{\mathscr{C}})\right) \longrightarrow \eta_{\mathrm{B},*}(n\boldsymbol{\mathscr{C}})$$

obtained via adjunction from the counit n-functor

$$\mathrm{cSpec}(\eta)^*(\mathrm{cSpec}(\eta)_*(n\mathscr{C})) \simeq \eta_{\mathrm{B}}^*\left(\mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\mathscr{C})\right)\right) \longrightarrow n\mathscr{C}$$

is an equivalence. Since both $\mathrm{cSpec}(\eta)^*$ and η_{B}^* are conservative, we can reduce ourselves to check whether the n-functor at the level of local sections over $\mathrm{Spec}(\Bbbk)$

$$n\boldsymbol{\Gamma}\left(\mathrm{Spec}(\Bbbk), \mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\mathscr{C})\right)\right) \longrightarrow n\boldsymbol{\Gamma}\left(\mathrm{Spec}(\Bbbk), \eta_{\mathrm{B},*}(n\mathscr{C})\right) \tag{5.36}$$

is an equivalence. Under the equivalence

$$\mathrm{LocSysCat}^n(X;\Bbbk) \simeq \mathrm{LMod}_{n\mathbf{LocSysCat}^{n-1}(\Omega_*X;\Bbbk)}\left(\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\right),$$

the codomain of the functor (5.36) can be written as

$$n\boldsymbol{\Gamma}(\mathrm{Spec}(\Bbbk), \eta_{\mathrm{B},*}(n\mathscr{C})) \simeq n\mathscr{C} \otimes_{n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{LocSysCat}^{n-1}(\Omega_*X;\Bbbk).$$

The left hand side, using Remark 5.34 and Proposition 5.28, can be instead described as

$$n\boldsymbol{\Gamma}\left(\mathrm{Spec}(\Bbbk), \mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\mathscr{C})\right)\right) \simeq \operatorname*{colim}_{\substack{\mathrm{Spec}(R)\to\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))\\ R\in\mathrm{CAlg}_{\Bbbk}^{\mathrm{disc}}}} n\mathscr{C} \otimes_{n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{Lin}_R\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}.$$

Since the tensor product of presentable n-categories is compatible with colimits, we can swap the tensor product and the colimit and using once again Remark 4.2.2 we can write

$$\begin{aligned} n\boldsymbol{\Gamma}\left(\mathrm{Spec}(\Bbbk), \mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\mathscr{C})\right)\right) &\simeq n\mathscr{C} \otimes_{n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} \left(\operatorname*{colim}_{\substack{\mathrm{Spec}(R)\to\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))\\ R\in\mathrm{CAlg}_{\Bbbk}^{\mathrm{disc}}}} n\mathbf{Lin}_R\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\right) \\ &\simeq n\mathscr{C} \otimes_{n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))). \end{aligned}$$

Therefore, the n-functor (5.36) can be interpreted as the tensor product over $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ of the affinization $(n-1)$-functor

$$\mathrm{aff}_{\Omega_*X}^*: n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))) \longrightarrow n\mathbf{LocSysCat}^{n-1}(\Omega_*X;\Bbbk)$$

with the identity functor of $n\mathscr{C}$. Since Ω_*X is $(n-1)$-Koszul (Lemma 5.26) the n-functor (5.36) is an equivalence because of the inductive hypothesis, as desired.

So, Barr–Beck–Lurie's monadicity theorem allows us to conclude. □

Remark 5.37. The reason why the above proof does not extend straightforwardly to the case when $n = 1$, forcing us to tackle the latter in a somewhat more convoluted way, is that in this case the functor $\mathrm{cSpec}(\eta)^*$ can only be proved to be comonadic – i.e., it is not obvious that the map $\mathrm{aff}_X: X_{\mathrm{B}} \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ is affine schematic, which is what guarantees that $\mathrm{cSpec}(\eta)_*$ is both a left and right adjoint to $\mathrm{cSpec}(\eta)^*$ ([Ste21, Corollary 14.2.10]). In particular, it is not obvious how to check that the natural functor

$$\mathscr{C} \otimes_{\mathrm{Mod}_{\Bbbk}} \left(\lim_{\substack{\mathrm{Spec}(R) \to \mathrm{cSpec}(\mathrm{C}^{\bullet}(\Omega_* X;\Bbbk)) \\ R \in \mathrm{CAlg}_{\Bbbk}^{\geqslant 0}}} \mathrm{Mod}_R \right) \longrightarrow \lim_{\substack{\mathrm{Spec}(R) \to \mathrm{cSpec}(\mathrm{C}^{\bullet}(\Omega_* X;\Bbbk)) \\ R \in \mathrm{CAlg}_{\Bbbk}^{\geqslant 0}}} \mathscr{C} \otimes_{\mathrm{Mod}_{\Bbbk}} \mathrm{Mod}_R$$

is an equivalence. This is true, *a posteriori*, because of Proposition 5.33.

We conclude this section with some explicit examples to which Theorem 5.22 applies.

Example 5.38. Let $X := \mathbf{B}\mathbb{CP}^{\infty}$. When $\Bbbk$ is a field of characteristic 0, its $\Bbbk$-cochain algebra $\mathrm{C}^{\bullet}(X;\Bbbk)$ is the symmetric $\Bbbk$-algebra on the $\Bbbk$-module $\Bbbk[-3]$. In particular, as a stack, $\mathrm{C}^{\bullet}(X;\Bbbk) \simeq \mathbf{B}^3(\mathbb{G}_{\mathrm{a},\Bbbk})$. The latter is known to be 1-affine ([Gai15, Theorem 2.5.7.(b)]). Notice that $\Omega_* \mathbf{B}^3\mathbb{G}_{\mathrm{a},\Bbbk} \simeq \mathbf{B}^2\mathbb{G}_{\mathrm{a},\Bbbk} \simeq \mathrm{cSpec}(\mathrm{C}^{\bullet}(\mathbb{CP}^{\infty};\Bbbk))$, which is again 1-affine ([Gai15, Theorem 2.5.7.(a)]). In particular, $\mathrm{Loc}_{\mathbf{B}^3\mathbb{G}_{\mathrm{a},\Bbbk}}$ and $\mathrm{Loc}_{\mathbf{B}^2\mathbb{G}_{\mathrm{a},\Bbbk}}$ are both trivially fully faithful.
So, we can argue as in Proposition 5.33 to describe $\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk)))$ as

$$\begin{aligned} \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))) &\simeq \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{C}^{\bullet}(\mathbb{CP}^{\infty};\Bbbk))}\Big(\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big) \\ &\simeq \mathrm{Lin}_{\mathrm{LocSys}(\mathbb{CP}^{\infty};\Bbbk)}\Big(\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big) \simeq \mathrm{LocSysCat}(X;\Bbbk). \end{aligned}$$

Applying the same strategy to $Y := \mathbf{B}X$ and to $\mathrm{cSpec}(\mathrm{C}^{\bullet}(Y;\Bbbk))$, we obtain an analogous equivalence

$$\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(\mathbf{B}^2\mathbb{CP}^{\infty};\Bbbk))) \xrightarrow{\simeq} \mathrm{LocSysCat}(\mathbf{B}^2\mathbb{CP}^{\infty};\Bbbk).$$

We now present some curious consequences of the above computation.

Corollary 5.39. *For all $n \geqslant 1$, the Betti stack $(\mathbf{B}^n\mathbb{CP}^{\infty})_{\mathrm{B}}$ is n-affine, while the Betti stack $\left(\mathbf{B}^{n+1}\mathbb{CP}^{\infty}\right)_{\mathrm{B}}$ is not n-affine.*

Proof. The case $n = 1$ is obvious from Example 5.38, since $\mathbf{B}^3\mathbb{G}_{\mathrm{a},\Bbbk} \simeq \mathrm{C}^{\bullet}(\mathbf{B}\mathbb{CP}^{\infty};\Bbbk)$ is 1-affine but $\mathbf{B}^4\mathbb{G}_{\mathrm{a},\Bbbk} \simeq \mathrm{C}^{\bullet}(\mathbf{B}^2\mathbb{CP}^{\infty};\Bbbk)$ is not ([Gai15, Theorem 2.5.7.(c)]). Then, a simple inductive argument using Theorem 4.2.9 yields the result. □

Remark 5.40. For all $n \geqslant 1$, Corollary 5.39 offers an example of an n-affine Betti stack corresponding to a non-n-truncated space X. However, as predicted by Corollary 4.2.24, the $(n+1)$-th homotopy group of X is always trivial.

Corollary 5.41. *The Betti stack $(\mathbb{CP}^{\infty})_{\mathrm{B}}$ is almost 0-affine.*

Proof. Just combine Corollary 5.39 (in the $n = 1$ case) with Proposition 4.1.18. □

Remark 5.42. To our knowledge, Corollary 5.41 is a novel result. Via private communication, Y. Harpaz showed us that the global sections functor

$$\Gamma(\mathbb{CP}^{\infty}, -) \colon \mathrm{LocSys}(\mathbb{CP}^{\infty};\Bbbk) \longrightarrow \mathrm{Mod}_{\Bbbk}$$

is indeed obtained by composing two monadic functors – namely, the monadic fully faithful Koszul duality functor $\mathrm{LMod}_{\mathrm{C}_{\bullet}(S^1;\Bbbk)} \subseteq \mathrm{Mod}_{\mathrm{C}^{\bullet}(\mathbb{CP}^{\infty};\Bbbk)}$ and the forgetful functor $\mathrm{Mod}_{\mathrm{C}^{\bullet}(\mathbb{CP}^{\infty};\Bbbk)} \to$

$\mathrm{Mod}_{\Bbbk}$. In particular, even with standard (i.e., de-categorified) arguments it is obvious that the global sections functor must be conservative. However, it is not clear how to prove that it preserves colimits of $\Gamma(\mathbb{CP}^\infty,-)$-split simplicial objects.

Remark 5.43. Suppose that X is a pointed n-Koszul space over a field $\Bbbk$ of characteristic 0. Then, since X is in particular $(n-1)$-Koszul, one can expect to recover the $\mathbb{E}_{n-1}$-Koszul duality equivalence between $(n-2)$-categorical modules by "delooping" $\mathbb{E}_n$-Koszul duality between $(n-1)$-categorical modules. This is indeed the case: notice that the unit for the monoidal structure on $(n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ is the sheaf $n\underline{\mathbf{ShvCat}}^{n-1}(-)$ whose global sections are precisely $n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$. So we have an equivalence of mapping n-categories between

$$n\underline{\mathbf{Fun}}^{\mathrm{L}}_{(n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\Big(n\underline{\mathbf{ShvCat}}^{n-1}(-),\, n\underline{\mathbf{ShvCat}}^{n-1}(-)\Big)$$

and

$$n\underline{\mathbf{Fun}}^{\mathrm{L}}_{(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}\Big(n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)},\, n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))\Big),$$

which is just $n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ because $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$ is the monoidal unit inside $(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$.

Similarly, the monoidal unit for $(n+1)\mathbf{LocSysCat}^n(X;\Bbbk)$ is the trivial local system

$$n\underline{\mathbf{LocSysCat}}^{n-1}(-) := \mathrm{const}\Big(n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\Big),$$

and so we have an equivalence of mapping n-categories between

$$n\underline{\mathbf{Fun}}^{\mathrm{L}}_{(n+1)\mathbf{LocSysCat}^n(X;\Bbbk)}\Big(n\underline{\mathbf{LocSysCat}}^{n-1}(-),\, n\underline{\mathbf{LocSysCat}}^{n-1}(-)\Big)$$

and

$$n\underline{\mathbf{Fun}}^{\mathrm{L}}_{(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}\Big(n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)},\, n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)\Big) \simeq n\mathbf{LocSysCat}^{n-1}(X;\Bbbk).$$

The $(n+1)$-functor aff_X^* sends $n\underline{\mathbf{ShvCat}}^{n-1}(-)$ to $n\underline{\mathbf{LocSysCat}}^{n-1}(-)$, because it is strongly monoidal and hence preserves the monoidal unit. Since aff_X^* is also an equivalence of $(n+1)$-categories, it induces an equivalence at the level of mapping n-categories, and so it recovers the $\mathbb{E}_{n-1}$-Koszul duality equivalence for $(n-2)$-categorical modules. Applying iteratively this argument, we recover the $\mathbb{E}_k$-Koszul duality equivalence for modules for all $k \leqslant n$, up to the classical $\mathbb{E}_1$-Koszul duality for modules of Corollary 5.20.

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