HIGHER KOSZUL DUALITY AND n-AFFINENESS

JAMES PASCALEFF, EMANUELE PAVIA, AND NICOLÒ SIBILLA

Abstract

We study $\mathbb{E}_n$-Koszul duality for pairs of algebras of the form $\mathrm{C}_\bullet(\Omega^n_* X;\Bbbk) \longleftrightarrow \mathrm{C}^\bullet(X;\Bbbk)$, and the closely related question of n-*affineness* for Betti stacks. It was expected, but not known, that $\mathbb{E}_n$-Koszul duality should induce a kind of Morita equivalence between categories of iterated modules. We establish this rigorously by proving that the (∞,n)-category of iterated modules over $\mathrm{C}_\bullet(\Omega^{n+1}_* X;\Bbbk)$ is equivalent to the (∞,n)-category of quasi-coherent sheaves of $(\infty,n-1)$-categories on $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$, where $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ is the *cospectrum* of $\mathrm{C}^\bullet(X;\Bbbk)$. By the *monodromy equivalence*, these categories are also equivalent to the category of higher local systems on X, $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$. Our result is new already in the classical case $n=1$, although it can be seen to recover well known formulations of $\mathbb{E}_1$-Koszul duality as a Morita equivalence of module categories (up to appropriate completions of the t-structures). We also investigate (higher) affineness properties of Betti stacks. We give a complete characterization of n-affine Betti stacks, in terms of the 0-affineness of their iterated loop space. As a consequence, we prove that n-truncated Betti stacks are n-affine; and that $\pi_{n+1}(X)$ is an obstruction to n-affineness.

CONTENTS

(James Pascaleff) UNIVERSITY OF ILLINOIS, WEST GREEN STREET, 1409, 61801, URBANA, IL, UNITED STATES
(Emanuele Pavia) SISSA, VIA BONOMEA 265, 34136 TRIESTE, TS, ITALY
(Nicolò Sibilla) SISSA, VIA BONOMEA 265, 34136 TRIESTE, TS, ITALY
E-mail addresses: jpascale@illinois.edu, epavia@sissa.it, nsibilla@sissa.it.

Introduction

In this paper we study local systems of higher categories over spaces. Our main goal is obtaining a formulation of $\mathbb{E}_n$-Koszul duality between the algebras of chains on the iterated loop space of a space X, and the algebra of cochains on X, as an equivalence of (∞, n)-categories. This will lead us to investigate the closely related problem of *n-affineness* for Betti stacks.

We wish to generalize and extend to higher categories the following well-known classical story. Let X be a connected space and let $\Bbbk$ be a field of characteristic 0. Local systems of $\Bbbk$-vector spaces over X are determined by monodromy data, in the sense that the abelian category of such local systems is equivalent to the category of representations of $\pi_1(X)$. Understanding the higher cohomology of local systems requires more information that is not captured by $\pi_1(X)$, and in fact depends on full homotopy type of X. More precisely, the stable category of complexes of sheaves of vector spaces on X whose cohomology sheaves are local systems is equivalent to the stable category of modules over $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$, the algebra of chains on the based loop space of X. In formulas, there is an equivalence

$$\mathrm{LocSys}(X;\Bbbk) \simeq \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \tag{I.0.1}$$

which we refer to as the *monodromy equivalence*.

The $\mathbb{E}_1$-Koszul dual of $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ is $\mathrm{C}^\bullet(X;\Bbbk)$, the algebra of cochains on X; under certain finiteness hypotheses, the reciprocal duality also holds, and moreover there is a tight relationship between the categories of modules over these two algebras. Hence, under these hypotheses, local systems over X also admit a description in terms of $\mathrm{C}^\bullet(X;\Bbbk)$. Passing to the n-categorical level, local systems of vector spaces are replaced by local systems of $\Bbbk$-linear (∞, n)-categories, the loop space is replaced by the $(n+1)$-fold iterated loop space $\Omega_*^{n+1}X$, and Koszul duality of $\mathbb{E}_1$-algebras is replaced by Koszul duality of the $\mathbb{E}_{n+1}$-algebras $\mathrm{C}_\bullet(\Omega_*^{n+1}X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$. This paper seeks to sort out how the classical relationships generalize to this setting.

In the companion work [PPS25] we set the basis for the present investigation by generalizing the monodromy equivalence to local systems of *presentable* (∞, n)-categories, in the sense of Stefanich [Ste20]. We refer the reader to Section 1 in the main text for a definition of presentable (∞, n)-categories; here will limit ourselves to say that they are a categorification of presentable $(\infty, 1)$-categories, and enjoy many of the same favourable formal properties. The categorified monodromy equivalence, proved as Theorem 3.2.24 in [PPS25], states that there is an equivalence of $(\infty, n+1)$-categories

$$(n+1)\mathbf{LocSysCat}^n(X) \simeq (n+1)\mathbf{LMod}_{n\mathbf{LMod}_{\Omega_*^{n+1}X}(\mathcal{S})}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}. \tag{I.0.2}$$

Here, the category on the left hand side is the $(\infty, n+1)$-category of local systems of presentable (∞, n)-categories over X; the category on the right hand side is the $(\infty, n+1)$-category of presentable (∞, n)-categories with an action of the presentable (∞, n)-category of iterated left modules over the grouplike topological $\mathbb{E}_{n+1}$-monoid $\Omega_*^{n+1}X$. We note that the categorified monodromy equivalence holds also when we incorporate linearity over a presentably symmetric monoidal $(\infty, 1)$-category $\mathscr{A}$: above, we stated it in the absolute case when $\mathscr{A} = \mathscr{S}$ is the $(\infty, 1)$-category of spaces. In this paper, we will mostly work over an algebraically closed field $\Bbbk$ of characteristic 0, i.e. we shall set $\mathscr{A} = \mathrm{Mod}_{\Bbbk}$.

In the next section of this introduction we explain in greater detail the ideas surrounding our contributions to $\mathbb{E}_n$-Koszul duality and n-affineness for Betti stacks. For clarity, we will mostly explain the first non-trivial case, namely $\mathbb{E}_2$-Koszul duality. Next, in Section I.2, we shall give an analytic description of the structure of the paper and state our main results.

I.1. **Koszul duality and n-affineness.** Our main goal is to study higher Koszul duality, and the closely related question of *n-affineness* of Betti stacks. To explain the context of our work, we start recalling in some greater detail classical Koszul duality.

Let $\Bbbk$ be an algebraically closed field of characteristic 0. Classical Koszul duality is a duality between certain augmented *associative* $\Bbbk$-algebras, the most well-known example of which is the duality between symmetric and exterior algebras. Topology and the theory of local systems are the source of one of the most important classes of Koszul dual algebras. Let us explain how this works. Let X be a pointed and simply connected finite CW complex. The natural map $X \to \{*\}$ equips the algebra $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ with an augmentation

$$\mathrm{C}_\bullet(\Omega_* X;\Bbbk) \longrightarrow \mathrm{C}_\bullet(\Omega_*\{*\},\Bbbk) \simeq \Bbbk.$$

The dg algebra of singular cochains $\mathrm{C}^\bullet(X;\Bbbk)$ is augmented via the pointing $\{*\} \to X$

$$\mathrm{C}^\bullet(X;\Bbbk) \longrightarrow \mathrm{C}^\bullet(\{*\},\Bbbk) \simeq \Bbbk.$$

Then the algebras $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$ are *Koszul dual*. Classically this means that we have the following two closely related statements.

(1) The algebra of endomorphisms of the augmentation of $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ is equivalent to $\mathrm{C}^\bullet(X;\Bbbk)$, and vice versa. In symbols:

$$\mathrm{C}_\bullet(\Omega_* X;\Bbbk) \simeq \underline{\mathrm{Map}}_{\mathrm{C}^\bullet(X;\Bbbk)}(\Bbbk,\Bbbk) \quad \text{and} \quad \mathrm{C}^\bullet(X;\Bbbk) \simeq \underline{\mathrm{Map}}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}(\Bbbk,\Bbbk).$$

(2) The functor between $\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}$ and $\mathrm{LMod}_{\mathrm{C}^\bullet(X;\Bbbk)}$ given by

$$\underline{\mathrm{Map}}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}(\Bbbk,-) : \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \longrightarrow \mathrm{LMod}_{\mathrm{C}^\bullet(X;\Bbbk)} \tag{I.1.3}$$

is *almost*, but not quite, a Morita equivalence.

Under the equivalence $\mathrm{LocSys}(X;\Bbbk) \simeq \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}$ the augmentation module is sent to the constant local system $\underline{\Bbbk}_X$, and the functor $\underline{\mathrm{Map}}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}(\Bbbk,-)$ corresponds to the enhanced global sections

$$\Gamma(X,-) : \mathrm{LocSys}(X;\Bbbk) \longrightarrow \mathrm{Mod}_{\mathrm{C}^\bullet(X;\Bbbk)}. \tag{I.1.4}$$

As it turns out, the enhanced global section functor (I.1.4), and thus functor (I.1.3), are almost never equivalences. Using the terminology of algebraic geometry, we can express this by saying that finite CW complexes, or more precisely their *Betti stacks*, are almost never *affine*. Here we understand affineness precisely as the property that global sections define an equivalence between the stable $(\infty,1)$-category of quasi-coherent sheaves, and the stable $(\infty,1)$-category of modules over the global sections of the structure sheaf. Now, the Betti stack of a space X, denoted X_{B}, is the constant stack with values X (see Section 2 in the main text for more details). The $(\infty,1)$-category $\mathrm{QCoh}(X_{\mathrm{B}})$ is naturally equivalent to $\mathrm{LocSys}(X;\Bbbk)$ and under this identification $\mathcal{O}_{X_{\mathrm{B}}}$ goes to the constant local system $\underline{\Bbbk}_X$. So the failure of (I.1.4) to give rise to an equivalence means precisely that X_{B} is not affine.

The failure of Koszul duality to give rise to an actual Morita equivalence is one the main subtleties of the theory. There are several ways to obviate this, and turn (2) into a rigorous mathematical statement. It is possible to show that functor (I.1.3) does restrict to an equivalence between categories of appropriately bounded modules: more precisely, there is an equivalence

$$\mathrm{LMod}^{-}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{LMod}^{+}_{\mathrm{C}^\bullet(X;\Bbbk)} \tag{I.1.5}$$

between *bounded above* $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$-modules, and *bounded below* $\mathrm{C}^\bullet(X;\Bbbk)$-modules (compare with [BGS96, Theorem 12.6]). Alternatively, we can modify the notion of module we work with. Namely, the functor (I.1.3) induces an equivalence

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{IndCoh}_{\mathrm{C}^\bullet(X,\Bbbk)} \tag{I.1.6}$$

where the right-hand side is the $(\infty,1)$-category of *ind-coherent* modules over $\mathrm{C}^\bullet(X;\Bbbk)$, which we define formally in Section 3 of the main text: suffice it to say for the moment that, in this setting, this is the $(\infty,1)$-category generated by the augmentation module. It is this latter formulation of $\mathbb{E}_1$-Koszul duality which will be particularly relevant for our approach to $\mathbb{E}_n$-Koszul duality.

Now let X be a pointed and n-connected finite CW complex. Much as before, we can associate to X two augmented algebras: except now these will be $\mathbb{E}_n$- rather than $\mathbb{E}_1$-algebras. On the one hand, the n-th iterated loop space

$$\Omega^n_* X \coloneqq \Omega_* \dots \Omega_* X$$

is a $\mathbb{E}_n$-space; thus, $\mathrm{C}_\bullet(\Omega^n_* X;\Bbbk)$ carries a $\mathbb{E}_n$-product. On the other hand, the algebra of $\Bbbk$-valued cochains $\mathrm{C}^\bullet(X;\Bbbk)$ on X is naturally a $\mathbb{E}_\infty$-algebra, so we can regard it in particular as an $\mathbb{E}_n$-algebra. The key claim is that these two algebras are $\mathbb{E}_n$-Koszul dual to each other:

$$\mathrm{C}_\bullet\big(\Omega^n_* X,\Bbbk\big) \longleftrightarrow \mathrm{C}^\bullet(X;\Bbbk).$$

Applying [Lur11b, Theorem 4.4.5] one can *almost* deduce an $\mathbb{E}_n$-analogue of statement (1). Indeed, using [Lur17, Example 5.3.1.5 and Lemma 5.3.1.11], one can prove that the Koszul dual of an augmented $\mathbb{E}_n$-algebra $A \to \Bbbk$ is the morphism object

$$A^! := \underline{\mathrm{Map}}_{\mathrm{Mod}_A^{\mathbb{E}_n}}(A, \Bbbk).$$

However, no analogue of statement (2) has been established in the literature. In fact, as far as we are aware of, even how to properly formulate (2) in the $\mathbb{E}_n$-setting was not known. No doubt one of the reasons for this gap in the literature is due to the subtle nature of the equivalence: as we discussed this is not a straightforward equivalence between $(\infty,1)$-categories of modules; its formulation requires sophisticated ingredients which are not easily adapted to the $\mathbb{E}_n$-setting. In this paper we prove an $\mathbb{E}_n$-analogue of statement (2), and this will yield in particular an $\mathbb{E}_n$-analogue of statement (1). Conceptually, our main innovation consists in reinterpreting statement (2), and in particular equivalence (I.1.6), from a novel perspective which makes categorification possible.

To explain our results, we shall focus on the case $n = 2$. We will give a more complete summary of our main results, for all n, in section (I.2) of this introduction. Consider the following diagram of $(\infty,2)$-categories.

$$\begin{array}{ccc} \mathbf{2LMod}_{\mathrm{LMod}_{C_\bullet(\Omega^2_* X;\Bbbk)}}\left(\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}\right) & \overset{A}{\longleftrightarrow} & \mathbf{2LMod}_{\mathrm{LMod}_{C^\bullet(X;\Bbbk)}}\left(\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}\right) \\ {\scriptstyle B}\updownarrow & & \updownarrow{\scriptstyle C} \\ \mathbf{2LocSysCat}(X;\Bbbk) & \underset{D}{\longleftrightarrow} & \mathbf{2LMod}_{\mathrm{LocSys}(X;\Bbbk)}\left(\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}\right) \end{array} \tag{I.1.7}$$

Here $\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}$ denotes the $(\infty,2)$-category of presentable categories. All categories appearing in the diagram, except $\mathbf{2LocSysCat}(X;\Bbbk)$, are defined as $(\infty,2)$-category of modules for appropriate $\mathbb{E}_1$-algebra objects (i.e. monoidal categories) in $\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}$. These four categories all play a role in a categorification of Koszul duality.

Not all arrows in the diagram stand for equivalences. Let us briefly comment on each of them separately.

- The category $\mathbf{2LocSysCat}(X;\Bbbk)$ is the $(\infty,2)$-category of local systems of k-linear presentable categories over X. Arrow B categorifies the presentation of local systems in terms of monodromy data (I.0.1). We have proved that B is an equivalence in [PPS25], where we extend the monodromy equivalence to local systems of presentable (∞,n)-categories for all n.
- Using the terminology of [Gai15], arrow D is an equivalence when the Betti stack X_{B} associated to X is 1-*affine*. This is a natural categorification of the notion of affineness (see the discussion after (I.1.4) above) which is due to Gaitsgory. In Section 2 we shall prove that 1-truncated Betti stacks are 1-affine; and that the non-vanishing of

$\pi_2 \otimes \Bbbk$ is an obstruction to 1-affineness; although not a complete one, as there exist spaces with vanishing $\pi_2 \otimes \Bbbk$ which are nonetheless not 1-affine. In fact, recent work of Stefanich allows us to make sense of the notion of n-affineness for all n, and we will also establish analogues of these results in the context of n-affineness.

We stress that the fact that Betti stacks typically fail to be *n-affine* is the main source of difficulties in higher Koszul duality theory, just as in the classical story. As we discussed, it is precisely because Betti stacks are virtually never affine that $\mathbb{E}_1$-Koszul duality fails to be a Morita equivalence. The question of affineness of Betti stacks is therefore closely related to Koszul duality, and this is why we devote Section 2 to an in-depth investigation of it.

- Arrow C is almost never an equivalence. This boils down to the failure of $\mathbb{E}_1$-Koszul duality to induce a Morita equivalence. For the same reason, arrow A is almost never an equivalence. Note that if A were an equivalence, then the two $\mathbb{E}_2$-algebras $C_\bullet(\Omega^2_* X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$ would actually be $\mathbb{E}_2$-Morita equivalent on the nose, in the sense that their categories of iterated modules would be equivalent.

  Based on classical $\mathbb{E}_1$-Koszul duality we should not expect such a straightforward statement to hold, and indeed it is typically false. However, in Section 3 we explain how to modify the 2-category $2\mathbf{LMod}_{\mathrm{LMod}_{\mathrm{C}^\bullet(X;\Bbbk)}}\left(2\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$ in such a way that A becomes an equivalence. We regard the resulting equivalence as the analogue of (I.1.6) in the setting of $\mathbb{E}_2$-Koszul duality. Also, we show how to obtain analogous results in the context of $\mathbb{E}_n$-Koszul duality for all n.

This last point is one of our main contributions in this article, so it is worthwhile to explain it in some more detail. Instead of viewing $\mathrm{C}^\bullet(X;\Bbbk)$ merely as a $\mathbb{E}_\infty$-algebra, we can do algebraic geometry with it. The algebra of cochains $\mathrm{C}^\bullet(X;\Bbbk)$ can be endowed with a structure of a commutative dg-algebra, but it does not fall within the range of ordinary derived geometry because it is not connective: its homology vanishes in positive degrees, and is concentrated in negative degrees; the contrary of what we require of a derived affine scheme. Toën ([Toë06]) and Lurie ([Lur11a]), have explained that we can view such an algebra as the algebra of functions on a *coaffine stack*, which is called its *cospectrum*. The cospectrum of $\mathrm{C}^\bullet(X;\Bbbk)$ is denoted $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$.

Quasi-coherent sheaves on $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ can be viewed as a *renormalization* of the category of $\mathrm{C}^\bullet(X;\Bbbk)$-modules. Under our assumptions on X, they coincide with ind-coherent modules

$$\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \mathrm{IndCoh}_{\mathrm{C}^\bullet(X,\Bbbk)}. \tag{I.1.8}$$

This yields a new formulation of the *almost* Morita equivalence which is at the heart of $\mathbb{E}_1$-Koszul duality. Namely, combining (I.1.6) and (I.1.8) we obtain an equivalence

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))). \tag{I.1.9}$$

In this way, the notion of ind-coherent $\mathrm{C}^\bullet(X;\Bbbk)$-module required to turn Koszul duality into an actual Morita equivalence is encoded in the geometry of $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. The great advantage over other formulations of Koszul duality is that equivalence (I.1.9) is well adapted to categorification.

Our main result in Section 3 is that, if X is a 2-connected finite CW complex,[1] there is an equivalence of $(\infty,2)$-categories between iterated modules over $\mathrm{C}_\bullet(\Omega^2_* X;\Bbbk)$ and *quasi-coherent sheaves of categories* over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. This is the analogue of equivalence (I.1.6) in the $\mathbb{E}_2$-setting: as in the classical story, this means in particular that if X is a 2-connected finite CW complex the theory of categorified local systems over X only depends on the algebra of cochains $\mathrm{C}^\bullet(X;\Bbbk)$.

This equivalence fits as the top arrow in the following commutative diagram of equivalences, which should be viewed as a better behaved replacement of diagram (I.1.7).

$$\begin{array}{ccc} \mathbf{2LMod}_{\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega^2_* X;\Bbbk)}}\left(\mathbf{2Pr}^{\mathrm{L}}_{(\infty,1)}\right) & \xleftrightarrow{\;\simeq\;} & \mathbf{2ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \\ \updownarrow \wr & & \updownarrow \wr \\ \mathbf{2LocSysCat}(X;\Bbbk) & \xleftrightarrow{\;\simeq\;} & \mathbf{2ShvCat}(X_{\mathrm{B}}) \end{array} \tag{I.1.10}$$

Let us explain our notations: here $\mathbf{2ShvCat}(-)$ denotes the symmetric monoidal $(\infty,2)$-category of *quasi-coherent sheaves of categories*, which was first introduced by Gaitsgory. It is a categorification of quasi-coherent sheaves in the precise sense that it is a delooping of $\mathrm{QCoh}(-)$: i.e. $\mathrm{QCoh}(-)$ can be recovered as the endomorphisms of the unit object in $\mathbf{2ShvCat}(-)$. In Section 3 we also prove analogous results for n-connected finite CW complex in the context of $\mathbb{E}_n$-Koszul duality.

I.2. Main results. We shall give next a more analytical description of the contents of the paper, and state our main results. In Section 1 we survey briefly all preliminary material which will be required in the remainder of the paper. This includes the theory of presentable (∞,n)-categories, which was recently introduced by Stefanich in [Ste20], and forms the technical backbone of many of our constructions. Additionally we recall the main results we obtained in the companion paper [PPS25], and most importantly the categorified monodromy equivalence (I.0.2).

In Section 2 we study the question of n-affineness for Betti stacks. We obtain a complete characterization of n-affine Betti stacks, which has however the drawback of not being explicit: it reduces the question of n-affineness of a Betti stack X_{B}, which is n-categorical in nature, to a purely 1-categorical condition on the Betti stack of the iterated loop space $\Omega^n_* X$. This condition is however difficult to check in practice, see Theorem A below. To obviate

[1]Our results hold in fact in greater generality, see Section 3 for the precise assumptions we need.

this shortcoming we extract from Theorem A one necessary condition, and one sufficient condition, which are both easily verifiable.

Theorem A (Theorem 2.26). *Let X be a space with a choice of a base point. Then its Betti stack X_{B} is n-affine if and only if the global section functor*

$$\Gamma(\Omega^n_* X, -)\colon \mathrm{LocSys}(\Omega^n_* X; \Bbbk) \longrightarrow \mathrm{Mod}_{\Bbbk}$$

is monadic.

Theorem B (Theorem 2.25 and Corollary 2.41). *Let X be a space, and let $\Bbbk$ be a field of characteristic* 0.

- *If X is n-truncated, then its Betti stack X_{B} is n-affine.*
- *Suppose that $\pi_{n+1}(X)$ does not vanish for some choice of a base point in X. Then the Betti stack X_{B} is not n-affine over $\Bbbk$.*

In Section 3 we turn our attention to $\mathbb{E}_n$-Koszul duality. In addition to our results proper, we believe that one of our main contributions in this section is of a conceptual nature. We propose that the *cospectrum* of the coconnective cdga of cochains on X, $\mathrm{C}^\bullet(X;\Bbbk)$, should play a key role in the study of Koszul duality for this class of algebras. We test this idea first in the classical case, where we show that if X is simply connected (and sufficiently finite) there is an equivalence of categories

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))). \tag{I.2.11}$$

As we explained, the $(\infty,1)$-category $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ is not equivalent to the $(\infty,1)$-category of $\mathrm{C}^\bullet(X;\Bbbk)$-modules (for which the equivalence above does not hold), though it is closely related: as noted in [Lur11a], $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ is the left completion of the natural t-structure on $\mathrm{C}^\bullet(X;\Bbbk)$-modules. Equivalence (I.2.11) shows that if we replace $\mathrm{C}^\bullet(X;\Bbbk)$-modules with quasi-coherent sheaves on $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ we can formulate Koszul duality as an actual equivalence of $(\infty,1)$-categories. Our main result in Section 3 is a categorification of (I.2.11).

Theorem C (Theorem 3.22). *Let $n \geqslant 1$ be an integer, let $\Bbbk$ be a field of characteristic* 0, *and let X be a pointed $(n+1)$-connected space satisfying appropriate finiteness conditions. Then there is a natural equivalence of $(\infty, n+1)$-categories*

$$(n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq (n+1)\mathbf{LocSysCat}^n(X;\Bbbk). \tag{I.2.12}$$

Combining this with the categorified monodromy equivalence Eq. (I.0.2)*, we obtain an equivalence of $(\infty, n+1)$-categories*

$$(n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq (n+1)\mathbf{Mod}_{n\mathbf{Mod}^{n-1}_{\mathrm{C}_\bullet(\Omega^{n+1}_* X;\Bbbk)}}\Big((n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\Big)$$

which is an n-fold categorification of equivalence (I.2.11).

In the statement of Theorem C, the $(n+1)$-category $n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ is the $(n+1)$-category of quasi-coherent sheaves of (presentably $\Bbbk$-linear) n-categories over the cospectrum of the $\Bbbk$-valued cochains $\mathrm{C}^\bullet(X;\Bbbk)$ of X. This is an n-categorification of the usual category of quasi-coherent sheaves: when $n=2$, this was defined in [Gai15], while for arbitrary n it has been recently introduced in [Ste21].

Before proceeding, let us comment further on equivalence (I.2.12), which we consider to be the deepest result of this paper. The appearance of $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ in the statement might seem only a technical artefact of our approach; on the contrary, we believe that our work clarifies the true nature of Koszul duality in the topological setting. There is a canonical map of stacks

$$\mathrm{aff}_X : X_{\mathrm{B}} \longrightarrow \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$$

called the *affinization map*. Equivalence (I.2.12) is given precisely by the pull back along aff_X. The real content of Koszul duality in this setting is therefore that, under appropriate connectivity assumptions on X, the theory of (higher) local systems does not distinguish between X_{B} and $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. We believe this to be a more transparent statement already in the classical case $n=1$ where, as we discussed, the standard formulation of the duality between $\mathrm{C}^\bullet(X;\Bbbk)$ and $\mathrm{C}_\bullet(\Omega_*X;\Bbbk)$ requires otherwise artificial size restrictions, or t-structure renormalizations.

Notations and conventions.

- We will use throughout the language of $(\infty,1)$-categories and higher homotopical algebra, as developed in [Lur09; Lur17], from which we borrow most of the notations and conventions.
- Since our work heavily relies on intrinsically derived and homotopical concepts, we shall simply write “limits", “colimits", “tensor product", suppressing adjectives such as “homotopy" or “derived" in our notations. Similarly, we shall simply write “categories" instead of “$(\infty,1)$-categories", and “n-categories" instead of “(∞,n)-categories".
- We will work with *local systems* and *sheaves* of categories, and it will be important pay attention to size issues. We fix a sequence of nested universes $\mathcal{U}\in\mathcal{V}\in\mathcal{W}\in\ldots$. We shall say that a category $\mathcal{C}$ is *small* if it is $\mathcal{U}$-small, that $\mathcal{C}$ is *large* if it is $\mathcal{V}$-small without being $\mathcal{U}$-small, that $\mathcal{C}$ is *very large* if it is $\mathcal{W}$-small without being $\mathcal{V}$-small, and that $\mathcal{C}$ is *huge* if it is not even $\mathcal{W}$-small. When dealing with categories of (possibly decorated) categories, we shall adopt the following notations in order to distinguish the size: large categories of categories will be denoted with a normal font; very large categories of categories will be denoted with $\widehat{(-)}$; huge categories of categories will be denoted with $\widehat{(-)}$ and capital letters.
  For example, $\mathrm{Cat}_{(\infty,1)}$ is the large category of small categories, while $\widehat{\mathrm{Cat}}_{(\infty,1)}$ is the very large category of large categories, and $\widehat{\mathrm{CAT}}_{(\infty,1)}$ is the huge category of very large categories.

- We shall denote the large category of small spaces by $\mathcal{S}$. In particular, by *space* we always mean *small space*.
- The large category $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ of large presentable categories and the very large category of $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ of large cocomplete categories are both symmetric monoidal categories: $\mathbb{E}_k$-algebras inside $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ and $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ are (respectively) presentable and cocomplete categories endowed with an $\mathbb{E}_k$-monoidal structure that commutes with colimits separately in each variable. In order to compactify our notations, in the rest of our paper we shall refer to an $\mathbb{E}_k$-algebra in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ as a *presentably* $\mathbb{E}_k$*-monoidal category*, and to an $\mathbb{E}_k$-algebra in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ as a *cocompletely* $\mathbb{E}_k$*-monoidal category*; in the case $k=\infty$ we shall simply write *symmetric monoidal* in place of $\mathbb{E}_\infty$-monoidal. The notation for $\mathbb{E}_k$-algebras in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ can sound ambiguous, since an $\mathbb{E}_k$-monoidal structure on a cocomplete category can fail to be compatible with colimits: for an easy counterexample, just consider the category of pointed spaces endowed with the Cartesian symmetric monoidal structure. However, we shall never be interested in such kind of monoidal structures in this work.
- In a similar fashion, for any $k\in\mathbb{N}_{\geqslant 1}\cup\{\infty\}$ and a cocompletely (resp. presentably) $\mathbb{E}_k$-monoidal ∞-category $\mathcal{A}$, we shall say that a category $\mathcal{C}$ is *cocompletely* (resp. *presentably*) left tensored over $\mathcal{A}$ if it is a left $\mathcal{A}$-module in $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ (resp. in $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$). This formula amounts to the datum of a cocomplete (or presentable) category $\mathcal{C}$ which is left tensored over $\mathcal{A}$ in such a way that the tensor action functor commutes with colimits separately in each argument.
- We shall deal with higher (i.e., n-)categories, and in particular with $(n+1)$-categories of (possibly decorated) n-categories: we follow the conventions introduced and adopted in [PPS25]. We shall denote n-categories with a bold font, and in order to avoid confusion concerning the “categorical height" we are working at, we shall adopt the following highly non-standard notation as well: if we want to refer to the (very large) higher category of large m-categories seen as a n-category, we shall write $n\widehat{\mathbf{Cat}}_{(\infty,m)}$. In the particular case $n=1$, we shall drop both the bold font and the 1 before our notations, and simply write $\widehat{\mathrm{Cat}}_{(\infty,m)}$. For example, $3\widehat{\mathbf{Cat}}_{(\infty,2)}$ is the very large 3-category of all large 2-categories, while $2\widehat{\mathbf{Cat}}_{(\infty,2)}$ is its underlying 2-category, and $\widehat{\mathrm{Cat}}_{(\infty,2)}$ is its underlying 1-category.
- Most of the times we will consider categories which are enriched over some preferred category (e.g., modules in spectra which are enriched over themselves, or presentably enriched categories which are enriched over themselves, and so forth). At the same time, we will need to consider the underlying spaces of maps between objects in such categories. For this reason, when $\mathcal{C}$ is enriched over a category $\mathcal{A}$, we will denote as $\mathrm{Map}_{\mathcal{C}}(-,-)$ the space of maps in $\mathcal{C}$, and as $\underline{\mathrm{Map}}_{\mathcal{C}}(-,-)$ the morphism object of $\mathcal{A}$ providing the enrichment, so as to to highlight whether we are seeing a morphism object as a space or as something more structured. If $\mathcal{C}$ is a higher

category of categories (e.g., $\mathscr{C} = \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$ or $\mathscr{C} = \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$) we will also use $\underline{\mathrm{Fun}}(-,-)$, possibly with decorations, to mean the category of structure-preserving functors which serves as the category of morphisms in $\mathscr{C}$.

Acknowledgements. This project took quite a long time to complete, and along the way we benefited enormously from conversations and email exchanges with friends and colleagues that helped us navigate the many subtle issues in the theory of Koszul duality. We owe special thanks to Francesco Battistoni, John Francis, Yonatan Harpaz, David Nadler, Mauro Porta, Pavel Safronov and German Stefanich.
The second author wishes to dedicate this work to the loving memory of his father, who passed away during the completion of this project.

1. Main setup

In this section we collect the main constructions and definitions concerning presentable n-categories ([Ste20]) and n-affineness of stacks ([Gai15; Ste21]). Both concepts have been introduced only very recently, so it is the opportunity to recall the main properties of presentable n-categories and of local systems of presentable n-categories over spaces (Section 1.1) and fix the main notations concerning n-affineness of stacks (Section 1.2).

1.1. Miscellanea on higher categories.

In this section, we fix our notations and collect the main results established in [PPS25] concerning presentable n-categories and local systems of presentable n-categories. which will be used extensively throughout the paper. We try to avoid the subtler technicalities and try to convey the main ideas – that is, that presentable n-categories provide the right generalization of the concept of presentable categories to the n-categorical setting, and that local systems of presentable n-categories are controlled by monodromy data just like ordinary local systems.

In what follows, κ_0 is the first large cardinal of our theory (i.e., the cardinal such that κ_0-small objects are precisely ordinary small objects).

1.1. Let $\mathscr{A}$ be a presentably symmetric monoidal category. Following Stefanich [Ste20], we can define for any $n \geqslant 1$ an $(n+1)$-category of presentable $\mathscr{A}$-linear n-categories that we denote as $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$. This is explicitly constructed as the sub-category of κ_0-compact objects inside the cocomplete category of iterated modules over $\mathscr{A}$ inside $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}$. This means that, if $\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}$ is the category of cocomplete n-categories, then one has that the underlying category of $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ is described as

$$\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} := \mathrm{Mod}_{n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}}\left(\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n)}\right)^{\kappa_0}.$$

Using the fact that $\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ is actually an n-category, one can produces an enhancement of $\mathrm{Lin}_{\mathscr{A}}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ to our desired $(n+1)$-category $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$.

When $\mathscr{A} = \mathrm{Mod}_{\Bbbk}$ is the category of modules over a commutative ring spectrum $\Bbbk$, we shall simply write $(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$.

Fact 1.2 ([Ste20, Section 5])**.** The $(n+1)$-category $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ enjoys the following properties.

1) For $n = 1$, this is just the 2-categorical incarnation of the category of $\mathscr{A}$-modules inside the 2-category $\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ of presentable categories. In particular, this agrees with the usual notion of presentably $\mathscr{A}$-linear categories.
2) For all $n \geqslant 1$, the $(n+1)$-category $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ is a commutative algebra object inside $(n+2)\widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,n+1)}$. This means that it admits all colimits and it is equipped with a symmetric monoidal structure which preserves them separately in each variable. The unit of such symmetric monoidal structure is provided by the presentable $\mathscr{A}$-linear n-category $n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$.
3) It admits all limits of left adjointable diagrams. That is, given a diagram of presentably $\mathscr{A}$-linear n-categories
$$I \longrightarrow (n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$$
such that every arrow $i \to j$ in I yields an n-functor $n\mathscr{C}(i) \to n\mathscr{C}(j)$ which admits a left adjoint, then the limit of such diagram exists and agrees with the colimit of opposite diagram
$$I^{\mathrm{op}} \longrightarrow (n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$$
where each n-functor is replaced with its left adjoint.
4) For A an $\mathbb{E}_k$-algebra object in a presentably symmetric monoidal category $\mathscr{A}$, one has for all integers $n \leqslant k-1$ an $(n+1)$-category of n-fold categorical modules inductively defined as
$$(n+1)\mathbf{LMod}^n_A(\mathscr{A}) := (n+1)\mathbf{LMod}_{n\mathbf{LMod}^{n-1}_A(\mathscr{A})}\Big((n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\Big),$$
where $1\mathbf{LMod}^0_A(\mathscr{A})$ is understood to be the ordinary category $\mathrm{LMod}_A(\mathscr{A})$ of left A-modules inside $\mathscr{A}$. This is again a presentable $\mathscr{A}$-linear $(n+1)$-category.
   When A is a $\mathbb{E}_k$ ring spectrum, then the $(n+1)$-category $(n+1)\mathbf{Mod}^n_A(\mathcal{S}\mathrm{p})$ agrees with the definition of $(n+1)\mathbf{Lin}_A\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ provided above.

1.3. Recall now from [PPS25, Section 1] that for any small space X and for any cocomplete category $\mathscr{C}$ we have a well defined category of *$\mathscr{C}$-valued local systems on X*
$$\mathrm{LocSys}(X;\mathscr{C}) := \mathrm{Fun}(X,\mathscr{C}),$$
which is equivalent to the category of hypersheaves over X which locally hyperconstant ([HPT23]). Applying this machinery to the cocomplete category $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$, we obtain an $(n+1)$-category of local systems of presentable $\mathscr{A}$-linear n-categories that we denote as
$$(n+1)\mathbf{LocSysCat}^n(X;\mathscr{A}) := (n+1)\mathbf{Fun}\Big(X, (n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\Big).$$

When $\mathscr{A} := \mathrm{Mod}_{\Bbbk}$ is the category of modules over some commutative ring spectrum $\Bbbk$, we shall trim our notations and denote such $(n+1)$-category simply as $(n+1)\mathbf{LocSysCat}^n(X;\Bbbk)$.

1.4. Whenever a category $\mathscr{A}$ is cocomplete, it is naturally tensored over the category of spaces: the action of a topological monoid G on an object A of $\mathscr{A}$ is expressed through the colimit of the diagram of shape G and constant value in A. It is known that for a presentable category $\mathscr{C}$, local systems over a connected space X with values in $\mathscr{C}$ are equivalently described as Ω_*X-modules inside $\mathscr{C}$. This can be generalized to all cocomplete categories, obtaining for an arbitrary cocomplete category $\mathscr{C}$ the equivalence

$$\mathrm{LocSys}(X;\mathscr{C}) \simeq \mathrm{LMod}_{\Omega_*X}(\mathscr{C}). \tag{1.5}$$

The main result of [PPS25] relates n-categorical local systems on an n-connected space X and monodromy data, expressed as an $(n+1)$-fold $\Omega_*^{n+1}X$-module structure on the fiber.

Theorem 1.6 ([PPS25, Theorem 3.2.24])**.** *Let $n \geqslant 1$ be an integer, let X be a pointed n-connected space (i.e., $\pi_k(X) \cong 0$ for every $k \leqslant n$), and let $\mathscr{A}$ be a presentably symmetric monoidal category. Then there exist equivalences of $(n+1)$-categories*

$$\begin{aligned}(n+1)\mathbf{LocSysCat}^n(X;\mathscr{A}) &\simeq (n+1)\mathbf{LMod}_{\Omega_*X}\big((n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}{}_{(\infty,n)}\big)\\ &\simeq (n+1)\mathbf{LMod}_{n\mathbf{LMod}^{n-1}_{\Omega_*^{n+1}X}(\mathscr{A})}\big((n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}{}_{(\infty,n)}\big).\end{aligned}$$

Remark 1.7 ([PPS25, Lemma 2.2])**.** When the category $\mathscr{A}$ is cocompletely symmetric monoidal with unit $\mathbb{1}_{\mathscr{A}}$ then the action of the space G can be expressed in terms of an algebra inside $\mathscr{A}$. Indeed, an object A of $\mathscr{A}$ is a left G-module if and only if it is a left $(G\otimes \mathbb{1}_{\mathscr{A}})$-module.

For example, when $\mathscr{A} = \mathrm{Mod}_{\Bbbk}$ is the category of $\Bbbk$-modules over a commutative ring spectrum $\Bbbk$, then the $\Bbbk$-algebra $G\otimes\Bbbk$ agrees with the $\Bbbk$-algebra of $\Bbbk$-valued chains $\mathrm{C}_\bullet(G;\Bbbk)$ with its Pontrjagin product. Therefore, a left G-module structure on a $\Bbbk$-module M is the same as a $\mathrm{C}_\bullet(G;\Bbbk)$-module structure.

When $\mathscr{A} = \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is the category of presentable $\Bbbk$-linear n-categories, then the presentable monoidal n-category $G\otimes n\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ is equivalent to the n-category $n\mathbf{LocSysCat}^{n-1}(G;\Bbbk)$ of presentable $\mathscr{A}$-linear local systems over G, equipped with the Day convolution tensor product ([PPS25, Lemma 3.2.28]).

Applying these observations to Theorem 1.6, we see that when X is n-connected then one can express local systems of presentable $\Bbbk$-linear n-categories over X as presentable $\mathrm{C}_\bullet(\Omega_*^{n+1}X;\Bbbk)$-linear n-categories.

For future reference, we also report the following key result on the ambidexterity of limits and colimits of presentable n-categories indexed over groupoids/spaces.

Lemma 1.8 ([PPS25, Lemma 3.2.29]))**.** *Let $\mathscr{A}$ be a presentably symmetric monoidal category, let X be a space, and let $F\colon X \to (n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ be a diagram of presentable $\mathscr{A}$-linear n-categories of shape X. Then, there is a natural equivalence* $\lim F \simeq \operatorname{colim} F$ *in* $(n+1)\mathbf{Lin}_{\mathscr{A}}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$.

1.2. **The notion of n-affineness.** Here we introduce the problem of n-affineness for arbitrary stacks. We shall use our results in this section as a stepping stone for our study of general n-affineness properties of Betti stacks in Section 2.2 below. We refer the reader to the Introduction for a thorough discussion of the relationship between n-affineness and higher Koszul duality. Before introducing the general case for arbitrary n, we start by reviewing basic definitions and results established in [Gai15].

Construction 1.9 ([Gai15, Section 1.1])**.** Let $\Bbbk$ be an $\mathbb{E}_\infty$-ring spectrum, and denote by $\mathrm{Aff}_\Bbbk$ the category of affine schemes over $\Bbbk$, i.e., the opposite category of the category $\mathrm{CAlg}_\Bbbk^{\geqslant 0}$ of connective and commutative $\Bbbk$-algebras. Let $\mathrm{PSt}_\Bbbk$ be the category of *prestacks over* $\Bbbk$, i.e., the category of accessible presheaves over the category $\mathrm{Aff}_\Bbbk$. The functor

$$\mathrm{ShvCat}\colon \mathrm{PSt}_\Bbbk \longrightarrow \mathrm{Lin}_\Bbbk \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$$

is by definition the right Kan extension of the functor

$$\mathrm{Lin}_{(-)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\colon \mathrm{Aff}_\Bbbk^{\mathrm{op}} \simeq \mathrm{CAlg}_\Bbbk \longrightarrow \mathrm{Lin}_\Bbbk \widehat{\mathrm{Cat}}^{\mathrm{rex}}_{(\infty,1)}$$

along the Yoneda embedding $よ\colon \mathrm{Aff}_\Bbbk^{\mathrm{op}} \to \mathrm{PSt}_\Bbbk^{\mathrm{op}}$.

Definition 1.10. Let $\mathcal{X}$ be a prestack. Then the category $\mathrm{ShvCat}(\mathcal{X})$ is the *category of quasi-coherent sheaves of ($\Bbbk$-linear) categories over* $\mathcal{X}$.

If $\mathcal{F}$ is a quasi-coherent sheaf of categories over $\mathcal{X}$, we have a well defined functor

$$\Gamma(-,\mathcal{F})\colon \left(\mathrm{Aff}_{\Bbbk/\mathcal{X}}\right)^{\mathrm{op}} \longrightarrow \mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

which we can right Kan extend to get the functor

$$\Gamma(-,\mathcal{F})\colon \mathrm{PSt}^{\mathrm{op}}_{\Bbbk/\mathcal{X}} \longrightarrow \mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

By fixing the prestack to be $\mathcal{X}$ itself, for any quasi-coherent sheaf of categories over $\mathcal{X}$ the $\Bbbk$-linear category of its global section is actually acted on by the stable category $\mathrm{QCoh}(\mathcal{X})$. Hence, we deduce the existence of a *global section functor*

$$\Gamma^{\mathrm{enh}}(\mathcal{X},-)\colon \mathrm{ShvCat}(\mathcal{X}) \longrightarrow \mathrm{Lin}_{\mathrm{QCoh}(\mathcal{X})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \tag{1.11}$$

which is right adjoint to the sheafification functor

$$\mathrm{Loc}_\mathcal{X}\colon \mathrm{Lin}_{\mathrm{QCoh}(\mathcal{X})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{ShvCat}(\mathcal{X}). \tag{1.12}$$

The latter acts on objects by sending a presentably $\mathrm{QCoh}(\mathcal{X})$-linear category $\mathcal{C}$ to the quasi-coherent sheaf of categories obtained by sheafifying the assignment

$$\mathrm{Spec}(S) \mapsto \mathrm{QCoh}(S) \otimes_{\mathrm{QCoh}(\mathcal{Y})} \mathcal{C}.$$

Definition 1.13 ([Gai15, Definition 1.3.7])**.** A prestack $\mathcal{X}$ is 1-*affine* if $\Gamma^{\mathrm{enh}}(\mathcal{X},-)$ and $\mathrm{Loc}_\mathcal{X}$ are mutually inverse equivalences.

Remark 1.14. Definition 1.13 has to be interpreted as a generalization of affineness, in the following sense. When $X = \operatorname{Spec}(R)$ is an affine scheme, then there is a canonical equivalence of stable categories

$$\operatorname{QCoh}(X) \simeq \operatorname{Mod}_{\Gamma(X,\mathscr{O}_X)}. \tag{1.15}$$

In [Gai15], stacks for which the equivalence (1.15) holds are called *weakly* 0-*affine*. Let us remark that, actually, the class of weakly 0-affine stacks (in this sense) sits between the class of affine schemes and an even weaker notion of 0-affineness. Indeed, if $\mathscr{X}$ is an arbitrary stack one could define a notion of 0-affineness by asking that the global sections functor

$$\Gamma(\mathscr{X},-)\colon \operatorname{QCoh}(\mathscr{X}) \to \operatorname{Mod}_{\mathbb{k}}$$

is monadic. If $\mathscr{X}$ is a weakly 0-affine stack in the sense of Gaitsgory, then the global sections are trivially monadic over $\operatorname{Mod}_{\mathbb{k}}$. However, this latter condition is *weaker*. For the rest of this Remark, we shall call a stack $\mathscr{X}$ that satisfies this condition *almost 0-affine*.

Let us explain the difference between Gaitsgory's weak 0-affineness, and our notion of almost 0-affineness. In virtue of the Schwede–Shipley recognition principle for stable categories of modules in spectra ([Lur17, Proposition 7.1.2.6]), if $\mathscr{X}$ is weakly 0-affine then $\mathscr{O}_{\mathscr{X}}$ has to be a compact generator of $\operatorname{QCoh}(\mathscr{X})$. This means that the functor

$$\Gamma(\mathscr{X},-) \simeq \underline{\operatorname{Map}}_{\operatorname{QCoh}(\mathscr{X})}(\mathscr{O}_{\mathscr{X}},-)$$

has to reflect equivalences and preserve all colimits. But for $\Gamma(\mathscr{X},-)$ to be monadic it is sufficient that it reflects equivalences and preserves only a special class of colimits – namely, colimits of $\Gamma(\mathscr{X},-)$-split simplicial objects ([Lur17, Theorem 4.7.3.5]). This implies that if $\mathscr{X}$ is almost 0-affine then $\mathscr{O}_{\mathscr{X}}$ has to be a generator (although this is not a sufficient condition): but it might very well fail to be a *compact* generator.

More generally, if $\mathscr{C}$ is a stable symmetric monoidal category, it can happen that the monoidal unit $\mathbb{1}$ is not compact, but the functor it corepresents preserves colimits of $\underline{\operatorname{Map}}_{\mathscr{C}}(\mathbb{1}_{\mathscr{C}},-)$-split simplicial objects. A particularly easy example is the following: take $\mathscr{C}$ to be a countable product of copies of $\operatorname{Mod}_{\mathbb{k}}$ (which can be interpreted as the category of local systems over the discrete space $\mathbb{Z}$). We will deduce from Lemma 2.12 that the functor of global sections is monadic, yet the monoidal unit $\mathbb{1}_{\mathbb{Z}}$ (which consists of the constant sequence $(\mathbb{k})_{n\in\mathbb{Z}}$) is not compact. Indeed, the global sections functor is equivalent to the functor

$$(M_n)_{n\in\mathbb{Z}} \mapsto \underline{\operatorname{Map}}_{\mathscr{C}}(\mathbb{1}_{\mathbb{Z}}, M_n) \simeq \prod_{n\in\mathbb{Z}} \underline{\operatorname{Map}}_{\mathbb{k}}(\mathbb{k}, M_n) \simeq \prod_{n\in\mathbb{Z}} M_n$$

and in general infinite colimits do not commute with infinite products. Another, highly non-trivial example of the difference between these two notions is provided by $\mathbb{CP}^\infty$ (Corollary 3.41).

This issue persists in the categorified setting. This means that Definition 1.13 has to be interpreted as a "strong" notion of 1-affineness, as is a direct categorification of Gaitsgory's weak 0-affineness. By the same token, we could define *almost* 1-affineness by requiring the mere monadicity of the global sections functor. These two notions are, in general, genuinely different. However, as we shall explain in Porism 2.6 below, for Betti stacks the situation is simpler, as almost 1-affineness implies 1-affineness. This will entail a significant simplification of some of our arguments.

We now present the general definition of n-affineness for arbitrary n. First, using Fact 1.2.(4), for any $n \geqslant 0$ we can construct a functor

$$\begin{aligned}(n+1)\mathbf{Mod}^n_{(-)}\colon \mathrm{Aff}^{\mathrm{op}}_{\Bbbk} \simeq \mathrm{CAlg}^{\geqslant 0}_{\Bbbk} &\longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n+1)}\\ \mathrm{Spec}(R) &\mapsto (n+1)\mathbf{Mod}^n_R\end{aligned}$$

sending an affine scheme $\mathrm{Spec}(R)$ to its $(n+1)$-category of n-fold R-modules (or equivalently of R-linear presentable n-categories, when $n \geqslant 1$).

Definition 1.16 ([Ste21, Definition 14.2.4])**.** Let $\mathcal{X}$ be a prestack defined over a commutative ring spectrum $\Bbbk$, and let $n \geqslant 1$ be an integer. The $(n+1)$*-category* $(n+1)\mathbf{ShvCat}^n(\mathcal{X})$ *of quasi-coherent sheaves of ($\Bbbk$-linear presentable) n-categories* over $\mathcal{X}$ is defined as the right Kan extension of the functor $(n+1)\mathbf{Mod}^n_{(-)}$ along the inclusion $\mathrm{Aff}^{\mathrm{op}}_{\Bbbk} \subseteq \mathrm{PSt}^{\mathrm{op}}_{\Bbbk}$.

This means that $(n+1)\mathbf{ShvCat}^n(\mathcal{X})$ is the limit computed inside $\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n+1)}$

$$(n+1)\mathbf{ShvCat}^n(\mathcal{X}) := \lim_{\substack{\mathrm{Spec}(R)\to\mathcal{X}\\ R\in\mathrm{CAlg}^{\geqslant 0}_{\Bbbk}}} (n+1)\mathbf{Mod}^n_R \simeq \lim_{\substack{\mathrm{Spec}(R)\to\mathcal{X}\\ R\in\mathrm{CAlg}^{\geqslant 0}_{\Bbbk}}} (n+1)\mathbf{Lin}_R\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}.$$

In particular, Definition 1.16 agrees with Construction 1.9 when $n=1$.

Remark 1.17. For $n \geqslant 2$, the right Kan extension defining the $(n+1)$-category of quasi-coherent sheaves of n-categories in Definition 1.16 is also a *left* Kan extension. Indeed, for any morphism of prestacks $f\colon \mathcal{X} \to \mathcal{Y}$ and for any $n \geqslant 2$ the pullback functor $f^*\colon \mathrm{ShvCat}^n(\mathcal{Y}) \to \mathrm{ShvCat}^n(\mathcal{X})$ is part of an ambidextrous adjunction ([Ste21, Corollary 14.2.10]). In particular, [Ste20, Theorem 5.5.14] implies that the limit inside $(n+2)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n+1)}$ along the pullback $(n+1)$-functors corresponds to the colimit inside $(n+2)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n+1)}$ along the colimit-preserving pushforward $(n+1)$-functors. This also holds for $n=1$ if the morphism f is assumed to be affine schematic.

Just like for the case $n=1$, we have a naturally defined global sections $(n+1)$-functor

$$(n+1)\mathbf{\Gamma}(\mathcal{X},-)\colon (n+1)\mathbf{ShvCat}^n(\mathcal{X}) \longrightarrow (n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$$

which for any quasi-coherent sheaf of n-categories $n\mathcal{F}$ over $\mathcal{X}$ computes the limit over all local sections

$$\lim_{\substack{\mathrm{Spec}(R)\to\mathcal{X}\\ R\in\mathrm{CAlg}^{\geqslant 0}_{\Bbbk}}} (n+1)\mathbf{\Gamma}(\mathrm{Spec}(R), n\mathcal{F})$$

inside $(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$.

Definition 1.18. Let $\mathcal{X}$ be a prestack. We say that $\mathcal{X}$ is *n-affine* if the global sections $(n+1)$-functor

$$(n+1)\mathbf{\Gamma}(\mathcal{X},-)\colon (n+1)\mathbf{ShvCat}^{n}(\mathcal{X})\longrightarrow (n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$$

is a monadic morphism in the $(\infty,2)$-category $(n+2)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n+1)}$ of $\Bbbk$-linear presentable $(n+1)$-categories.

Remark 1.19. Thanks to [Ste21, Proposition 14.3.6], we know that for $n\geqslant 2$ the monadicity requirement for n-affineness can be checked at the level of the underlying categories – i.e., the $(n+1)$-functor $(n+1)\mathbf{\Gamma}(\mathcal{X},-)$ is a monadic morphism of presentable $\Bbbk$-linear $(n+1)$-categories if and only if the underlying functor of ordinary categories

$$\Gamma(\mathcal{X},-)\colon\ \mathrm{ShvCat}^{n}(\mathcal{X})\longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

is monadic.

However, in the case $n=1$ Definition 1.18 recovers Definition 1.13 ([Ste21, Remark 14.3.8]). As explained in Remark 1.14, this is is a stronger requirement than asking simply for the monadicity of the global sections functor: indeed, this is equivalent to asking for it to be a *colimit-preserving* monadic right adjoint. The fact that, for $n\geqslant 2$, this issue does not arise boils down to the fact that pullbacks and pushforwards between presentable $(n+1)$-categories of quasi-coherent sheaves of n-categories form an *ambidextrous* adjunction in $(n+2)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n+1)}$, as already mentioned in Remark 1.17. In particular, if $n\geqslant 2$ the monadicity requirement of Definition 1.18 yields a $\Bbbk$-linear equivalence of presentable $(n+1)$-categories

$$(n+1)\mathbf{ShvCat}^{n}(\mathcal{X})\simeq (n+1)\mathbf{Mod}_{n\mathbf{ShvCat}^{n-1}(\mathcal{X})}\Big((n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}\Big).$$

2. Betti stacks and n-affineness

2.1. **Betti stacks and 1-affineness.** Let $\mathrm{St}_{\Bbbk}$ be the category of *stacks over* $\Bbbk$, i.e., the full subcategory of $\mathrm{PSt}_{\Bbbk}$ spanned by those prestacks which are hypercomplete sheaves with respect to the étale topology over $\mathrm{Aff}_{\Bbbk}$. The natural functor $\mathrm{Aff}_{\Bbbk}\longrightarrow\{*\}$ induces a functor

$$(-)_{\mathrm{B}}\colon\ \mathrm{Shv}(\{*\})\simeq\mathcal{S}\longrightarrow \mathrm{St}_{\Bbbk}$$

which corresponds to sending a space X to the sheafification of the constant prestack

$$\mathrm{Aff}^{\mathrm{op}}_{\Bbbk}\longrightarrow\{*\}\xrightarrow{X}\mathcal{S}.$$

Definition 2.1. The functor $(-)_{\mathrm{B}}\colon\mathcal{S}\to\mathrm{St}_{\Bbbk}$ is the *Betti stack functor*.

2.2. Betti stacks are intimately linked to the theory of local systems over spaces. Indeed, for every space X we can consider the category $\mathrm{QCoh}\,(X_{\mathrm{B}})$ of quasi-coherent sheaves over its associated Betti stack, and for every affine scheme $\mathrm{Spec}(R)$ over $\Bbbk$ one has a symmetric monoidal equivalence of stable categories

$$\mathrm{QCoh}\,(X_{\mathrm{B}} \times \mathrm{Spec}(R)) \simeq \mathrm{LocSys}(X;R),$$

where on the right hand side we are considering the point-wise tensor product, as proved in [PS20, Proposition 3.1.1]. In particular, taking R to be $\Bbbk$ in the above formula yields

$$\mathrm{QCoh}\,(X_{\mathrm{B}} \times \mathrm{Spec}(\Bbbk)) \simeq \mathrm{QCoh}\,(X_{\mathrm{B}}) \simeq \mathrm{LocSys}(X;\Bbbk).$$

Analogously, at a categorified level, it turns out that quasi-coherent sheaves of categories over X_{B} recover categorical local systems over X.

Lemma 2.3. *For any space X and for any base commutative ring spectrum $\Bbbk$, we have an equivalence of categories*

$$\mathrm{ShvCat}\,(X_{\mathrm{B}}) \simeq \mathrm{LocSysCat}\,(X;\Bbbk),$$

where X_{B} is seen as a stack over $\Bbbk$.

Proof. The functor $(-)_{\mathrm{B}} \colon \mathcal{S} \to \mathrm{St}_{\Bbbk}$ is a pullback functor between categories of sheaves, hence it obviously commutes with colimits. Presenting X as a colimit of its contractible cells yields hence equivalences

$$X_{\mathrm{B}} \simeq \operatorname*{colim}_{\{*\}\to X} \{*\}_{\mathrm{B}} \simeq \operatorname*{colim}_{\mathrm{Spec}(\Bbbk)\to X_{\mathrm{B}}} \mathrm{Spec}(\Bbbk),$$

where the last equivalence is due to [PS20, Proposition 3.1.1]. Since the functor ShvCat sends colimits of prestacks to limits of categories ([Gai15, Lemma 1.1.3]), we would like to conclude that

$$\mathrm{ShvCat}\,(X_{\mathrm{B}}) \simeq \mathrm{ShvCat}\left(\operatorname*{colim}_{\mathrm{Spec}(\Bbbk)\to X_{\mathrm{B}}} \mathrm{Spec}(\Bbbk)\right) \simeq \lim_{\mathrm{Spec}(\Bbbk)\to X_{\mathrm{B}}} \mathrm{ShvCat}\,(\mathrm{Spec}(\Bbbk)),$$

but the colimit inside the brackets is a colimit of *stacks*, which is in general different from the colimit of *prestacks*: the former is computed by sheafifying the latter, i.e., by applying the left adjoint to the inclusion $\mathrm{St}_{\Bbbk} \subseteq \mathrm{PSt}_{\Bbbk}$. However, the functor ShvCat is a sheaf for the fppf topology ([Gai15, Theorem 1.5.7]), hence for the étale topology; in particular, it factors through the sheafification functor $\mathrm{PSt}_{\Bbbk} \to \mathrm{St}_{\Bbbk}$. So, we indeed deduce that

$$\mathrm{ShvCat}\,(X_{\mathrm{B}}) \simeq \lim_{X} \mathrm{ShvCat}\,(\mathrm{Spec}(\Bbbk)) \simeq \lim_{X} \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)},$$

where in the second equivalence we used the fact that every affine scheme is tautologically 1-affine. On the other hand, we already know that

$$\mathrm{LocSysCat}\,(X;\Bbbk) \simeq \lim_{X} \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)},$$

hence the two expressions match. □

Combining Lemma 2.3 with Remark 1.7, we obtain the following.

Corollary 2.4. *For any simply connected space X and any $\mathbb{E}_\infty$-ring spectrum $\Bbbk$ we have an equivalence of categories*

$$\mathrm{ShvCat}\,(X_{\mathrm{B}}) \simeq \mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}\Big(\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big) \simeq \mathrm{Lin}_{\mathrm{C}_\bullet(\Omega^2_* X;\Bbbk)}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\,.$$

Our main result in this section is a characterization of 1-affine Betti stacks. We start by proving, in the next Proposition, that for a Betti stacks X_{B} the functor $\mathrm{Loc}_{X_{\mathrm{B}}}$ is always fully faithful.

Proposition 2.5. *Let X be a space. Then*

$$\mathrm{Loc}_{X_{\mathrm{B}}}\colon\ \mathrm{Lin}_{\mathrm{QCoh}(X_{\mathrm{B}})}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{ShvCat}(X_{\mathrm{B}})$$

is fully faithful.

Proof. Consider the following diagram of categories.

$$\begin{array}{ccc}
\mathrm{ShvCat}(X_{\mathrm{B}}) \simeq \mathrm{LocSysCat}(X;\Bbbk) & \xrightarrow{\ \Gamma^{\mathrm{enh}}(X_{\mathrm{B}},-)\ } & \mathrm{Lin}_{\mathrm{QCoh}(X_{\mathrm{B}})}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \simeq \mathrm{Lin}_{\mathrm{LocSys}(X;\Bbbk)}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \\
 & \searrow^{\Gamma(X,-)} \qquad \swarrow_{\mathrm{oblv}_{\mathrm{LocSys}(X;\Bbbk)}} & \\
 & \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} &
\end{array}$$

Here, $\Gamma(X,-)\colon \mathrm{LocSysCat}(X) \to \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ is the functor which takes global sections of a local system of $\Bbbk$-linear presentable categories: this amounts to taking the limit over the diagram of presentable categories of shape X defined by a local system of categories. Since limits and colimits over spaces in $\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ agree (Lemma 1.8), this functor is both a right and left adjoint to the constant local system functor. In particular, it is a right adjoint that preserves *all* geometric realizations.

Analogously,

$$\mathrm{oblv}_{\mathrm{LocSys}(X;\Bbbk)}\colon\ \mathrm{Lin}_{\mathrm{LocSys}(X;\Bbbk)}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

simply forgets the $\mathrm{LocSys}(X;\Bbbk)$-module structure of a $\Bbbk$-linear presentable category. Again, this obviously commutes with both limits and colimits (hence, with geometric realizations) and it is a right adjoint to the base change functor

$$-\otimes_{\mathrm{Mod}_{\Bbbk}} \mathrm{LocSys}(X;\Bbbk)\colon \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \longrightarrow \mathrm{Lin}_{\mathrm{LocSys}(X;\Bbbk)}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\,.$$

Moreover, such functor is conservative (it is actually monadic).

Notice that this diagram does commute. Indeed, under the equivalence of Lemma 2.3, the global sections of a quasi-coherent sheaf of categories $\mathcal{F}$ over the Betti stack X_{B} correspond to the "enhanced" global sections of a local system of categories over X, taking into account the natural tensor action of $\mathrm{LocSys}(X;\Bbbk)$ over them. This is simply a categorification of the fact that global sections of local systems of $\Bbbk$-modules over a space are endowed with an action of the algebra of its $\Bbbk$-cochains. In particular, forgetting such action recovers the underlying $\Bbbk$-linear presentable category $\Gamma(X,\mathcal{F})$.

Summarizing, the above diagram is a commutative diagram of categories where the arrow on the left is a right adjoint which preserves geometric realizations and the arrow on the right is a *conservative* right adjoint which preserves geometric realizations. Moreover, for any $\mathbb{k}$-linear presentable category $\mathscr{C}$ we have that the unit map for the adjunction $\mathrm{const} \dashv \Gamma(X,-)$

$$\mathscr{C} \longrightarrow \Gamma(X, \mathrm{const}(\mathscr{C})) \simeq \mathrm{oblv}_{\mathrm{LocSys}(X;\mathbb{k})} \Gamma^{\mathrm{enh}}(X_{\mathrm{B}}, \mathrm{const}(\mathscr{C}))$$

produces by adjunction the map

$$\mathscr{C} \otimes_{\mathrm{Mod}_{\mathbb{k}}} \mathrm{LocSys}(X;\mathbb{k}) \longrightarrow \Gamma^{\mathrm{enh}}(X_{\mathrm{B}}, \mathrm{const}(\mathscr{C}))\,.$$

If this was an equivalence, then invoking [Lur17, Corollary 4.7.3.16] one would deduce the existence of a left adjoint to $\Gamma^{\mathrm{enh}}(X_{\mathrm{B}},-)$ (which of course has to be $\mathrm{Loc}_{X_{\mathrm{B}}}$) and this left adjoint is moreover *fully faithful*. Since forgetting the $\mathrm{LocSys}(X;\mathbb{k})$-module structure is conservative, we can check whether this map is an equivalence at the level of the underlying $\mathbb{k}$-linear categories. On one side, we can write the domain of the above map as

$$\begin{aligned}\mathscr{C} \otimes_{\mathrm{Mod}_{\mathbb{k}}} \mathrm{LocSys}(X;\mathbb{k}) &\simeq \mathscr{C} \otimes_{\mathrm{Mod}_{\mathbb{k}}} \lim_X \mathrm{Mod}_{\mathbb{k}} \simeq \mathscr{C} \otimes_{\mathrm{Mod}_{\mathbb{k}}} \operatorname*{colim}_X \mathrm{Mod}_{\mathbb{k}} \\ &\simeq \operatorname*{colim}_X \mathscr{C} \otimes_{\mathrm{Mod}_{\mathbb{k}}} \mathrm{Mod}_{\mathbb{k}} \simeq \operatorname*{colim}_X \mathscr{C} \simeq \lim_X \mathscr{C},\end{aligned}$$

thanks to the fact that one can swap limits and colimits of presentable categories indexed by groupoids (Lemma 1.8). On the other hand, the codomain of this map is

$$\mathrm{oblv}_{\mathrm{LocSys}(X)} \Gamma^{\mathrm{enh}}(X_{\mathrm{B}}, \mathrm{const}(\mathscr{C})) \simeq \lim_X \Gamma(\{*\}_{\mathrm{B}}, \mathrm{const}(\mathscr{C})) \simeq \lim_X \mathscr{C},$$

so we immediately deduce our claim. □

Porism 2.6. The proof of Proposition 2.5 relies crucially on [Lur17, Corollary 4.7.3.16]. As an immediate consequence of that result, it follows that the fully faithful left adjoint $\mathrm{Loc}_{X_{\mathrm{B}}}$ is an equivalence precisely if the global sections functor is conservative, because this is the only obstruction to its monadicity.

Moreover, the proof of Proposition 2.5 identifies the global sections functor $\Gamma(X_{\mathrm{B}},-)$ with the global sections functor for local systems of categories over X, which in turn is realized as a limit of presentable stable categories indexed by the space X. Since limits and colimits of presentable categories over diagrams indexed by spaces are canonically equivalent (Lemma 1.8) it follows that in this case the global sections *always* commute with *all* colimits. This simplifies greatly the discussion in Remark 1.14 and implies that for Betti stacks the weak and strong notions of 1-affineness agree.

In order to give a characterization of 1-affine Betti stacks we need a few preliminary results. First, we show that we can reduce to consider connected spaces.

Lemma 2.7. *Let X be a disjoint union of (possibly infinitely many) connected components*

$$X = \bigcup_{\alpha \in A} X_\alpha$$

such that the Betti stack of each connected component is 1*-affine. Then* X_{B} *is* 1*-affine as well.*

Proof. Under our assumptions, the global sections functor

$$\Gamma(X,-)\colon \operatorname{LocSysCat}(X;\Bbbk) \longrightarrow \operatorname{Lin}_{\Bbbk}\operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}$$

is simply the product of all the global sections functors

$$\Gamma(X_\alpha,-)\colon \operatorname{LocSysCat}(X_\alpha;\Bbbk) \longrightarrow \operatorname{Lin}_{\Bbbk}\operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

By Porism 2.6, it follows that we only need to check that taking products of conservative functors is conservative. So suppose

$$F \simeq (F_\alpha)\colon \mathscr{F} \simeq (\mathscr{F}_\alpha) \longrightarrow \mathscr{G} \simeq (\mathscr{G}_\alpha)$$

is a map between categorical local systems on X, and suppose it becomes an equivalence after applying $\Gamma(X,-)$. Since $\operatorname{Lin}_{\Bbbk}\operatorname{Pr}^{\mathrm{L}}_{(\infty,1)}$ is pointed, we have a commutative diagram

$$\begin{array}{ccc}
\Gamma(X,\mathscr{F}_\alpha) & \xrightarrow{\Gamma(X_\alpha,F_\alpha)} & \Gamma(X,\mathscr{G}_\alpha) \\
\downarrow{\scriptstyle \iota_\alpha} & & \downarrow{\scriptstyle \iota_\alpha} \\
\prod_\alpha \Gamma(X,\mathscr{F}_\alpha) & \xrightarrow{\prod_\alpha \Gamma(X_\alpha,F_\alpha)} & \prod_\alpha \Gamma(X,\mathscr{G}_\alpha) \\
\downarrow{\scriptstyle \pi_\alpha} & & \downarrow{\scriptstyle \pi_\alpha} \\
\Gamma(X,\mathscr{F}_\alpha) & \xrightarrow{\Gamma(X_\alpha,F_\alpha)} & \Gamma(X,\mathscr{G}_\alpha)
\end{array}$$

that exhibits each $\Gamma(X_\alpha,F_\alpha)$ as a retract of the equivalence $\Gamma(X,F) \simeq \prod \Gamma(X_\alpha,F_\alpha)$; in particular, they are equivalences as well. But since each $(X_\alpha)_{\mathrm{B}}$ is 1-affine, it follows that each $\Gamma(X_\alpha,-)$ is conservative; in particular, F_α is an equivalence for each α. So, F must be an equivalence as well. □

Using Lemma 2.7, we can restrict attention to connected spaces. The next result give a complete characterization of Betti stacks that are 1-affine. A drawback of our result is that the necessary and sufficient condition we find is not very easy to verify in practice. In the remainder of the Section we shall complement this result by providing more explicit criteria for 1-affineness and its failure.

Theorem 2.8. *Let X be a pointed and connected space. Then* X_{B} *is* 1*-affine if and only if the global section functor*

$$\Gamma(\Omega_* X,-) : \operatorname{LocSys}(\Omega_* X;\Bbbk) \longrightarrow \operatorname{Mod}_{\Bbbk}$$

is monadic.

Remark 2.9. Before proceeding with the proof of Theorem 2.8 let us comment on its statement. Under what conditions is the global section functor monadic? In Theorem 2.8 we look at the loop space of a connected space X, but this question makes sense for a general space Y. We touched upon this question in the general setting of stacks in Remark 1.14

above, where we called a stack for which this property holds *almost* 0-*affine*. We can say something more in the case of Betti stacks. It turns out that the answer is quite subtle. As explained in Remark 1.14, we have the following simple facts:

(1) The functor $\Gamma(Y,-)$ is monadic, i.e. Y_{B} is almost 0-affine, *only if* the trivial local system $\underline{\Bbbk}_Y$ is a generator of $\mathrm{LocSys}(Y;\Bbbk)$
(2) The functor $\Gamma(Y,-)$ is monadic *if* the trivial local system $\underline{\Bbbk}_Y$ is a compact generator of $\mathrm{LocSys}(Y;\Bbbk)$. Indeed, in this case Y_{B} is weakly 0-affine in the sense of Gaitsgory, and therefore in particular almost 0-affine. In turn, by Schwede–Shipley ([Lur17, Proposition 7.1.2.6]) $\underline{\Bbbk}_Y$ is a compact generator if and only if the global section functor induces an equivalence

$$\mathrm{LocSys}(Y;\Bbbk) \simeq \mathrm{Mod}_{\mathrm{C}^\bullet(Y;\Bbbk)}. \tag{2.10}$$

In [BN12, Corollary 3.18] it is stated that equivalence (2.10) holds for all simply connected and finite spaces. However, as pointed out to us by Y. Harpaz and confirmed by D. Nadler in private communication, this statement is wrong. Consider for example $Y := S^2$ to be the sphere. The algebra of $\Bbbk$-chains on its based loop space $\Omega_* Y$ is a free associative algebra $\Bbbk\langle u\rangle$ generated by a variable u lying in homological degree 1. Then the functor $\underline{\mathrm{Map}}_{\mathrm{C}_\bullet(\Omega_* Y;\Bbbk)}(\Bbbk,-)$ cannot be conservative, because the non-trivial $\Bbbk\langle u\rangle$-module $\Bbbk\langle u,u^{-1}\rangle$ is right orthogonal to $\Bbbk$. In particular, $\Bbbk$ cannot be a *generator* of $\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* Y;\Bbbk)}$: and so in particular it cannot be a *compact generator*.

Equivalence (2.10) holds for 0-truncated spaces, and we believe that in fact this might be a necessary condition. We do not know of any non-trivial space satisfying (2.10). On the other hand a characterization of almost 0-affine Betti stacks would be very interesting, but it seems difficult to achieve, and we have only partial results in this direction. In Lemma 2.17, we show that the non-triviality of $\pi_1(Y,y)$ at any base point obstructs the monadicity of the global sections functor of local systems over Y, which is perhaps an expected result. In Corollary 3.41 we will also show that the Betti stack of $\mathbb{CP}^\infty$ is almost 0-affine, so we do have non-trivial examples. We leave the further exploration of these questions to future work.

Let us now proceed with the proof of Theorem 2.8. The key ingredient in the proof is the following result of Gaitsgory.

Proposition 2.11 ([Gai15, Proposition 11.2.1])**.** *Let G be a group prestack. Assume that* Loc_G *is fully faithful, that* $\mathrm{QCoh}(G)$ *is dualizable as a* $\Bbbk$*-linear presentable category, and that the convolution tensor product on* $\mathrm{QCoh}(G)$ *turns it into a rigid monoidal category. The following are equivalent.*

(1) The stack $\mathbf{B}G$ *is 1-affine.*
(2) The global sections functor $\mathrm{QCoh}(G) \to \mathrm{Mod}_{\Bbbk}$ *is monadic.*

Proof of Theorem 2.8. Let $G := \Omega_* X$ be the based loop space of X. Since the Betti stack functor is the left adjoint in a geometric morphism between categories of sheaves and as such it preserves products, we have that $(\Omega_* X)_{\mathrm{B}}$ is still a group stack (hence a group prestack, since products are preserved by the inclusion $\mathrm{St}_{\Bbbk} \subseteq \mathrm{PSt}_{\Bbbk}$ and

$$\Omega_*(X_{\mathrm{B}}) \simeq (\Omega_* X)_{\mathrm{B}}\,.$$

We set $G := \Omega_*(X_{\mathrm{B}}) \simeq (\Omega_* X)_{\mathrm{B}}$ and note that X_{B} is realized as the delooping of G in the category $\mathrm{PSt}_{\Bbbk}$. In formulas, we can write

$$X_{\mathrm{B}} \simeq \mathbf{B}G.$$

Let us check that also the other assumptions of Proposition 2.11 hold in our situation. Observe first that

$$\mathrm{QCoh}(G) \simeq \mathrm{LocSys}(\Omega_* X; \Bbbk)$$

is compactly generated. If $\Omega_* X$ is connected this follows because $\mathrm{LocSys}(\Omega_* X; \Bbbk)$ is equivalent to the category of modules over $\mathrm{C}_\bullet(\Omega_* \Omega_* X; \Bbbk)$, and is therefore compactly generated. Since compactly generated categories are stable under products ([Lur09, Proposition 5.5.7.6]), we conclude that compact generation holds in general also if $\Omega_* X$ fails to be connected. Thus $\mathrm{QCoh}(G)$ is in particular a dualizable $\Bbbk$-linear presentable categories (thanks to [Lur18, Theorem D.7.0.7]).

Moreover, the classical monodromy equivalence (1.5) implies that $\mathrm{QCoh}(G)$ decomposes (non-canonically) as a product of categories of left modules

$$\mathrm{QCoh}(G) \simeq \prod_{\alpha \in \pi_0(\Omega_* X)} \mathrm{LMod}_{\Bbbk \otimes \Omega^2_* X}.$$

The convolution tensor product on the left hand side translates into a "Künneth-like" relative tensor product

$$(M_\alpha)_\alpha \otimes (N_\alpha)_\alpha \simeq \left(\bigoplus_{\beta \cdot \gamma = \alpha} M_\beta \otimes_{\Omega^2_* X} N_\gamma \right)_\alpha,$$

where $\cdot$ denotes the group law on $\pi_0(\Omega_* X) \cong \pi_1(X)$. Since we already proved that $\mathrm{QCoh}(G)$ is compactly generated, in order to prove that it is rigid it is sufficient to prove that every compact object is fully dualizable ([Gai15, Section D.1.3]). Notice that each factor $\mathrm{LMod}_{\Bbbk \otimes \Omega^2_* X}$ is a rigid monoidal category, since in a category of left modules over an $\mathbb{E}_2$-ring spectrum equipped with its relative tensor product compact objects are known to be precisely the class of fully dualizable objects. Since

$$\underline{\mathrm{Map}}_{\mathrm{QCoh}(G)}((M_\alpha)_\alpha, (N_\alpha)_\alpha) \simeq \prod_{\alpha \in \pi_1(X)} \underline{\mathrm{Map}}_{\Bbbk \otimes \Omega^2_* X}(M_\alpha, N_\alpha),$$

is not difficult to see that a collection of $\Omega^2_* X$-modules $(M_\alpha)_\alpha$ is compact if and only if each M_α is compact as a $(\Bbbk \otimes \Omega^2_* X)$-module and the collection of indices for which M_α is not trivial is finite. In particular, a compact object $(M_\alpha)_\alpha$ in $\prod_\alpha \mathrm{LMod}_{\Bbbk \otimes \Omega^2_* X} \simeq \mathrm{QCoh}(G)$ admits a both left and right dual, which is described by the collection of $(\Bbbk \otimes \Omega^2_* X)$-modules

$$\big((M_\alpha)_{\alpha\in\pi_1(X)}\big)^\vee := \big(M^\vee_{\alpha^{-1}}\big)_{\alpha\in\pi_1(X)}$$

where $M^\vee_{\alpha^{-1}}$ is the $(\Bbbk \otimes \Omega^2_* X)$-linear dual of $M_{\alpha^{-1}}$.

Finally, Proposition 2.5 implies that Loc_G is fully faithful, so in particular the hypotheses of Proposition 2.11 are satisfied by the based loop space on any space X. □

Theorem 2.8 guarantees that we can check the 1-affineness of the Betti stack of a pointed and connected space X by looking at the monadicity of the de-categorified global sections over its based loop space $\Omega_* X$. In virtue of Lemma 2.7, we know that we can always assume X to be connected. The following Lemma, which is a de-categorified analogue of the aforementioned result, allows us to reduce ourselves without loss of generality to the case when X is even *simply* connected.

Lemma 2.12. *Let X be a disjoint union of (possibly infinitely many) connected components*

$$X = \bigcup_{\alpha\in\pi_0(X)} X_\alpha$$

such that each $\Gamma(X_\alpha, -)\colon \mathrm{LocSys}(X_\alpha;\Bbbk) \to \mathrm{Mod}_\Bbbk$ is monadic. Then $\Gamma(X,-)\colon \mathrm{LocSys}(X) \to \mathrm{Mod}_\Bbbk$ is monadic as well.

Proof. By Barr–Beck–Lurie's theorem ([Lur17, Theorem 4.7.3.5]), the functor

$$\Gamma(X,-)\colon \mathrm{LocSys}(X;\Bbbk) \longrightarrow \mathrm{Mod}_\Bbbk$$

is monadic if and only if it is conservative and preserves colimits of $\Gamma(X,-)$-split simplicial objects. Under the equivalence

$$\mathrm{LocSys}(X;\Bbbk) \simeq \prod_{\alpha\in\pi_0(X)} \mathrm{LocSys}(X_\alpha;\Bbbk)$$

the functor $\Gamma(X,-)$ is equivalent to the functor $\prod_{\alpha\in\pi_0(X)} \Gamma(X_\alpha,-)$, and because of our assumptions each $\Gamma(X_\alpha,-)$ is monadic. So $\Gamma(X,-)$ is conservative, thanks to an analogous argument to the one used for the proof of Lemma 2.7.

So we only need to check that taking the global sections of a local system of $\Bbbk$-modules over X preserves colimits of $\Gamma(X,-)$-split simplicial objects. We argue as follows: let

$$\mathscr{F}_\bullet\colon \mathbf{\Delta}_+^{\mathrm{op}} \longrightarrow \mathrm{LocSys}(X;\Bbbk) \simeq \prod_{\alpha\in\pi_0(X)} \mathrm{LocSys}(X_\alpha;\Bbbk)$$

be an augmented simplicial object which becomes split after taking global sections. In particular, we can interpret it as a collection of augmented simplicial objects

$$\mathscr{F}_\bullet \simeq \big(\mathscr{F}^\alpha_\bullet\big)_{\alpha\in\pi_0(X)},$$

and we have

$$\Gamma(X,\mathscr{F}_\bullet)\simeq\prod_{\alpha\in\pi_0(X)}\Gamma\left(X_\alpha,\mathscr{F}_\bullet^\alpha\right).$$

We claim that such product is split because each $\Gamma(X_\alpha,\mathscr{F}_\bullet^\alpha)$ is already a split simplicial object of $\mathrm{Mod}_{\Bbbk}$.

Let

$$\rho\colon\mathbf{\Delta}_+\longrightarrow\mathbf{\Delta}_+$$

be the functor defined via the construction

$$[n]\mapsto[0]\star[n]\simeq[n+1].$$

For any category $\mathscr{C}$, pre-composition with ρ^{op} produces a functor at the level of categories of simplicial objects of $\mathscr{C}$

$$\mathrm{T}:=-\circ\rho^{\mathrm{op}}\colon\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathscr{C})\longrightarrow\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathscr{C}).$$

Applying this general machinery to our case, we obtain an augmented simplicial $\Bbbk$-module

$$\mathrm{T}\Gamma(X,\mathscr{F}_\bullet):=\Gamma(X,-)\circ\mathscr{F}_\bullet\circ\rho^{\mathrm{op}}\colon\mathbf{\Delta}_+^{\mathrm{op}}\longrightarrow\mathrm{LocSys}(X;\Bbbk)\longrightarrow\mathrm{Mod}_{\Bbbk},$$

such that for all $n\geqslant 0$ one has

$$\mathrm{T}\Gamma(X,\mathscr{F}_n)\simeq\Gamma(X,\mathrm{T}\mathscr{F}_n)\simeq\Gamma(X,\mathscr{F}_{n+1}).$$

The natural inclusion $[n]\subseteq\rho([n])$ defines a natural transformation from ρ to $\mathrm{id}_{\mathbf{\Delta}_+}$, hence a canonical map

$$\varphi_\bullet\colon\mathrm{T}\Gamma(X,\mathscr{F}_\bullet)\longrightarrow\Gamma(X,\mathscr{F}_\bullet).$$

Since the simplicial local system $\mathscr{F}_\bullet$ is a $\Gamma(X,-)$-split simplicial object, [Lur17, Corollary 4.7.2.9] tells us that one has a right homotopy inverse

$$\psi_\bullet\colon\Gamma(X,\mathscr{F}_\bullet)\longrightarrow\mathrm{T}\Gamma(X,\mathscr{F}_\bullet).$$

Unraveling all definitions, this means that we have a sequence of maps of $\Bbbk$-modules

$$\left(\varphi_\bullet^\alpha\colon\Gamma\left(X_\alpha,\mathrm{T}\mathscr{F}_\bullet^\alpha\right)\longrightarrow\Gamma\left(X_\alpha,\mathscr{F}_\bullet^\alpha\right)\right)_{\alpha\in\pi_0(X)}.$$

and after taking their product the composition

$$\prod_{\alpha\in\pi_0(X)}\Gamma\left(X_\alpha,\mathscr{F}_\bullet^\alpha\right)\xrightarrow{\psi_\bullet}\prod_{\alpha\in\pi_0(X)}\Gamma\left(X_\alpha,\mathrm{T}\mathscr{F}_\bullet^\alpha\right)\xrightarrow{\prod\varphi_\bullet^\alpha}\prod_{\alpha\in\pi_0(X)}\Gamma\left(X_\alpha,\mathscr{F}_\bullet^\alpha\right)$$

is homotopic to the identity.

We claim that $\psi_\bullet$ yields a right homotopy inverse to *each* map $\varphi_\bullet^\alpha$. Indeed, for each $\overline{\alpha}\in\pi_0(X)$ define

$$\psi_\bullet^{\overline{\alpha}}\colon\Gamma\left(X_\alpha,\mathscr{F}_\bullet^{\overline{\alpha}}\right)\xhookrightarrow{\iota_\bullet^{\overline{\alpha}}}\prod_{\alpha\in\pi_0(X)}\Gamma\left(X_\alpha,\mathscr{F}_\bullet^\alpha\right)\xrightarrow{\psi_\bullet}\prod_{\alpha\in\pi_0(X)}\Gamma\left(X_\alpha,\mathrm{T}\mathscr{F}_\bullet^\alpha\right)\xrightarrow{\pi_\bullet^{\overline{\alpha}}}\Gamma\left(X_\alpha,\mathrm{T}\mathscr{F}_\bullet^{\overline{\alpha}}\right).$$

Since we have by assumption homotopies making the diagram

$$\begin{array}{ccccc} & & \overset{\mathrm{id}}{\frown} & & \\ \prod \Gamma\left(X_\alpha, \mathcal{F}^\alpha_\bullet\right) & \xrightarrow{\psi_\bullet} & \prod \Gamma\left(X_\alpha, \mathrm{T}\mathcal{F}^\alpha_\bullet\right) & \xrightarrow{\varphi_\bullet} & \prod \Gamma(X_\alpha, \mathcal{F}^\alpha_\bullet) \\ \big\uparrow{\scriptstyle \iota^{\overline{\alpha}}_\bullet} & & \big\downarrow{\scriptstyle \pi^{\overline{\alpha}}_\bullet} & & \big\downarrow{\scriptstyle \pi^{\overline{\alpha}}_\bullet} \\ \Gamma\left(X_\alpha, \mathcal{F}^{\overline{\alpha}}_\bullet\right) & \xrightarrow[\psi^\alpha_\bullet]{} & \Gamma\left(X_\alpha, \mathrm{T}\mathcal{F}^{\overline{\alpha}}_\bullet\right) & \xrightarrow[\varphi^\alpha_\bullet]{} & \Gamma\left(X_\alpha, \mathcal{F}^{\overline{\alpha}}_\bullet\right) \end{array}$$

commute in every direction, and since $\pi^{\overline{\alpha}}_\bullet \circ \iota^{\overline{\alpha}}_\bullet$ is homotopic to the identity of $\mathcal{F}^{\overline{\alpha}}_\bullet$, it follows that each morphism

$$\psi^\alpha_\bullet \colon \Gamma\left(X_\alpha, \mathcal{F}^\alpha_\bullet\right) \longrightarrow \Gamma\left(X_\alpha, \mathrm{T}\mathcal{F}^\alpha_\bullet\right)$$

produces a right homotopy inverse to the natural map $\varphi^\alpha_\bullet$ of sections over X_α for each $\alpha \in \pi_0(X)$.

So applying the opposite direction of the criterion of [Lur17, Corollary 4.7.2.9] we obtain that a $\Gamma(X,-)$-split simplicial object in $\mathrm{LocSys}(X;\Bbbk)$ corresponds to a sequence of $\Gamma(X_\alpha,-)$-split simplicial objects of $\mathrm{LocSys}(X_\alpha;\Bbbk)$. This discussion implies that we have a chain of equivalences

$$\begin{aligned} \operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}} \Gamma\left(X, \mathcal{F}_\bullet\right) &\simeq \operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}} \prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha, \mathcal{F}^\alpha_\bullet\right) \\ &\simeq \prod_{\alpha\in\pi_0(X)} \operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}} \Gamma\left(X_\alpha, \mathcal{F}^\alpha_\bullet\right) \\ &\simeq \prod_{\alpha\in\pi_0(X)} \Gamma\left(X_\alpha, \operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}} \mathcal{F}^\alpha_\bullet\right) \simeq \Gamma\left(X, \operatorname*{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}} \mathcal{F}_\bullet\right). \end{aligned}$$

All these equivalence hold because we were assuming that all $\Gamma(X_\alpha,-)$ were monadic functors (hence, they all preserves colimit of $\Gamma(X_\alpha,-)$-split simplicial objects), together with the fact that colimits of split simplicial objects are universal ([Lur17, Remark 4.7.2.4]) and with the observation that colimits in products of categories are computed component-wise. □

Lemma 2.12 will provide a useful tool in proving that Betti stacks of certain spaces are, or are not, 1-affine.

Corollary 2.13. *Let X be a 1-truncated topological space. Then the Betti stack X_{B} is 1-affine.*

Proof. We can assume that X is connected thanks to Lemma 2.7; in particular, we have that $X \simeq K(\pi, 1)$ is an Eilenberg-MacLane space for the discrete group $\pi := \pi_1(X)$. We apply Theorem 2.8: X_{B} is 1-affine precisely if

$$\Gamma(\pi,-)\colon \mathrm{LocSys}(\pi;\Bbbk) \longrightarrow \mathrm{Mod}_{\Bbbk}$$

is monadic. But π is a disjoint union of contractible components

$$\pi \simeq \bigcup_{\alpha\in\pi_1(X)} \{*\},$$

so Lemma 2.12 immediately implies our statement. □

Proposition 2.14. *Let X be a space, and assume $\Bbbk$ to be a semisimple commutative ring. If there exists a base point x such that $\pi_2(X,x)$ contains an element g either of infinite order, or such that the order of g is a unit in $\Bbbk$, then the Betti $\Bbbk$-stack X_{B} is not 1-affine.*

Remark 2.15. In particular, if $\Bbbk$ is a field of characteristic 0, the second homotopy group of a space X always provides an obstruction to the 1-affineness of its Betti stack over $\Bbbk$.

Proof of Proposition 2.14. Let us write $\pi := \pi_2(X,x)$. In virtue of Lemma 2.7, we can assume without loss of generality that X is connected. So, taking the space of loops based at x and using Theorem 2.8, our statement will follow once we prove that the global sections functor on $\mathrm{LocSys}(\Omega_* X;\Bbbk)$ is not monadic over $\mathrm{Mod}_\Bbbk$. Lemma 2.12 allows us to assume that $\Omega_* X$ is itself connected, so under the equivalence of (1.5) we are asked to check whether the functor

$$\underline{\mathrm{Map}}_{\Bbbk\otimes\Omega^2_*X}(\Bbbk, -)\colon \mathrm{LMod}_{\Bbbk\otimes\Omega^2_*X} \longrightarrow \mathrm{Mod}_\Bbbk$$

is not monadic.

Notice that $\Bbbk\otimes\pi \simeq \pi_0(\Bbbk\otimes\Omega^2_*X)$ is isomorphic as a $\Bbbk$-algebra to the group ring $\Bbbk[\pi]$, and the obvious projection

$$\Omega^2_*X \longrightarrow \pi_0(\Omega^2_*X) \simeq \pi$$

turns $\Bbbk[\pi]$ into a $(\Bbbk\otimes\Omega^2_*X)$-module. Let $g\in\pi$ be as in the statement: then $(1-g)$ is a non-nilpotent element. Indeed, consider the subgroup $\langle g\rangle$ generated by g: the inclusion $\langle g\rangle \subseteq \pi$ induces an inclusion of commutative rings $\Bbbk[\langle g\rangle] \subseteq \Bbbk[\pi]$, so if $(1-g)$ is not nilpotent in $\Bbbk[\langle g\rangle]$ it will automatically be not nilpotent in $\Bbbk[\pi]$ as well. We can therefore reduce ourselves to prove the statement in the case π is cyclic and generated by g.

(1) If g has infinite order, then $\Bbbk[\pi]\cong\Bbbk[t,t^{-1}]$ is a domain, hence $(1-g)$ is not nilpotent.
(2) If g has finite order n and n is a unit in $\Bbbk$, then $\Bbbk[\pi]$ is a semisimple algebra because of Maschke's theorem (see for example [PS02, Theorem 3.4.7]). In particular, $\Bbbk[\pi]$ is reduced and $(1-g)$ is not nilpotent.

It follows that the set $S := \{(1-t)^n \mid n\geqslant 0\} \subseteq \Bbbk[\pi]$ satisfies the Ore conditions in the graded commutative ring $\pi_\bullet(\Bbbk\otimes\Omega^2_*X)$, hence there exists the localization $(\Bbbk\otimes\Omega^2_*X)[(1-g)^{-1}]$ ([Lur17, Section 7.2.3]) and since S does not contain the 0 element such localization is not trivial. Since $(1-g)$ is an element in the fiber of the map

$$\Bbbk\otimes\Omega^2_*X \longrightarrow \Bbbk\otimes\pi_0(\Omega^2_*X) \simeq \Bbbk[\pi] \longrightarrow \Bbbk,$$

it follows that $\Bbbk$ is S-nilpotent. In particular, [Lur17, Proposition 7.3.2.14] guarantees that $(\Bbbk \otimes \Omega^2_* X)[(1-g)^{-1}]$ is right orthogonal to $\Bbbk$ as a $(\Bbbk \otimes \Omega^2_* X)$-module, so the global sections cannot be conservative. □

Porism 2.16. Notice that, in the setting of the proof of Proposition 2.14, the functor

$$\underline{\mathrm{Map}}_{\Bbbk\otimes\Omega^2_* X}(\Bbbk,-)\colon \mathrm{LMod}_{\Bbbk\otimes\Omega^2_* X} \longrightarrow \mathrm{Mod}_{\Bbbk}$$

is simply the global sections functor

$$\Gamma(\Omega_* X,-)\colon \mathrm{LocSys}(\Omega_* X;\Bbbk) \longrightarrow \mathrm{Mod}_{\Bbbk}$$

under the equivalence $\mathrm{LocSys}(\Omega_* X;\Bbbk)\simeq \mathrm{LMod}_{\Bbbk\otimes\Omega^2_* X}$ of (1.5). So, the proof of Proposition 2.14 actually passes through the proof of the following de-categorified analogue.

Lemma 2.17. *Let X be a space, and assume $\Bbbk$ to be a semisimple commutative ring. If there exists $\Bbbk$ a base point x such that $\pi_1(X,x)$ contains an element g either of infinite order, or such that the order of g is a unit in $\Bbbk$, then the Betti stack X_{B} is not almost 0-affine.*

Remark 2.18. Proposition 2.14 is *not* a necessary condition. As we will explain in Remark 3.40, Example 3.38 shows that $\mathbf{B}\mathbb{CP}^\infty \simeq \mathbf{B}^3\mathbb{Z}$ is 1-affine.

We conclude this section by studying in detail two examples. The infinite projective space $\mathbb{CP}^\infty$ has non-trivial, free second homotopy group. Thus, by Proposition 2.14, it is not 1-affine. We will give a direct proof of this fact, which essentially already appeared in Teleman [Tel14]. The second example we will discuss is the circle S^1. Since S^1 is 1-truncated it is 1-affine by Corollary 2.13. We will give a different argument for the 1-affineness of S^1, following [Tel14] and [GHM23]. In fact, both examples can be viewed as instances of the emerging picture of 3d Homological Mirror Symmetry: we refer the reader to [Tel14], [GHM23] and references therein for additional information.

Example 2.19. Let $X := \mathbb{CP}^\infty$ be the infinite-dimensional projective space. It is a simply connected CW complex whose based loop space is homotopy equivalent to the circle, i.e.,

$$\Omega_* X \simeq S^1, \quad \text{and} \quad \Omega^2_* X \simeq \mathbb{Z}.$$

In particular, there is an equivalence of $\mathbb{E}_2$-algebras $\mathrm{C}_\bullet(\Omega^2_* X;\Bbbk)\simeq \Bbbk[t,t^{-1}]$. This yields an equivalence of monoidal categories

$$\mathrm{LMod}_{\Omega^2_* X}(\mathrm{Mod}_{\Bbbk}) \simeq \mathrm{Mod}_{\Bbbk[t,t^{-1}]}.$$

We can use this to obtain an interesting alternative description of $\mathrm{LocSysCat}(X;\Bbbk)$. Indeed, we can write a chain of equivalences

$$\psi\colon \mathrm{LocSysCat}(X;\Bbbk) \xrightarrow[1.6]{\simeq} \mathrm{Lin}_{\Bbbk[t,t^{-1}]}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \xrightarrow{\simeq} \mathrm{Lin}_{\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \xrightarrow{\simeq} \mathrm{ShvCat}(\mathbb{G}_{\mathrm{m},\Bbbk})$$

where the last equivalence follows from the fact that $\mathbb{G}_{\mathrm{m},\Bbbk}$, being affine, is obviously 1-affine. Thus categorical local systems on X can be described equivalently as quasi-coherent quasi-coherent sheaves of categories over $\mathbb{G}_{\mathrm{m},\Bbbk}$.

We remark that ψ can be seen as an instance of 3d Homological Mirror Symmetry. Let us briefly explain why this is the case. The pair of spaces

$$\mathrm{T}^*\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk} \longleftrightarrow \mathrm{T}^*\mathbb{G}_{\mathrm{m},\Bbbk}$$

is one of the basic examples of 3d mirror partners. At least if $\Bbbk = \mathbb{C}$, the category $\mathrm{LocSysCat}(X;\Bbbk)$ can be viewed as (a subcategory of) the category of 3d A-branes on $\mathrm{T}^*\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}$. The key observation here is that topologically we have a homotopy equivalence

$$\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}(\mathbb{C}) \simeq \mathbb{CP}^\infty$$

and the category of 3d A-branes of a cotangent stack is expected to contain local systems of categories over the base; we refer to [Tel14] for a fuller discussion of this point. Conversely, $\mathrm{ShvCat}(\mathbb{G}_{\mathrm{m},\Bbbk})$ is (a subcategory of) the category of 3d B-branes on $\mathrm{T}^*\mathbb{G}_{\mathrm{m},\Bbbk}$. From this perspective, equivalence ψ implements a dictionary relating A-branes on $\mathrm{T}^*\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}$ and B-branes on its mirror, thus paralleling closely the classical 2d HMS story.

Next, let us show that X is not 1-affine; see also [Tel14] for a similar discussion. Note that it is enough to show that the global sections functor

$$\Gamma(X,-)\colon \mathrm{LocSysCat}(X;\Bbbk) \longrightarrow \mathrm{Lin}_{\Bbbk}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

is not conservative. This is easily done directly using the equivalences provided by Theorem 1.6

$$\mathrm{LocSysCat}(X;\Bbbk) \simeq \mathrm{LMod}_{S^1}\left(\mathrm{Lin}_{\Bbbk}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \simeq \mathrm{Lin}_{\Bbbk[t,t^{-1}]}\,\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}. \tag{2.20}$$

We can easily classify the $\Bbbk[t,t^{-1}]$-linear structures on $\mathrm{Mod}_{\Bbbk}$: these are equivalently described as characters of $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$ (see [PPS25, Proposition 2.16 and Proposition 2.21]). So, an S^1-action on $\mathrm{Mod}_{\Bbbk}$ corresponds to the choice of a non-trivial scalar $\lambda \in \Bbbk^\times$, which is the image of t under an $\mathbb{E}_2$-morphism $\Bbbk[t,t^{-1}] \to \Bbbk$. Let us denote by $\mathrm{Mod}_{\Bbbk}(\lambda)$ the corresponding categorical S^1-module.

Now, the global section functor on $\mathrm{LocSysCat}(X;\Bbbk)$ is corepresented by the trivial local system, which is the unit with respect to the ordinary monoidal structure on $\mathrm{LocSysCat}(X;\Bbbk)$. We denote this object by

$$\underline{\mathrm{LocSys}}(-) \in \mathrm{LocSysCat}(X;\Bbbk).$$

Under equivalence (2.20), the object $\underline{\mathrm{LocSys}}(-)$ is mapped to $\mathrm{Mod}_{\Bbbk}(1)$.

Notice that, for any invertible $\lambda \in \Bbbk^\times$, $\mathrm{Mod}_{\Bbbk}(\lambda)$ can be seen as the category of $\Bbbk(\lambda)$-modules inside $\mathrm{Mod}_{\Bbbk[t,t^{-1}]}$, where $\Bbbk(\lambda)$ is the commutative $\Bbbk[t,t^{-1}]$-algebra on the underlying $\Bbbk$-module $\Bbbk$ determined by the evaluation $\mathrm{ev}_\lambda\colon \Bbbk[t,t^{-1}] \to \Bbbk$; in other words:

$$\mathrm{Mod}_{\Bbbk}(\lambda) \simeq \mathrm{Mod}_{\Bbbk(\lambda)}\left(\mathrm{Mod}_{\Bbbk[t,t^{-1}]}\right) \simeq \mathrm{Mod}_{\Bbbk},$$

where in the last equivalence we used [Lur17, Remark 7.1.3.7]. Invoking the categorical Eilenberg-Watts theorem, we have then

$$\begin{aligned}\underline{\mathrm{Fun}}^{\mathrm{L}}_{\Bbbk[t,t^{-1}]}(\mathrm{Mod}_{\Bbbk}(1), \mathrm{Mod}_{\Bbbk}(\lambda)) &\simeq \underline{\mathrm{Fun}}^{\mathrm{L}}_{\Bbbk[t,t^{-1}]}\big(\mathrm{Mod}_{\Bbbk(1)}\big(\mathrm{Mod}_{\Bbbk[t,t^{-1}]}\big), \mathrm{Mod}_{\Bbbk}(\lambda)\big)\\ &\simeq {}_{\Bbbk(1)}\mathrm{BMod}_{\Bbbk(\lambda)}\big(\mathrm{Mod}_{\Bbbk[t,t^{-1}]}\big)\\ &\simeq \mathrm{Mod}_{\Bbbk(1)\otimes_{\Bbbk[t,t^{-1}]}\Bbbk(\lambda)}.\end{aligned}$$

But now an easy homological computation shows that $\Bbbk(1)\otimes_{\Bbbk[t,t^{-1}]}\Bbbk(\lambda)$ is 0 whenever $\lambda \neq 1$, and so the S^1-fixed points are trivial.

It might be useful to revisit the previous calculation from a geometric standpoint using equivalence ψ. Let

$$\iota_\lambda : \mathrm{Spec}(\Bbbk) \longrightarrow \mathbb{G}_{\mathrm{m},\Bbbk}$$

be the $\Bbbk$-rational point $\lambda \in \mathbb{G}_{\mathrm{m},\Bbbk}(\Bbbk)$. Under ψ, the object $\mathrm{Mod}_{\Bbbk}(\lambda)$ becomes a categorified *skyscraper sheaf*. That is, it is the quasi-coherent sheaf of categories obtained by pushing-forward the unit along the functor

$$\iota_{\lambda,*} : \mathrm{ShvCat}(\mathrm{Spec}(\Bbbk)) \longrightarrow \mathrm{ShvCat}(\mathbb{G}_{\mathrm{m},\Bbbk})$$

The computation above shows that, as expected, skyscraper sheaves at different points are mutually orthogonal.

Example 2.21. Let us consider next the case $X = S^1 \simeq K(\mathbb{Z}, 1)$. We can prove directly that its Betti stack is 1-affine as follows. Note first that $\mathrm{LocSysCat}(S^1;\Bbbk)$ is the same as the category of $\Bbbk$-linear presentable categories equipped with a choice of an autoequivalence $F\colon \mathscr{C} \simeq \mathscr{C}$. The latter, in turn, is equivalent to the category of $\mathrm{QCoh}(\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk})$-linear presentable categories ([GHM23, Example 0.4]). Since $\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}$ is 1-affine ([Gai15, Remark 2.5.2]), it follows that

$$\mathrm{ShvCat}(S^1_{\mathrm{B}}) \simeq \mathrm{LocSysCat}(S^1;\Bbbk) \simeq \mathrm{Lin}_{\mathrm{QCoh}(\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)} \simeq \mathrm{ShvCat}(\mathbf{B}\mathbb{G}_{\mathrm{m},\Bbbk}). \tag{2.22}$$

The latter category is also equivalent to the category $\mathrm{Lin}_{\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ of $\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})$-linear presentable categories, where now $\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})$ is seen as a symmetric monoidal category via the convolution tensor product induced by the $\mathbb{E}_\infty$-group structure on $\mathbb{G}_{\mathrm{m},\Bbbk}$. This can be deduced by concatenating the equivalences (10.1) and (10.4) in [Gai15]; an earlier proof of this fact can be found in [BFN12].

Equipped with this monoidal structure, $\mathrm{QCoh}(\mathbb{G}_{\mathrm{m},\Bbbk})$ is monoidally equivalent to the category of representations of $\mathbb{Z}$ inside $\mathrm{Mod}_{\Bbbk}$, which is in turn equivalent to the category $\mathrm{LocSys}(S^1;\Bbbk)$ equipped with its point-wise monoidal structure. This is precisely $\mathrm{QCoh}(S^1_{\mathrm{B}})$. We deduce that there is an equivalence

$$\mathrm{ShvCat}\left(S^1_{\mathrm{B}}\right) \simeq \mathrm{Lin}_{\mathrm{QCoh}(S^1_{\mathrm{B}})}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$$

i.e., S^1_{B} is 1-affine, as we wanted to show. We remark that equivalence (2.22) can be viewed as the opposite direction of 3d HMS with respect to the one considered in Example 2.19, see [GHM23] for more information.

2.2. **The case for arbitrary** n**.** In this Section we establish an n-analogue version of Corollary 2.13 for all positive integers n. Namely, we will show that if X is a n-truncated space then its Betti stack X_{B} is n-affine in the sense of Definition 1.18. We start by observing the following easy fact, which is an n-categorical analogue of Lemma 2.3.

Lemma 2.23. *For any space X and for any commutative ring spectrum $\Bbbk$, we have an equivalence of $(n+1)$-categories*

$$(n+1)\mathbf{ShvCat}^n(X_{\mathrm{B}}) \simeq (n+1)\mathbf{LocSysCat}^n(X;\Bbbk).$$

Proof. Since $(n+1)\mathbf{ShvCat}^n$ is a sheaf for the étale topology ([Ste21, Corollary 14.3.5]), the proof of Lemma 2.3 carries out *verbatim* also in the n-categorical setting. □

Proposition 2.24. *Let X be any space. Then the functor*

$$\Gamma^{\mathrm{enh}}(X_{\mathrm{B}},-)\colon \mathrm{LocSysCat}^n(X;\Bbbk) \longrightarrow \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

admits a fully faithful left adjoint.

Proof. The Proposition is proved in the same way as Proposition 2.5. The key observation is that limits of left adjointable diagrams inside $(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$ exist and agree with colimits of the corresponding diagram of left adjoints (Fact 1.2.(3)). □

So, just like in the 2-categorical case (Porism 2.6), the only obstruction to n-affineness for Betti stacks is the conservativity of the global sections functor $\Gamma(X,-)\colon \mathrm{LocSysCat}^n(X;\Bbbk) \to \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$.

Theorem 2.25. *Let X be a space, and suppose that X is n-truncated. Then its Betti stack X_{B} is n-affine.*

In order to prove Theorem 2.25, we need to establish the following categorified variant of Proposition 2.11 – at least for Betti stacks.

Theorem 2.26. *Let X be a space with a choice of a base point. Then its Betti stack X_{B} is n-affine if and only if the Betti stack of the based loop space $\Omega_* X_{\mathrm{B}}$ is $(n-1)$-affine.*

Remark 2.27. Theorem 2.26 can be also interpreted as a topological analogue of the n-affineness criterion of [Ste21, Theorem 14.3.9].

Before proceeding, let us remark that Theorem 2.26 immediately implies Theorem 2.25. Indeed, for any $n \geqslant 1$, a simple induction shows that the n-affineness of X_{B} boils down to the 1-affineness of $\Omega^{n-1}_* X$. If $\Omega^{n-1}_* X$ is 1-truncated (which is equivalent to asking that X is n-truncated) then its Betti stack is 1-affine in virtue of Corollary 2.13. So, all is left to do is showing that Theorem 2.26 holds. We will devote to this task most of the remainder of this Section.

Lemma 2.28. *Let $n \geqslant 2$ be an integer. The assigment $X \mapsto \mathrm{Lin}_{\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ satisfies descent with respect to effective epimorphisms in $\mathcal{S}$ precisely if Ω_*X is $(n-1)$-affine.*

We shall split the proof of Lemma 2.28 in substeps for the convenience of the reader.

2.29. First, assume that X is connected: arguing as in Lemma 2.7, this is not a restrictive assumption. Take any homotopy effective epimorphism $U_\bullet \to X$. For every non-negative integer n the n-th space in the simplicial diagram $U_\bullet \to X$ is described as an n-fold fiber product

$$U_n \simeq \underbrace{U_0 \times_X U_0 \times_X \cdots \times_X U_0}_{n \text{ times}}.$$

Applying the functor $\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(-;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$, we obtain a cosimplicial diagram of categories

$$\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \longrightarrow \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_\bullet;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)},$$

hence a natural functor

$$\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \longrightarrow \lim_{[m]\in\mathbf{\Delta}} \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}, \tag{2.30}$$

where the limit on the right hand side is computed in the category of $(n+1)$-categories.

2.31. Such functor admits a right adjoint: this can be described at the level of objects by the assignment

$$\mathcal{F}_\bullet := (\mathcal{F}_m)_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \mapsto \lim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \iota_{m,*}\mathcal{F}_m,$$

where $\mathcal{F}_\bullet$ is a cosimplicial system of presentable $n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)$-linear n-categories. Notice that this limit of $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-linear n-categories *does exist*. Indeed, for $n \geqslant 2$ and for an arbitrary morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ the canonical pullback n-functor

$$f^* : (n+1)\,\mathrm{ShvCat}^n(\mathcal{Y}) \longrightarrow (n+1)\,\mathrm{ShvCat}^n(\mathcal{X})$$

is part of an *ambidextrous* adjunction: this is [Ste21, Corollary 14.2.10]. So, the existence of limits of left adjointable diagrams of presentable n-categories allows us to conclude that this formula does make sense.

To see that the functor $\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \to \lim_{[m]}\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is an equivalence, we check that both the unit and the counit of the aforementioned adjunction are equivalences of n-categories. The unit of an adjunction is an equivalence if and only if the functor (2.30) is fully faithful.

Lemma 2.32. *For any space X, for any choice of a effective epimorphism $U_\bullet$ and for any integer $n \geqslant 1$, the functor (2.30) is fully faithful.*

Proof. We have a commutative diagram of functors

$$\begin{array}{ccc}
\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} & \xrightarrow{(2.30)} & \lim_{[m]}\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \\
{\scriptstyle \mathrm{Loc}^n_{X_{\mathrm{B}}}}\big\downarrow & & \big\downarrow{\scriptstyle \lim_{[m]}\mathrm{Loc}^n_{(U_m)_{\mathrm{B}}}} \\
\mathrm{LocSysCat}^n(X;\Bbbk) & \xrightarrow{\simeq} & \lim_{[m]}\mathrm{LocSysCat}^{n-1}(U_m;\Bbbk).
\end{array}$$

The bottom arrow is an equivalence because for any integer $n \geqslant 2$ the assignment $X \mapsto n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$ satisfies descent in X; this is true also for $n=1$ if we interpret $1\mathbf{LocSysCat}^0(X;\Bbbk)$ to be $\mathrm{LocSys}(X;\Bbbk)$. Moreover, the two vertical arrows are fully faithful thanks to Proposition 2.24. It follows that the upper arrow has to be fully faithful as well. □

We are only left to check that the counit is an equivalence. We argue as follows: since we are assuming X to be connected, we can choose an effective epimorphism $\{*\} \to X$ given by the inclusion of any base point in the only connected component of X: the fact that this is an effective epimorphism is due to [Lur09, Proposition 7.2.1.14]. Setting $V_n := U_n \times_X \{*\}$, we obtain a commutative diagram of categories

$$\begin{array}{ccc}
\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} & \hookrightarrow & \lim_{[m]}\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \\
\big\downarrow & & \big\downarrow \\
\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} & \longrightarrow & \lim_{[m]}\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\,.
\end{array} \tag{2.33}$$

The bottom functor is an equivalence. Indeed, effective epimorphisms of spaces are stable under pullbacks (because colimits are universal in any topos), so under base change we obtain an effective epimorphism

$$V_\bullet := U_\bullet \times_X \{*\} \longrightarrow \{*\}\,.$$

Choosing the inclusion of any point $\{*\} \to V_0 \simeq U_0 \times_X \{*\}$ and using [Lur17, Corollary 4.7.2.9] we obtain a splitting of the above augmented simplicial diagram: it follows that its colimit is preserved by *any* functor, and in particular by the contravariant functor $\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(-;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$.

2.34. On the other hand, the vertical functor on the right is conservative. Indeed, under the equivalence

$$\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \simeq \lim_{[m]\in\Delta^{\mathrm{op}}}\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

such functor corresponds to the composition of the fully faithful embedding

$$\lim_{[m]\in\Delta^{\mathrm{op}}}\mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \hookrightarrow \lim_{[m]\in\Delta^{\mathrm{op}}}\mathrm{LocSysCat}^n(U_m;\Bbbk) \simeq \mathrm{LocSysCat}^n(X;\Bbbk)$$

with the pullback along the chosen base point $\{*\} \to X$. Since X is assumed to be connected, this pullback corresponds to forgetting the $\Omega_* X$ action on a presentably $\Bbbk$-linear n-category under the equivalence of Theorem 1.6, which is a conservative operation.

So the issue is whether the diagram (2.33) is horizontally right adjointable: if that was the case, then one could check whether the counit is an equivalence after pulling back over the base point, and there the answer is obvious because the bottom functor is an equivalence.

2.35. Notice that the vertical functor on the right hand side of diagram (2.33) can be computed in the following way: at each step of the simplicial diagram

$$n\mathscr{C}_m \mapsto n\mathscr{C}_m \otimes_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)$$

we take the base change induced by the symmetric monoidal pullback

$$n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk) \longrightarrow n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)$$

along the natural projection $V_m := U_m \times_X \{*\} \to U_m$. On the left hand side, the vertical functor is similarly described as a base change functor

$$n\mathscr{C} \mapsto n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}.$$

So, the right adjointability of the diagram (2.33) amounts to asking for the natural $\Bbbk$-linear n-functor

$$\lim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \iota_{m,*} n\mathscr{C}_m \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow$$
$$\longrightarrow \lim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \iota_{m,*}\left(n\mathscr{C}_m \otimes_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(V_m;\Bbbk)\right)$$

to be an equivalence.

The commutativity of the totalization of the cosimplicial diagram with the tensor product is not a problem: such limit is computed as a colimit of presentable n-categories along the simplicial diagram obtained by passing to the left adjoints. Since the tensor product of presentable n-categories commutes with colimits, we can bring the limit inside and outside of the tensor product without any harm. So we can rephrase our problem as follows: given a map of topological spaces $U \to X$ and setting $V := U \times_X \{*\}$, when is the n-functor

$$n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(V;\Bbbk) \qquad (2.36)$$

an equivalence for an arbitrary $n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)$-linear presentable n-category $n\mathscr{C}$?

2.37. We will show that (2.36) is an equivalence by proving the following stronger statement. Let $Y \to X \leftarrow Z$ be morphisms of spaces, such that the Betti stack $(\Omega_* X)_{\mathrm{B}}$ is $(n-1)$-affine. We will show that there is an equivalence

$$n\mathbf{LocSysCat}^{n-1}(Y) \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(Z;\Bbbk) \xrightarrow{\simeq} n\mathbf{LocSysCat}^{n-1}(Y \times_X Z;\Bbbk)$$

Note that by setting $Y := U$ and $Z := \{*\}$, we can recover (2.36) in the case $n\mathscr{C} = n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)$. Then, using the fact that for any other $n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)$-linear presentable n-category $n\mathscr{C}$ one has a canonical equivalence

$$n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(U;\Bbbk)} n\mathbf{LocSysCat}^{n-1}(U;\Bbbk) \simeq n\mathscr{C},$$

we can extend our result to an arbitrary $n\mathscr{C}$. Since categorical local systems satisfy hyperdescent, we can replace both Y and Z by some effective epimorphism $W_\bullet \to Y$ whose 0-th stage is described by a disjoint union of contractible spaces $\{*\}_{\alpha\in A}$, and where the m-th stage is described by the usual Čech formula

$$W_m := \coprod_{\alpha_1,\cdots,\alpha_m} \{*\}_{\alpha_1} \times_{W_{m-1}} \cdots \times_{W_{m-1}} \{*\}_{\alpha_m}.$$

In virtue of Lemma 1.8, such a disjoint union is sent via the functor $n\mathbf{LocSysCat}^{n-1}(-;\Bbbk)$ to a coproduct of presentably $\Bbbk$-linear n-categories. The latter distributes over tensor products: we can therefore reduce ourselves to the case in which W_0 is just a point. Using the fact that

$$\Omega^p_* X \times_X \Omega^q_* X \simeq \Omega^{p+q}_* X,$$

it is easy to see that W_m is described by an m-fold based loop space $\Omega^m_* X$. So, we are left to prove that there is an equivalence

$$n\mathbf{LocSysCat}^{n-1}(\Omega^m_* X;\Bbbk) \simeq \underbrace{n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} \cdots \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}}_{m \text{ times}}.$$

Arguing by induction, and using the fact that

$$n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

can be written as

$$\Big(n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\Big) \otimes \Big(n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\Big),$$

we are reduced to prove that there is an equivalence

$$n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk) \simeq n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}.$$

In the formulas above, we are writing $\otimes$ for $\otimes_{n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}}$.

The symmetric monoidal pullback n-functor $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk) \to n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ induced by the inclusion of the base point in X turns $n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ into an $\mathbb{E}_\infty$-$n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-algebra inside $\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$. In particular, we have a $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-linear n-functor

$$n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \tag{2.38}$$

corresponding to such symmetric monoidal operation.

Lemma 2.39. *The underlying functor of the action $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-linear functor*

$$n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

is a monadic functor of categories.

Proof. The tensor product $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ is computed as a geometric realization of a simplicial diagram of n-categories $n\mathscr{C}_\bullet$, whose i-th term is described as

$$n\mathscr{C}_i\simeq n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\otimes n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)^{\otimes i}\otimes n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\simeq n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)^{\otimes i},$$

where the faces and degeneracies are induced by pullback n-functors. Here, the tensor product is understood as the tensor product of $\Bbbk$-linear presentable n-categories, whose monoidal unit is $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$. Under the equivalences

$$n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\otimes n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)^{\otimes i}\otimes n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\simeq n\mathbf{LocSysCat}^{n-1}\left(X^{\times i};\Bbbk\right),$$

we can describe such simplicial object in more detail.

(1) The degeneracy morphisms of such simplicial diagram correspond to pulling back categorical local systems along projections $X^{\times i}\to X^{\times i-1}$.
(2) The face morphisms correspond either to pulling back categorical local systems along the extremal inclusions $X^{i-1}\simeq\{*\}\times X^{i-1}\subseteq X^i$ and $X^{i-1}\simeq X^{i-1}\times\{*\}\subseteq X^i$ (these are the face morphisms ∂_0 and ∂_i), or to pulling back categorical local systems along the morphism $\Delta_p\colon X^{i-1}\to X^i$ described informally by $(x_1,\ldots,x_{p-1},x_p,x_{p+1},\ldots,x_{i-1})\mapsto(x_1,\ldots,x_{p-1},x_p,x_p,x_{p+1},\ldots,x_{i-1})$ (these are the face morphisms ∂_p, for $p\in\{1,\ldots,i-1\}$).

Thanks to this description of the faces n-functors in this simplicial n-category, we see immediately that for any morphism $\alpha\colon[i]\to[j]$ the diagram

$$\begin{array}{ccc}
n\mathbf{LocSysCat}^{n-1}\left(X^{\times(j+1)};\Bbbk\right) & \xrightarrow{\partial_{i+1,*}} & n\mathbf{LocSysCat}^{n-1}\left(X^{\times j};\Bbbk\right)\\
\big\downarrow{\scriptstyle(\alpha\star\mathrm{id}_{[0]})^*} & & \big\downarrow{\scriptstyle\alpha^*}\\
n\mathbf{LocSysCat}^{n-1}\left(X^{\times(i+1)};\Bbbk\right) & \xrightarrow[\partial_{i,*}]{} & n\mathbf{LocSysCat}^{n-1}\left(X^{\times i};\Bbbk\right)
\end{array}$$

is commutative. This means that such simplicial diagram of n-categories (or better, the underlying simplicial diagram of categories) satisfies the monadic Beck-Chevalley condition ([Gai15, Definition C.1.5]), after applying suitably [Gai15, Lemma C.1.6]. Hence, [Gai15, Lemma C.1.8] implies that such action functor is indeed monadic. □

Lemma 2.39 is what we need in order to apply [Gai15, Corollary C.2.3], which guarantees that we can compute the monad described by the action functor

$$\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}\otimes_{\mathrm{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}\longrightarrow\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

as the composition

$$\eta^* \circ \eta_* \colon \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

where $\eta \colon \{*\} \hookrightarrow X$ is the inclusion of the base point. This implies that the naturally defined functor

$$\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{\mathrm{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow \mathrm{LocSysCat}^{n-1}(\Omega_* X;\Bbbk),$$

obtained by taking the right adjoint to

$$\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \simeq \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{\mathrm{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

and then composing with the pullback n-functors induced by the two projections $\pi_1, \pi_2 \colon \Omega_* X \rightrightarrows \{*\}$, makes the diagram

$$\begin{array}{ccc}
\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} \otimes_{\mathrm{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} & \longrightarrow & \mathrm{LocSysCat}^{n-1}(\Omega_* X;\Bbbk) \\
 & \searrow \quad \swarrow & \\
 & \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)} &
\end{array}$$

commute. In the above picture, the horizontal arrow is the one described in this paragraph; the right-hand side arrow is the push-forward along the natural terminal morphism $\Omega_* X \to \{*\}$ (i.e., it is the global sections functor), and the left-hand side arrow is the functor (2.38) corresponding $\mathrm{LocSysCat}^{n-1}(X;\Bbbk)$-linear monoidal structure of $\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$.

So, by Barr–Beck–Lurie, the horizontal arrow is an equivalence precisely if

$$\Gamma(\Omega_* X_{\mathrm{B}}, -) \colon \mathrm{LocSysCat}^{n-1}(\Omega_* X;\Bbbk) \longrightarrow \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n-1)}$$

is a monadic functor. But this is equivalent to $n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)$ being monadic over $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ as n-categories, and this is precisely the definition of $(n-1)$-affineness for the Betti stack $(\Omega_* X)_{\mathrm{B}}$. Combining all the arguments of the last paragraphs, we obtain the proof of Lemma 2.28.

Proof of Theorem 2.26. First, using again [Ste21, Proposition 14.3.6], we reduce ourselves to check the monadicity at the level of the underlying category. Notice that the Seifert-Van Kampen Theorem implies that the sheafy side $\mathrm{LocSysCat}^n(X;\Bbbk)$ satisfies hyperdescent in X for any topological space X. When the side of presentable $\mathrm{LocSysCat}^{n-1}(X;\Bbbk)$-modules satisfies hyperdescent in X as well, we ccan conclude that the left adjoint to $\Gamma^{\mathrm{enh}}(X_{\mathrm{B}}, -)$ is an equivalence: indeed, it is be sufficient to choose a hypercover $U_\bullet \to X$ described in each degree by a disjoint union of contractible spaces, and then obtain that

$$\begin{aligned}
\mathrm{LocSysCat}^n(X;\Bbbk) &\simeq \lim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{LocSysCat}^n(U_m;\Bbbk) \\
&\simeq \lim_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Mod}_{n\mathbf{LocSysCat}^{n-1}(U_m;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)} \\
&\simeq \mathrm{Mod}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} \mathrm{Pr}^{\mathrm{L}}_{(\infty,n)},
\end{aligned}$$

using the fact that disjoint unions of contractible spaces are obviously n-affine.

We can reformulate questions concerning hyperdescent of local systems on X as questions concerning descent, thanks to the hypercompleteness of the topos $\mathrm{Fun}(X,\mathcal{S}) =: \mathrm{LocSys}(X)$: this is always hypercomplete without any assumptions on X ([Lur09, Example 7.2.1.9 and Corollary 7.2.1.12]). In particular, if

$$X \mapsto \mathrm{Mod}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

satisfies descent with respect to any effective epimorphism $U_\bullet \to X$ inside $\mathcal{S}$, it automatically satisfies hyperdescent as well, and hence we can conclude that the global sections functor $\Gamma(X_{\mathrm{B}},-)\colon \mathrm{LocSysCat}^n(X;\Bbbk) \to \mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$ is monadic. So we can conclude thanks to the n-affineness criterion for Betti stacks provided in Lemma 2.28. □

Remark 2.40. Theorem 2.25 yields, for any $n \geqslant 1$ and for any n-truncated space X, an equivalence of symmetric monoidal $(n+1)$-categories

$$(n+1)\mathbf{LocSysCat}^n(X;\Bbbk) \simeq (n+1)\mathbf{Mod}_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)},$$

where $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$ is seen as a symmetric monoidal n-category via the natural (point-wise) symmetric monoidal structure. On the other hand, if X is connected we have a symmetric monoidal equivalence

$$(n+1)\mathbf{LocSysCat}^n(X;\Bbbk) \simeq (n+1)\mathbf{Mod}_{n\mathbf{LocSysCat}^n(\Omega_*X;\Bbbk)}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$$

in virtue of Theorem 1.6. Here, however, we consider the monoidal structure on $n\mathbf{LocSysCat}^{n-1}(\Omega_*X;\Bbbk)$ provided by the Day convolution tensor product, which takes into account the $\mathbb{E}_1$-algebra structure of Ω_*X. Combining these two equivalences, we obtain that for any connected and n-truncated space X there is an equivalence between presentable n-categorical modules for the standard monoidal structure on $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$ and n-categorical modules for the convolution monoidal structure on $n\mathbf{LocSysCat}^{n-1}(\Omega_*X;\Bbbk)$. Unraveling all the constructions, we can see that the explicit equivalence is provided by sending a $n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)$-module $n\mathscr{C}$ to the presentable n-category

$$n\mathscr{C} \otimes_{n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)} n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)},$$

which inherits an $n\mathbf{LocSysCat}^{n-1}(\Omega_*X;\Bbbk)$-action from the one on $n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$. This can be seen as a topological analogue of the Morita equivalence for convolution categories of [BFN12, Theorem 1.3].

We end this section with another immediate consequence of Theorem 2.26.

Corollary 2.41. *Let X be a space, and assume $\Bbbk$ to be a semisimple commutative ring. If there exists a base point x such that $\pi_{n+1}(X,x)$ contains an element g either of infinite order, or such that the order of g is a unit in $\Bbbk$, then the Betti $\Bbbk$-stack X_{B} is not n-affine. In particular, if k is a field of characteristic 0, if $\pi_{n+1}(X,x)$ does not vanish for all base points x then X is not n-affine.*

Proof. In virtue of Theorem 2.26, we only need to check whether the based loop space $\Omega_*^{n-1}X$ is 1-affine. But its second homotopy group is isomorphic to $\pi_{n+1}(X,x)$, so the conclusion follows from Proposition 2.14. □

3. Categorified Koszul duality via coaffine stacks

This Section contains our main contribution to $\mathbb{E}_n$-Koszul duality, at least in the topological setting. Consider first the case $n=1$. If X is a pointed simply connected space with the same homotopy type of a CW complex of finite type, the algebras $\mathrm{C}_\bullet(\Omega_*X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$ are $\mathbb{E}_1$-Koszul dual ([DGI06, §4.22]). When $\Bbbk$ is a field of characteristic 0, there is a kind of Morita equivalence relating modules over $\mathrm{C}_\bullet(\Omega_*X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$ but the right statement is subtle: it requires to either restrict to appropriately bounded modules; or to change the notion of module we work with. In particular, if we work with *ind-coherent* modules, i.e. if we replace $\mathrm{LMod}_{\mathrm{C}^\bullet(X;\Bbbk)}$ with $\mathrm{IndCoh}_{\mathrm{C}^\bullet(X;\Bbbk)}$, we do obtain an equivalence

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_*X;\Bbbk)} \simeq \mathrm{IndCoh}_{\mathrm{C}^\bullet(X;\Bbbk)}\,. \tag{3.1}$$

In this Section, we will explain how to define a category that should be viewed as the category of iterated "ind-coherent" modules over $\mathrm{C}^\bullet(X;\Bbbk)$. This will allow us to prove an n-categorical Morita equivalence statement relating $\mathrm{C}_\bullet(\Omega_*^nX;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$, which in particular recovers (3.1) when $n=1$. In fact we will not attempt to define directly a categorification of the notion of ind-coherent module. Rather, the key idea in our approach is using the theory *coaffine stacks* introduced in [Toë06] and further studied in [Lur11a]. We stress that our approach is new even in the classical case of $\mathbb{E}_1$-Koszul duality, although in that setting it is ultimately equivalent to (3.1).

We start by recalling some fundamental results in the theory of coaffine stacks defined over a field $\Bbbk$ of characteristic 0. We will mostly adopt the conventions from [Lur11a]. In particular, as in [Lur11a], we will call these objects *coaffine* rather than *affine* stacks, to stress the difference with affine schemes (which in turn are the spectra of *connective* $\Bbbk$-algebras). We will then use this theory to revisit the classical $\mathbb{E}_1$-Koszul duality between $\mathrm{C}_\bullet(\Omega_*X;\Bbbk)$ and $\mathrm{C}^\bullet(X;\Bbbk)$, and in particular equivalence (3.1). We will conclude this Section proving our main result (Theorem 3.22), which provides an n-categorification of equivalence (3.1).

Definition 3.2. Let $n\geqslant 1$ be an integer.

1) We say that a $\Bbbk$-algebra A is *n-coconnective* if the structure morphism $\Bbbk\to A$ induces an isomorphism of abelian groups

$$\Bbbk \xrightarrow{\cong} \pi_0 A$$

and the homotopy groups $\pi_i A$ vanish for both $i\geqslant 1$ and $-n<i<0$. If A is 1-coconnective, we shall simply say that A is *coconnective*.

2) A *coaffine stack* is a stack X which is equivalent to the stack

$$\mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(A,-)\colon \mathrm{CAlg}_{\Bbbk}^{\geqslant 0} \longrightarrow \mathcal{S}$$

for some coconnective $\Bbbk$-algebra A. In this case, we shall say that X is the *cospectrum of* A, and we shall denote it as $\mathrm{cSpec}(A)$.

3.3. Coaffine stacks behave in a very similar way to affine schemes: for any stack $\mathcal{Y}$ defined over $\Bbbk$, giving a morphism $\mathcal{Y} \to \mathrm{cSpec}(A)$ is equivalent to giving a morphism of commutative $\Bbbk$-algebras $A \to \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ ([Lur11a, Theorem 4.4.1]). Moreover, any coaffine stack $\mathcal{X} \simeq \mathrm{cSpec}(A)$ can be realized as the left Kan extension of its restriction to *discrete* $\Bbbk$-algebras (i.e., coaffine stacks are 0-*coconnective* in the sense of [GR17, Chapter 2, §1.3.4]). In particular, defining the category of classical affine schemes

$$\mathrm{Aff}_{\Bbbk}^{\mathrm{cl}} := \left(\mathrm{CAlg}_{\Bbbk}^{\mathrm{disc}}\right)^{\mathrm{op}}$$

as the opposite of the category of discrete $\Bbbk$-algebras, we have that for any coaffine stack $\mathcal{X} \simeq \mathrm{cSpec}(A)$ the inclusion

$$\left(\mathrm{Aff}_{\Bbbk}^{\mathrm{cl}}\right)_{/\mathcal{X}} \simeq \left(\left(\mathrm{CAlg}_{\Bbbk}^{\mathrm{disc}}\right)_{A/}\right)^{\mathrm{op}} \subseteq \left(\left(\mathrm{CAlg}_{\Bbbk}\right)_{A/}\right)^{\mathrm{op}} \simeq \left(\mathrm{Aff}_{\Bbbk}\right)_{/\mathcal{X}}$$

is cofinal, as stated in the proof of [GR17, Chapter 3, Lemma 1.2.2].

However, contrarily to the case of affine schemes, the category

$$\mathrm{QCoh}(\mathcal{X}) := \lim_{\substack{\mathrm{Spec}(R)\to\mathcal{X} \\ R\in\mathrm{CAlg}_{\Bbbk}^{\geqslant 0}}} \mathrm{Mod}_R$$

of quasi-coherent sheaves over a coaffine stack $\mathcal{X} \simeq \mathrm{cSpec}(A)$ does not recover the category of A-modules. Rather, we have the following result.

Proposition 3.4 ([Lur11a, Propositions 3.5.2 and 3.5.4, Remark 3.5.6])**.** *Let A be a coconnective $\Bbbk$-algebra, let $\mathcal{X} := \mathrm{cSpec}(A)$ be its corresponding coaffine stack. Let $\eta \in \mathcal{X}(\Bbbk)$ be any $\Bbbk$-point of $\mathcal{X}$.*

(1) There exists a right complete t-structure on Mod_A defined as follows.
- *The coconnective objects are detected via the forgetful functor*

$$\mathrm{oblv}_A\colon \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\Bbbk}.$$

- *The connective objects are those A-modules such that, for any morphism of commutative $\Bbbk$-algebras $A \to R$ with R connective, the R-module $M \otimes_A R$ is connective.*

(2) There exists a both left and right complete t-structure on $\mathrm{QCoh}(\mathcal{X})$, whose heart is equivalent to the ordinary abelian category of algebraic representations of the prounipotent group scheme $\pi_1(\mathcal{X},\eta)$. Both connective and coconnective objects are detected via the pullback functor

$$\eta^*\colon \mathrm{QCoh}(\mathcal{X}) \longrightarrow \mathrm{QCoh}(\mathrm{Spec}(\Bbbk)) \simeq \mathrm{Mod}_{\Bbbk}.$$

(3) Let $F : \mathrm{Mod}_A \to \mathrm{QCoh}(\mathfrak{X})$ be the natural symmetric monoidal pullback functor. Then F exhibits the t-structure of $\mathrm{QCoh}(\mathfrak{X})$ as the left completion of the t-structure on Mod_A.

Definition 3.5 ([Lur11b, Definitions 3.0.1, 3.1.13 and 3.4.1])**.** Let A be an associative $\Bbbk$-algebra, let M be a left A-module.

1) We say that M is *locally small* if $\pi_k M$ is a finite dimensional $\Bbbk$-vector space for any integer k. We say that A is *locally small* if its underlying A-module is locally small.
2) We say that M is *small* if
$$\pi_\bullet M := \bigoplus_{k\in\mathbb{Z}} \pi_k M$$
is a finite dimensional $\Bbbk$-vector space.
3) We say that A is *small* if its underlying A-module is connective and small, and the structure morphism $\Bbbk \to A$ induces an isomorphism of discrete $\Bbbk$-algebras $\Bbbk \cong \pi_0 A/\mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\pi_0 A$.

Remark 3.6. Let A be an associative $\Bbbk$-algebra, and let $\mathrm{LMod}_A^{\mathrm{sm}}$ be the full sub-category of LMod_A spanned by small objects. We have a Cartesian diagram of categories

$$\begin{array}{ccc} \mathrm{LMod}_A^{\mathrm{sm}} & \xrightarrow{\mathrm{oblv}_A} & \mathrm{Perf}_{\Bbbk} \\ \big\downarrow & & \big\downarrow \\ \mathrm{LMod}_A & \xrightarrow[\mathrm{oblv}_A]{} & \mathrm{Mod}_{\Bbbk}. \end{array}$$

Forgetting the A-module structure commutes with all limits and colimits, and $\mathrm{Perf}_{\Bbbk}$ is stable under finite limits and colimits inside $\mathrm{Mod}_{\Bbbk}$. It follows that $\mathrm{LMod}_A^{\mathrm{sm}}$ is a stable (but of course not complete or cocomplete) sub-category of LMod_A. In particular, its ind-completion

$$\mathrm{IndCoh}_A^{\mathrm{L}} := \mathrm{Ind}\left(\mathrm{LMod}_A^{\mathrm{sm}}\right)$$

is stable ([Lur17, Proposition 1.1.3.6]). Moreover, since $\mathrm{LMod}_A^{\mathrm{sm}}$ admits all finite coproducts, it follows that $\mathrm{IndCoh}_A^{\mathrm{L}}$ admits *all* coproducts (since they are realized as filtered colimits of finite coproducts).

Definition 3.7. Let A be an associative $\Bbbk$-algebra. The category $\mathrm{IndCoh}_A^{\mathrm{L}}$ is the *category of left ind-coherent modules on A*.

Warning 3.8. The notation can be misleading: in [GR17], *ind-coherent sheaves* over an affine scheme are interpreted as bounded modules with coherent homology. Rather, our definition of ind-coherent modules matches the one in [Lur11b, Definition 3.4.4]. Yet, if A is a discrete local Artinian ring, or a small $\Bbbk$-algebra in the sense of Definition 3.5.(3), then the two notions coincide.

In the following, we revisit $\mathbb{E}_1$-Koszul duality for associative algebras and its formulation in terms of correspondences between categories of ind-coherent and quasi-coherent modules.

Recall that, over any field $\Bbbk$, an augmented associative $\Bbbk$-algebra A admits a $\mathbb{E}_1$-Koszul dual $A^!$ if and only if there exists a morphism

$$\mu : A \otimes A^! \longrightarrow \Bbbk$$

which exhibits $A^!$ as the classifying object $\underline{\mathrm{Map}}_A(\Bbbk, \Bbbk)$ of morphisms of left A-modules from $\Bbbk$ to itself, see [Lur11b, Remark 3.1.12].

Proposition 3.9. *Let A be an augmented associative $\Bbbk$-algebra such that the augmentation of A induces an isomorphism*

$$\pi_0 A \xrightarrow{\cong} \Bbbk.$$

Suppose that the Koszul dual $A^!$ is locally small as a $\Bbbk$-module. Then there is an equivalence of categories

$$\mathrm{IndCoh}^{\mathrm{L}}_A \simeq \mathrm{RMod}_{A^!}. \tag{3.10}$$

Proof. Using [Lur11b, Remark 3.4.2], we know that $\mathrm{LMod}^{\mathrm{sm}}_A$ is the thick sub-category spanned by $\Bbbk$ inside LMod_A – i.e., it is the smallest stable category sitting inside LMod_A containing $\Bbbk$ and closed under retracts. In particular, the Koszul duality functor for modules

$$\mathrm{LMod}^{\mathrm{op}}_A \longrightarrow \mathrm{LMod}_{A^!}$$

restricts to an equivalence between $\mathrm{LMod}^{\mathrm{sm}}_A$ and $\mathrm{Perf}^{\mathrm{op}}_{A^!}$ ([Lur11b, Proposition 3.5.6]), so applying the $A^!$-linear duality self-functor we have

$$\mathrm{LMod}^{\mathrm{sm}}_A \xrightarrow{\simeq} \mathrm{Perf}_{A^!}$$

hence an equivalence on their ind-completions. □

Remark 3.11. At first sight, [Lur11b, Proposition 3.5.2] would seem to imply the need for some smallness assumption on $A^!$ in Proposition 3.9. Actually, this is not the case: the smallness is only needed in order to have an equivalence of *functors* from $\mathrm{Alg}^{\mathrm{sm}}_{\Bbbk}$ to $\widehat{\mathrm{Cat}}_{(\infty,1)}$ between $\mathrm{IndCoh}^{\mathrm{L}}_{(-)}$ and $\mathrm{RMod}_{(-)^!}$. Of course, if we do not assume our algebras to be small the tensor product does not preserve small modules, so the functor $\mathrm{IndCoh}^{\mathrm{L}}_{(-)}$ is not even well-defined; yet, the *point-wise* equivalence (3.10) still applies under our, milder, assumptions on A.

We shall now equip $\mathrm{IndCoh}^{\mathrm{L}}_{A^!}$ with a t-structure using the following general recipe.

Lemma 3.12 ([GR17, Chapter IV, Lemma 1.2.4])**.** *Let $\mathscr{C}$ be a (non cocomplete) stable category, endowed with a t-structure. Then $\mathrm{Ind}(\mathscr{C})$ carries a unique t-structure which is compatible with filtered colimits (i.e., such that truncation functors commute with filtered colimits), and for which the tautological inclusion $\mathscr{C} \subseteq \mathrm{Ind}(\mathscr{C})$ is t-exact. Moreover:*

1) *The sub-categories $\mathrm{Ind}(\mathscr{C})_{\geqslant 0}$ and $\mathrm{Ind}(\mathscr{C})_{\leqslant 0}$ are compactly generated by $\mathscr{C}_{\geqslant 0}$ and $\mathscr{C}_{\leqslant 0}$, respectively.*

2) Given any other stable category $\mathscr{D}$ equipped with a t-structure compatible with filtered colimits, any functor $F\colon \mathrm{Ind}(\mathscr{C}) \to \mathscr{D}$ is t-exact if and only if $F|_{\mathscr{C}}$ is t-exact.

Proposition 3.13. *Let A be a connective associative $\Bbbk$-algebra. Then $\mathrm{IndCoh}^{\mathrm{L}}_A$ admits a right complete t-structure. Moreover, if A is locally small the t-exact functor*

$$\Phi_A\colon \mathrm{IndCoh}^{\mathrm{L}}_A \longrightarrow \mathrm{LMod}_A$$

induced by the natural inclusion $\mathrm{LMod}^{\mathrm{sm}}_A \subseteq \mathrm{LMod}_A$ exhibits LMod_A as the left completion of the t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_A$.

Warning 3.14. If one assumes A to be *small* rather than only locally small, Proposition 3.13 boils down to [Lur11b, Proposition 3.4.18]. One could be confused by the fact that there the t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_A$ fails to be *right* complete, but this is easily explained: in [Lur11b] the functor Φ_A is replaced by another functor Ψ_A, which is more compatible with base change. The functor Ψ_A is closely related to Φ_A but involves A-linear duality as well; in particular, it swaps connective and coconnective objects, and this explains why the t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_A$ described in [Lur11b, Definition 3.4.16 and Remark 3.4.17] is left but not right complete. Rather, our definition of the t-structure in Proposition 3.13 resembles the t-structure on ind-coherent sheaves over Noetherian schemes defined in [GR17, Proposition 1.2.2].

Proof of Proposition 3.13. For any connective associative $\Bbbk$-algebra the restriction of the ordinary t-structure on the category of left A-modules yields a t-structure on $\mathrm{LMod}^{\mathrm{sm}}_A$: this boils down to the fact that this is true for $\mathrm{Perf}_{\Bbbk}$, and that forgetting the A-module structure is a conservative operation which preserves all limits and colimits. Thus, the existence of the t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_A$ follows from Lemma 3.12.

We can easily see that such t-structure is right complete as follows. Since $\mathrm{IndCoh}^{\mathrm{L}}_A$ is stable and admits uncountable coproducts, and coconnective objects are stable under uncountable coproducts (because this is true in LMod_A), we can use the (dual of the) criterion [Lur11b, Proposition 1.2.1.19] for the right completeness of t-structures on stable categories. Indeed, we have that

$$\left(\mathrm{IndCoh}^{\mathrm{L}}_A\right)_{\leqslant -\infty} := \bigcap_{n\geqslant 0} \left(\mathrm{IndCoh}^{\mathrm{L}}_A\right)_{\leqslant -n} \simeq \bigcap_{n\geqslant 0} (\mathrm{LMod}_A)_{\leqslant -n} \simeq 0.$$

Moreover, the functor $\Phi_A\colon \mathrm{IndCoh}^{\mathrm{L}}_A \to \mathrm{LMod}_A$ is t-exact: this is an obvious consequence of Lemma 3.12.(2) because the inclusion $\mathrm{LMod}^{\mathrm{sm}}_A \subseteq \mathrm{LMod}_A$ is t-exact.

To prove the claim about the left completion, we simply need to check that for any integer n the functor Φ_A induces an equivalence of categories

$$\left(\mathrm{IndCoh}^{\mathrm{L}}_A\right)_{\leqslant n} \xrightarrow{\simeq} (\mathrm{LMod}_A)_{\leqslant n}\,. \tag{3.15}$$

Indeed, the equivalence (3.15) would yield an equivalence between the categories of eventually coconnective objects

$$\mathrm{IndCoh}_A^{\mathrm{L},+} := \bigcup_{n\in\mathbb{Z}} \left(\mathrm{IndCoh}_A^{\mathrm{L}}\right)_{\leqslant n} \simeq \bigcup_{n\in\mathbb{Z}} (\mathrm{LMod}_A)_{\leqslant n} =: \mathrm{LMod}_A^{+}.$$

Restriction to eventually coconnective objects does not alter the left completion of a t-structure ([Lur17, Remark 1.2.1.18]), so this implies that the left completion of $\mathrm{IndCoh}_A^{\mathrm{L}}$ and LMod_A are equivalent. But the canonical t-structure on LMod_A is left complete ([Lur17, Proposition 7.1.1.13]), so we conclude that it has to be the left completion of the t-structure on $\mathrm{IndCoh}_A^{\mathrm{L}}$ as well.

Since the functor Φ_A is exact, we can reduce ourselves to consider the case $n=0$. We first prove that any perfect and coconnective left A-module M is obtained as a colimit of small coconnective left A-modules. Write any such M as a colimit

$$M \simeq \operatorname*{colim}_{i\in I} A^{\oplus r_i}[n_i]$$

over some diagram I. Notice that, even if M is perfect, the diagram cannot be assumed to be finite because M could be obtained from A via shifts, finite direct sums or *retracts*, and the latter are only realized as *countably infinite* colimits ([Lur09, Section 4.4.5]). Since M is coconnective, we have

$$M \simeq \tau_{\leqslant 0} M \simeq \tau_{\leqslant 0}\left(\operatorname*{colim}_{i\in I} A^{\oplus r_i}[n_i]\right) \simeq \operatorname*{colim}_{i\in I} \tau_{\leqslant 0}\left(A^{\oplus r_i}[n_i]\right),$$

where in the last equivalence we used the fact the truncation functor $\tau_{\leqslant 0}$ is a left adjoint. Since A is locally small and connective, each $\tau_{\leqslant 0}(A^{\oplus r_i}[n_i])$ is a small A-module. Moreover, by the very same definition of the t-structure on $\mathrm{LMod}_A^{\mathrm{sm}}$, we conclude that $\tau_{\leqslant 0}(A^{\oplus r_i}[n_i])$ is coconnective inside $\mathrm{LMod}_A^{\mathrm{sm}}$. Next, we prove that the functor Φ_A is fully faithful when restricted to $\left(\mathrm{IndCoh}_A^{\mathrm{L}}\right)_{\leqslant 0}$. We will actually prove that for *any* small left A-module M (seen trivially as a left ind-coherent A-module) and for any coconnective left ind-coherent A-module N the map of spaces

$$\mathrm{Map}_{\mathrm{IndCoh}_A^{\mathrm{L}}}(M, N) \longrightarrow \mathrm{Map}_{\mathrm{LMod}_A}(\Phi_A(M), \Phi_A(N))$$

is an equivalence. Writing N as a filtered colimit $\mathrm{colim}_i N_i$, with each N_i small and coconnective, we have that

$$\begin{aligned}\mathrm{Map}_{\mathrm{IndCoh}_A^{\mathrm{L}}}(M, N) &\simeq \mathrm{Map}_{\mathrm{IndCoh}_A^{\mathrm{L}}}\left(M, \operatorname*{colim}_i N_i\right)\\ &\simeq \operatorname*{colim}_i \mathrm{Map}_{\mathrm{IndCoh}_A^{\mathrm{L}}}(M, N_i) \simeq \operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A^{\mathrm{sm}}}(M, N_i),\end{aligned}$$

because each small A-module is compact in $\mathrm{IndCoh}_A^{\mathrm{L}}$. The functor Φ_A sends M and N to their actual colimits in LMod_A, so Φ_A is fully faithful on coconnective objects if

$$\operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A}(M, N_i) \longrightarrow \mathrm{Map}_{\mathrm{LMod}_A}\left(M, \operatorname*{colim}_i N_i\right)$$

is an equivalence. As we already observed, $\mathrm{LMod}_A^{\mathrm{sm}}$ is the thick stable sub-category of LMod_A spanned by $\Bbbk$: so it sufficient to write M as a retract of shifts and direct sums of $\Bbbk$

$$M \simeq \operatorname*{colim}_{j\in J} \Bbbk^{\oplus r_j}[n_j],$$

and observe that the augmentation $A \to \Bbbk$ produces a map

$$f\colon \operatorname*{colim}_j A^{\oplus r_j}[n_j] \longrightarrow M$$

whose fiber is at least 1-connective. In particular, recalling that each N_i is coconnective, we obtain

$$\begin{aligned}\operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A}(M, N_i) &\simeq \operatorname*{colim}_i \mathrm{fib}\left(\mathrm{Map}_{\mathrm{LMod}_A}\left(\operatorname*{colim}_j A^{\oplus r_j}[n_j], N_i\right) \longrightarrow \mathrm{Map}_{\mathrm{LMod}_A}(\mathrm{fib}(f), N_i)\right)\\ &\simeq \operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A}\left(\operatorname*{colim}_j A^{\oplus r_j}[n_j], N_i\right)\\ &\simeq \operatorname*{colim}_i \lim_j \mathrm{Map}_{\mathrm{LMod}_A}\left(A^{\oplus r_j}[n_j], N_i\right).\end{aligned}$$

Using the fact that the diagram J is a limit over the category Idem^+, and such limits are universal because they are colimits as well, we obtain that

$$\begin{aligned}\operatorname*{colim}_i \lim_j \mathrm{Map}_{\mathrm{LMod}_A}\left(A^{\oplus r_j}[n_j], N_i\right) &\simeq \lim_j \operatorname*{colim}_i \mathrm{Map}_{\mathrm{LMod}_A}\left(A^{\oplus r_j}[n_j], N_i\right)\\ &\simeq \lim_j \mathrm{Map}_{\mathrm{LMod}_A}\left(A^{\oplus r_j}[n_j], N\right) \simeq \mathrm{Map}_{\mathrm{LMod}_A}(M, N),\end{aligned}$$

and this concludes the proof. □

Remark 3.16. The fact that LMod_A is the left completion of $\mathrm{IndCoh}_A^{\mathrm{L}}$ implies that such t-structure is left complete if and only if $\mathrm{IndCoh}_A^{\mathrm{L}}$ is equivalent to LMod_A. This cannot be true if small and compact objects are not the same – which is *never* the case, unless A is discrete and finite as a $\Bbbk$-module. Indeed, the equality between the smallness and compactness conditions implies that the perfect left A-module A has to be small, hence eventually coconnective; but if A is not discrete, it is easy to see via a homological computation that the small left A-module $\pi_0 A$ does not admit a finite resolution of semi-free A-modules, hence it cannot be perfect.

The previous discussion allows us to reformulate (3.10) in terms of algebraic geometry. The key ingredient is the concept of cospectrum of a coconnective algebra.

Proposition 3.17. *Let A be a coconnective and locally small commutative $\Bbbk$-algebra, which as a mere associative algebra admits a $\mathbb{E}_1$-Koszul dual $A^!$. Then we have an equivalence of categories*

$$\mathrm{LMod}_{A^!} \simeq \mathrm{QCoh}(\mathrm{cSpec}(A)).$$

Proof. The augmented associative algebra $A^!$ is connective ([Lur11b, Theorem 3.1.14]); if A is locally small, one can see that $A^! \simeq \underline{\mathrm{Map}}_A(\Bbbk, \Bbbk)$ is locally small as well. In particular, Propositions 3.4 and 3.13 provides us with the following characters.

(1) The t-structure on $\mathrm{IndCoh}^{\mathrm{L}}_{A^!}$.
(2) The t-structure on $\mathrm{LMod}_{A^!}$, which is the left completion of the one on $\mathrm{IndCoh}^{\mathrm{L}}_{A^!}$.
(3) The t-structure on Mod_A.
(4) The t-structure on $\mathrm{QCoh}(\mathrm{cSpec}(A))$, which is the left completion of the one on Mod_A.

Since A is coconnective and locally small, we are in the setting of Proposition 3.9 and we obtain the equivalence (3.10). If such equivalence is t-exact then we can deduce our statement from the universal property of the left completion. Again, Lemma 3.12 implies that we just need to check whether the restriction of this equivalence to the full sub-category $\mathrm{LMod}^{\mathrm{sm}}$ is t-exact.

(1) First, notice that the duality functor $\mathrm{IndCoh}^{\mathrm{L}}_{A^!} \to \mathrm{Mod}_A$ preserves coconnective objects. Indeed, let $M^!$ be a coconnective small left $A^!$-module. Its image inside Mod_A is the module
$$M := \underline{\mathrm{Map}}_{A^!}\left(\Bbbk, M^!\right),$$
and this mapping $\Bbbk$-module is immediately seen to be coconnective since it is a mapping spectrum from a connective object to a coconnective one.
(2) The duality functor also preserves connective objects. We can see it as follows: notice that, inside Mod_A, a module M is connective precisely if for any (or, equivalently, *one*) map of $\Bbbk$-algebras $A \to R$ where R is connective, the R-module $R \otimes_A M$ is connective ([Lur11a, Proposition 4.5.4]). So we can test whether for a connective small left $A^!$-module $M^!$ the $\Bbbk$-module
$$M \otimes_A \Bbbk := \underline{\mathrm{Map}}_{A^!}\left(\Bbbk, M^!\right) \otimes_A \Bbbk$$
is connective. But this is just the underlying $\Bbbk$-module of the $A^!$-module $M^!$, since the inverse to
$$\underline{\mathrm{Map}}_{A^!}(\Bbbk, -)\big|_{\mathrm{LMod}^{\mathrm{sm}}_{A^!}} : \mathrm{LMod}^{\mathrm{sm}}_{A^!} \longrightarrow \mathrm{Perf}_A$$
is realized precisely by its left adjoint $- \otimes_A \Bbbk$. So our claim follows from the fact that $M^!$ was assumed to be connective in the first place.

□

We shall now apply these results and construction to a certain class of spaces to which Koszul duality applies. We fix the following definition.

Definition 3.18. Let $n \geqslant 0$ be an integer. A space X is *n-Koszul (over a field $\Bbbk$ of characteristic* 0 if the following conditions hold.

1) The space X is cohomologically of finite type over $\Bbbk$: the commutative algebra $\mathrm{C}^\bullet(X; \Bbbk)$ is locally small in the sense of Definition 3.5.(1).
2) The space X is $(n-1)$-connected.
3) The n-th homotopy group $\pi_n(X)$ is finite.

4) The space X is nilpotent: for any choice of a base point x, the fundamental group $\pi_1(X,x)$ is a nilpotent group which acts nilpotently on every higher homotopy group $\pi_k(X)$ for $k \geqslant 2$.

Remark 3.19.

(1) The property of being n-Koszul is obviously closed under finite products, essentially because of the Künneth formula.
(2) When $n \geqslant 1$, then an n-Koszul space is in particular k-Koszul for all $0 \leqslant k \leqslant n$.
(3) When $n \geqslant 1$, n-connectedness implies simply connectedness. Therefore, in order to check whether an n-connected space X is n-Koszul over $\Bbbk$ for some $n \geqslant 1$ it is sufficient to check whether it is cohomologically of finite type over $\Bbbk$.
(4) When X is simply connected, then the homotopy groups $\pi_k(X)$ are finitely generated for all $k \in \mathbb{N}$ if and only if the homology groups $\mathrm{H}_k(X;\mathbb{Z})$ are finitely generated for all $k \in \mathbb{N}$ as well ([MP12, Theorem 4.5.4]). Flatness of fields of characteristic 0 over $\mathbb{Z}$ and the Künneth formula hence imply that $\mathrm{H}_k(X;\Bbbk)$ is finitely generated over $\Bbbk$ for all $k \in \mathbb{N}$. Finally, the universal coefficients theorem implies that $\mathrm{H}^k(X;\Bbbk)$ is finitely generated for all $k \in \mathbb{N}$. It follows that for all integers $n \geqslant 1$, any n-connected space with the same homotopy type as a CW complex of finite type is n-Koszul.
(5) Since the multiplication $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is an isomorphism, the discussion above carries on *verbatim* also to the case of simply connected spaces which are only of *rational* finite type.

In other words: whenever $n \geqslant 1$, if X is a pointed n-connected space with the same homotopy type as a CW complex which is either of finite type, or of rational finite type, then X is k-Koszul for all $0 \leqslant k \leqslant n$.

Whenever X is a pointed and $(n-1)$-connected space, the $\mathbb{E}_n$-Kozul dual of the $\mathbb{E}_n$-$\Bbbk$-algebra $\mathrm{C}_\bullet(\Omega^n_* X;\Bbbk)$ is computed by the underlying $\mathbb{E}_n$-algebra of the commutative algebra of $\Bbbk$-cochains on X, i.e.,

$$\mathrm{C}^\bullet(X;\Bbbk) \simeq \mathrm{C}_\bullet(\Omega^n_* X;\Bbbk)^{!_{\mathbb{E}_n}}.$$

The n-Koszul hypothesis is a sufficient condition for the reciprocal duality. That is, when X is pointed and n-Koszul we also have the equivalence

$$\mathrm{C}_\bullet(\Omega^n_* X;\Bbbk) \simeq \mathrm{C}^\bullet(X;\Bbbk)^{!_{\mathbb{E}_n}},$$

compare with [AF15, Proposition 5.3, ArXiv v6].

We can apply the machinery of Proposition 3.17 to deduce the following.

Corollary 3.20. *Let X be a pointed 1-Koszul space. Then we have an equivalence*

$$\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)} \simeq \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))).$$

Remark 3.21. If X is a pointed 1-Koszul space then the equivalence of Corollary 3.20 arises geometrically as follows. Thanks to (1.5), the category $\mathrm{LMod}_{\mathrm{C}_\bullet(\Omega_* X;\Bbbk)}$ can be equivalently described as $\mathrm{LocSys}(X;\Bbbk)$, which is the category of quasi-coherent sheaves over the Betti stack X_{B} (as already observed in Paragraph 2.2). Since $\Gamma(X_{\mathrm{B}},\mathcal{O}_{X_{\mathrm{B}}})\simeq \mathrm{C}^{\bullet}(X;\Bbbk)$, the identity map of $\mathrm{C}^{\bullet}(X;\Bbbk)$ induces an affinization map

$$\mathrm{aff}_X\colon X_{\mathrm{B}}\longrightarrow \mathrm{C}^{\bullet}(X;\Bbbk).$$

The equivalence of Corollary 3.20 is then realized by pulling back and pushing forward along aff_X. Indeed, the pullback functor aff_X^* is a functor between stable categories equipped with both left and right complete t-structures which is strongly monoidal and right t-exact (i.e., it preserves connective objects). Therefore, using [Lur11b, Corollary 4.6.18], we can deduce that it is uniquely determined by the symmetric monoidal and right t-exact functor

$$\widetilde{\mathrm{aff}}_X^*\colon \mathrm{Mod}_{\mathrm{C}^{\bullet}(X;\Bbbk)}\longrightarrow \mathrm{LMod}_{\mathrm{C}_{\bullet}(\Omega_*X;\Bbbk)}$$

which is obtained by pre-composing aff_X^* with the natural left completion functor $\mathrm{Mod}_{\mathrm{C}^{\bullet}(X;\Bbbk)}\to \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk)))$.

So, it will suffice to understand the behaviour of $\widetilde{\mathrm{aff}}^*$. Let $\eta\colon\{*\}\to X$ be the chosen base point in X, and let $\eta_{\mathrm{B}}\colon \mathrm{Spec}(\Bbbk)\to X_{\mathrm{B}}$ be its image under the Betti stack functor. Pullback along η_{B} yields a forgetful functor $\mathrm{LMod}_{\mathrm{C}_{\bullet}(\Omega_*X;\Bbbk)}\to \mathrm{Mod}_{\Bbbk}$, and using again [Lur11b, Corollary 4.6.18] we can see that the symmetric monoidal and right t-exact functor

$$\eta_{\mathrm{B}}^*\circ \mathrm{aff}_X^*\colon\ \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk)))\longrightarrow \mathrm{Mod}_{\Bbbk}$$

uniquely corresponds to the natural base change functor $\mathrm{Mod}_{\mathrm{C}^{\bullet}(X;\Bbbk)}\to \mathrm{Mod}_{\Bbbk}$ along the coaugmentation $\mathrm{C}^{\bullet}(X;\Bbbk)\to\Bbbk$ induced at the level of $\Bbbk$-cochains by η. In particular, for any $\mathrm{C}^{\bullet}(X;\Bbbk)$-module M the underlying $\mathrm{C}_{\bullet}(\Omega_*X;\Bbbk)$-module of $\widetilde{\mathrm{aff}}_X^*(M)$ is equivalent to $M\otimes_{\mathrm{C}^{\bullet}(X;\Bbbk)}\Bbbk$. This is just the left adjoint of the Koszul duality functor which induces the equivalence of Proposition 3.17.

Corollary 3.20 allows us to revisit the classical Koszul duality for modules (3.10), in a substantially equivalent formulation. However, this point of view has a considerable advantage. Namely, while the concept of n-categorical ind-coherent modules is somewhat mysterious and it is far from clear how to define it directly, quasi-coherent sheaves on coaffine stacks can be categorified in a natural way: that is, we can consider quasi-coherent sheaves of n-categories over $\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))$ (Definition 1.16). Hence, $\mathbb{E}_n$-Koszul duality for categorified modules over $\mathbb{E}_n$-Koszul dual algebras in the topological setting can be straightforwardly generalized as follows.

Theorem 3.22. *Let $n\geqslant 1$ be an integer, and let X be a pointed $(n+1)$-Koszul space over a field $\Bbbk$ of characteristic 0 whose homotopy groups $\pi_q(X)$ are finitely generated for each $q\geqslant 0$. Then the natural $(n+1)$-functor*

$$\mathrm{aff}_X^*\colon (n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk)))\longrightarrow (n+1)\mathbf{LocSysCat}^n(X;\Bbbk)$$

is an equivalence of $(n+1)$-categories.

Remark 3.23. If we set

$$\mathrm{ShvCat}^0(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) := \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$$

and

$$\mathrm{LocSysCat}^0(X;\Bbbk) := \mathrm{LocSys}(X;\Bbbk),$$

then we can extend Theorem 3.22 also to $n=0$: indeed, this reduces to the combination of Corollary 3.20 with Remark 3.21.

Remark 3.24. We stress that Theorem 3.22 does provide a generalization of the usual $\mathbb{E}_1$-Koszul duality equivalence between categories of modules. Let us briefly comment on the two characters appearing in the statement: the $(n+1)$-category $(n+1)\mathbf{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ provides the natural higher categorification of the concept of the category of quasi-coherent sheaves over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. On the other hand, if X is $(n+1)$-Koszul (hence, n-connected), Theorem 1.6 provides an equivalence

$$(n+1)\mathbf{LocSysCat}^n(X;\Bbbk) \simeq (n+1)\mathbf{Lin}_{\mathrm{C}_\bullet(\Omega_*^{n+1}X;\Bbbk)}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}.$$

Thus, for X an $(n+1)$-Koszul space, Theorem 3.22 does relate (the categorification of) quasi-coherent sheaves over the coaffine stack $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$, and (the categorification of) left modules over the $\mathbb{E}_n$-algebra $\mathrm{C}_\bullet(\Omega_*X;\Bbbk)$.

3.25. Even if the proof of Theorem 3.22 is essentially carried out via an inductive argument, proving the $n=1$ case is strikingly more technically-demanding than the $n\geqslant 2$ case. Indeed, for $n\geqslant 2$ the proof is somewhat formal and essentially depends on the general behaviour of pullbacks and pushforwards of sheaves of $(n+1)$-categories along maps of prestacks (Remark 1.17); however, the case $n=1$ requires some more *ad hoc* arguments. Therefore, we first study this latter case in full detail – which provides the base case for the induction, and then prove the theorem for an arbitrary $n\geqslant 2$.

We start by proving some helpful results concerning Koszul spaces and their cochain $\Bbbk$-algebras, which will be used extensively in the proof of Theorem 3.22. Lemma 3.26 allows us to set up the inductive argument, while Lemma 3.27 and Proposition 3.28 are pivotal in relating the based loop stack of a coaffine stack $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ and the coaffine stack $\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$.

Lemma 3.26. *Let $n\geqslant 1$ be an integer, and let X be a pointed $(n+1)$-Koszul space over a field $\Bbbk$ of characteristic 0 such that the homotopy groups $\pi_q(X)$ are finitely generated for all integers $q\geqslant 0$. Then for any $1\leqslant k<n$ the iterated based loop space Ω_*^kX is $(n-k)$-Koszul over $\Bbbk$.*

Proof. Fix $1\leqslant k<n$. Obviously, if X is $(n-1)$-connected with finite $\pi_n(X)$ then Ω_*^kX is $(n-k-1)$-connected with finite $\pi_{n-k}(\Omega_*^kX)\cong\pi_n(X)$. Moreover, all connected based loop spaces are connected H-spaces, hence they are nilpotent ([MP12, Pag. 49]). The only non-trivial part of the statement is proving that Ω_*^kX inherits the condition of being of

cohomological $\Bbbk$-finite type (Definition 3.18.(1)). Recall that for any field $\Bbbk$ of characteristic 0, and for any simply connected space X whose $\Bbbk$-algebra of $\Bbbk$-cochains is locally small (in the sense of Definition 3.5.(1)), we have an isomorphism of $\Bbbk$-algebras

$$\mathrm{H}_\bullet(\Omega_* X;\Bbbk) \cong \mathrm{U}(\pi_\bullet(\Omega_* X)\otimes_{\mathbb{Z}}\Bbbk)$$

between the graded $\Bbbk$-algebra of $\Bbbk$-chains on $\Omega_* X$ and the graded universal enveloping $\Bbbk$-algebra of the graded Lie algebra $\pi_\bullet(\Omega_* X)\otimes_{\mathbb{Z}}\Bbbk$ endowed with the Whitehead bracket (see for example [FHT01, Theorem 16.13]). Since X is n-connected with finite $\pi_{n+1}(X)$, the underlying graded $\Bbbk$-vector space of

$$\pi_\bullet(\Omega_* X)\otimes_{\mathbb{Z}}\Bbbk \cong \pi_{\bullet+1}(X)\otimes_{\mathbb{Z}}\Bbbk$$

has non-trivial generators lying only in degrees $q \geqslant n$. It follows that the homology $\mathrm{H}_0(\Omega_* X;\Bbbk)$ is isomorphic to $\Bbbk$, the homology $\mathrm{H}_q(\Omega_* X;\Bbbk)$ is trivial for $1 \leqslant q \leqslant n-1$, and for $q \geqslant n$ we have that

$$\dim_{\Bbbk}\mathrm{H}_q(\Omega_* X;\Bbbk) \leqslant \dim_{\Bbbk}\left(\bigoplus_{p\geqslant 0}\left(\bigoplus_{i_1+\cdots+i_p=q}\pi_{i_1}(\Omega_* X)\otimes_{\mathbb{Z}}\cdots\otimes_{\mathbb{Z}}\pi_{i_p}(\Omega_* X)\otimes_{\mathbb{Z}}\Bbbk\right)\right).$$

The right hand side is obviously finite, because $\pi_\bullet(\Omega_* X)\otimes_{\mathbb{Z}}\Bbbk$ is bounded below and finitely generated in each degree in the first place. It follows that the algebra $\mathrm{C}_\bullet(\Omega_* X;\Bbbk)$ is locally small, hence the commutative algebra $\mathrm{C}^\bullet(\Omega_* X;\Bbbk)$ is locally small as well because of the universal coefficients theorem. Since $\Omega_* X$ is now an n-Koszul space whose homotopy groups are once again finitely generated for all integers $q \geqslant 0$, the claim for the iterated based loop space follows by induction. □

Lemma 3.27. *Let A be a coconnective $\Bbbk$-algebra, and let $A \to R$ and $A \to S$ be two A-algebras. Assume that R and S are coconnective as $\Bbbk$-algebras. Then we have a natural equivalences of stacks*

$$\mathrm{cSpec}(R\otimes_A S) \simeq \mathrm{cSpec}(R)\times_{\mathrm{cSpec}(A)}\mathrm{cSpec}(S).$$

Proof. First, notice that, if $\Bbbk$ is a field and A is a coconnective $\Bbbk$-algebras, then coconnective A-modules are stable under tensor product over A ([Lur11a, Proposition 4.5.4.(6)]). Moreover, [Lur17, Proposition 7.2.1.19] yields that $\pi_0(R\otimes_A S)\cong\Bbbk$, so $R\otimes_A S$ is itself a coconnective $\Bbbk$-algebra and it does make sense to consider the associated coaffine stack over $\Bbbk$.

We first observe that the functor cSpec sends tensor products of coconnective $\Bbbk$-algebras to products of stacks. Indeed, for any stack $\mathcal{X}$ and for any couple of coconnective $\Bbbk$-algebras

A_1 and A_2, we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{St}_{\Bbbk}}(\mathcal{X}, \mathrm{cSpec}(A_1)\times\mathrm{cSpec}(A_2)) &\simeq \mathrm{Map}_{\mathrm{St}_{\Bbbk}}(\mathcal{X}, \mathrm{cSpec}(A_1))\times\mathrm{Map}_{\mathrm{St}_{\Bbbk}}(\mathcal{X}, \mathrm{cSpec}(A_2))\\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(A_1, \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}))\times\mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(A_2, \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}))\\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(A_1\otimes_{\Bbbk}A_2, \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}))\\ &\simeq \mathrm{Map}_{\mathrm{St}_{\Bbbk}}(\mathcal{X}, \mathrm{cSpec}(A_1\otimes_{\Bbbk}A_2)),\end{aligned}$$

where we used that the tensor product is the coproduct in the category of $\Bbbk$-commutative algebras. So, let A be a coconnective $\Bbbk$-algebra, and let R and S be A-algebras which are coconnective as $\Bbbk$-algebras. Again, for any stack $\mathcal{X}$ we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{St}_{\Bbbk}}(\mathcal{X}, \mathrm{cSpec}(R\otimes_A S)) &\simeq \mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}(R\otimes_A S, \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}))\\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}\Big(\operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} R\otimes_{\Bbbk}A^{\otimes n}\otimes_{\Bbbk}S, \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})\Big)\\ &\simeq \lim_{[n]\in\Delta^{\mathrm{op}}}\mathrm{Map}_{\mathrm{CAlg}_{\Bbbk}}\big(R\otimes_{\Bbbk}A^{\otimes n}\otimes_{\Bbbk}S, \Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}})\big)\\ &\simeq \lim_{[n]\in\Delta^{\mathrm{op}}}\mathrm{Map}_{\mathrm{St}_{\Bbbk}}\big(\mathcal{X}, \mathrm{Spec}(R)\times\mathrm{cSpec}(A)^{\times n}\times\mathrm{Spec}(S)\big)\\ &\simeq \mathrm{Map}_{\mathrm{St}_{\Bbbk}}\Big(\mathcal{X}, \lim_{[n]\in\Delta^{\mathrm{op}}}\mathrm{Spec}(R)\times\mathrm{cSpec}(A)^{\times n}\times\mathrm{Spec}(S)\Big)\\ &\simeq \mathrm{Map}_{\mathrm{St}_{\Bbbk}}\big(\mathcal{X}, \mathrm{Spec}(R)\times_{\mathrm{cSpec}(A)}\mathrm{Spec}(S)\big).\end{aligned}$$

□

Proposition 3.28. *Let X be a pointed 2-Koszul space over a field $\Bbbk$ of characteristic 0. Then there is an equivalence of stacks*

$$\mathrm{cSpec}(\mathrm{C}^{\bullet}(\Omega_* X;\Bbbk))\simeq\mathrm{Spec}(\Bbbk)\times_{\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))}\mathrm{Spec}(\Bbbk).$$

Proof. Since X is 2-Koszul, it is simply connected and Lemma 3.26 implies that the algebra of $\Bbbk$-cochains of $\Omega_* X$ is of finite type over $\Bbbk$. So, we can apply the Eilenberg-Moore theorem (see for example [Lur11c, Corollary 1.1.10]) and deduce the existence of a canonical equivalence

$$\Bbbk\otimes_{\mathrm{C}^{\bullet}(X;\Bbbk)}\Bbbk\simeq\mathrm{C}^{\bullet}(\Omega_* X;\Bbbk).$$

Applying the cospectrum functor and using Lemma 3.27 we deduce our claim. □

The following is the key lemma for the proof of Theorem 3.22 when $n=1$.

Lemma 3.29. *Let X be a pointed 1-Koszul space over a field $\Bbbk$ of characteristic 0. Then the functor*

$$\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))}\colon \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk)))}\big(\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\big)\longrightarrow\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk)))$$

is fully faithful.

Proof. Since X is 1-Koszul, Corollary 3.20 applies: so we obtain a commutative diagram of categories

$$\begin{array}{ccc} \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\big(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\big) & \xrightarrow{\ \simeq\ } & \mathrm{Lin}_{\mathrm{LocSys}(X;\Bbbk)}\big(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\big). \\ \Big\downarrow {\scriptstyle \mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}} & & \Big\downarrow {\scriptstyle \mathrm{Loc}_{X_{\mathrm{B}}}} \\ \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) & \xrightarrow{\ \mathrm{aff}_X^*\ } & \mathrm{LocSysCat}(X;\Bbbk) \end{array} \tag{3.30}$$

and Proposition 2.5 implies that the composition

$$\mathrm{aff}_X^* \circ \mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))} \colon \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\big(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\big) \longrightarrow \mathrm{LocSysCat}(X;\Bbbk)$$

is fully faithful. This means that for every categorical $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$-modules $\mathscr{C}$ and $\mathscr{D}$ and for all $k \geqslant 0$, any k-simplex

$$[\sigma] \in \pi_k\big(\mathrm{Map}_{\mathrm{LocSysCat}(X;\Bbbk)}\big(\mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{C}), \mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{D})\big)\big)$$

is homotopic to the image of an essentially unique k-simplex

$$[\widetilde{\sigma}] \in \pi_k\Big(\mathrm{Map}_{\mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)}(\mathscr{C}, \mathscr{D})\Big)$$

under the functor $\mathrm{Loc}_{X_{\mathrm{B}}}$. We will prove that this forces every k-simplex

$$[\tau] \in \pi_k\big(\mathrm{Map}_{\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\big(\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{C}), \mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{D})\big)\big)$$

to arise in the same way. In virtue of the commutativity of the diagram (3.30), if a k-simplex $[\tau]$ as above is the image under $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}$ of some k-simplex $[\widetilde{\tau}]$ then such $[\widetilde{\tau}]$ is unique up to homotopy. Indeed, $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}$ is the first map in the composition $\mathrm{aff}_X^* \circ \mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))} \simeq \mathrm{Loc}_{X_{\mathrm{B}}}$. Since the latter is a fully faithful functor, it induces a morphism between mapping spaces which is an isomorphism on all homotopy groups; therefore, composition with $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}$ produces a morphism between mapping spaces which is forced to be injective on all homotopy groups. So we only need to prove that for all categorical $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$-modules $\mathscr{C}$ we can lift every k-simplex

$$[\tau] \in \pi_k\big(\mathrm{Map}_{\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\big(\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{C}), \mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{D})\big)\big)$$

to a k-simplex

$$[\widetilde{\tau}] \in \pi_k\Big(\mathrm{Map}_{\mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)}(\mathscr{C}, \mathscr{D})\Big).$$

Since X is assumed to be 1-Koszul, the commutative $\Bbbk$-algebra $\mathrm{C}^\bullet(X;\Bbbk)$ is in particular a coconnective $\Bbbk$-algebra in the sense of Definition 3.2.(1). The chosen base point $\eta\colon \{*\} \to X$ induces an augmentation $\mathrm{C}^\bullet(X;\Bbbk) \to \Bbbk$: this augmentation is essentially unique up to – non-unique – homotopy because of [Lur11a, Corollary 4.1.7]. In turn, this augmentation yields an essentially unique pointing $\mathrm{cSpec}(\eta)\colon \mathrm{Spec}(\Bbbk) \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$, which therefore can be assumed to factor as a composition

$$\mathrm{cSpec}(\eta)\colon\ \mathrm{Spec}(\Bbbk) \xrightarrow{\eta_{\mathrm{B}}} X_{\mathrm{B}} \xrightarrow{\mathrm{aff}_X} \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)).$$

So, let $[\tau]$ be a k-simplex in the space of maps between $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{C})$ and $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{D})$, as before. Composing with aff_X^*, we obtain a k-simplex

$$[\mathrm{aff}_X^*(\tau)] \in \pi_k\big(\mathrm{Map}_{\mathrm{LocSysCat}(X;\Bbbk)}\big(\mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{C}), \mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{D})\big)\big).$$

Since $\mathrm{Loc}_{X_{\mathrm{B}}}$ is fully faithful, the k-simplex $[\tau]$ comes from a k-simplex $[\tilde{\tau}]$ in the space of morphisms between $\mathscr{C}$ and $\mathscr{D}$ as presentably $\mathrm{LocSys}(X;\Bbbk)$-linear categories (or, equivalently, as presentably $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$-linear categories). Since the space of morphisms in a limit of categories is the limit of the spaces of morphisms in each category, the fact that $[\mathrm{aff}_X^*(\tau)] \simeq [\mathrm{Loc}_{X_{\mathrm{B}}}(\tilde{\tau})]$ means that for every R-point $\mathrm{Spec}(R) \to X_{\mathrm{B}}$ of the Betti stack X_{B} there exists a homotopy

$$[\Gamma(\mathrm{Spec}(R), \mathrm{aff}_X^*(\tau))] \simeq [\Gamma(\mathrm{Spec}(R), \mathrm{Loc}_{X_{\mathrm{B}}}(\tilde{\tau}))]$$

and all such homotopies come equipped with a system of higher homotopies which are compatible with base change. In particular, for $R = \Bbbk$, we have a homotopy

$$[\Gamma(\mathrm{Spec}(\Bbbk), \mathrm{aff}_X^*(\tau))] \simeq [\tilde{\tau} \otimes_{\mathrm{LocSys}(X;\Bbbk)} \mathrm{id}_{\mathrm{Mod}_{\Bbbk}}]. \tag{3.31}$$

Since X is 1-Koszul and the pointing $\mathrm{cSpec}(\eta)\colon\ \mathrm{Spec}(\Bbbk) \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ can be assumed to factor through X_{B}, given any categorical $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$-modules $\mathscr{C}$ the local sections over $\mathrm{Spec}(\Bbbk)$ of $\mathrm{Loc}_{X_{\mathrm{B}}}(\mathscr{C})$ and $\mathrm{aff}_X^*(\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\mathscr{C}))$ are the same. Indeed, they both are equivalent to $\mathscr{C} \otimes_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))} \mathrm{Mod}_{\Bbbk} \simeq \mathscr{C} \otimes_{\mathrm{LocSys}(X;\Bbbk)} \mathrm{Mod}_{\Bbbk}$. Therefore, (3.31) yields also a homotopy

$$[\Gamma(\mathrm{Spec}(\Bbbk), \tau)] \simeq [\tilde{\tau} \otimes_{\mathrm{LocSys}(X;\Bbbk)} \mathrm{id}_{\mathrm{Mod}_{\Bbbk}}]. \tag{3.32}$$

Now, using [Lur11a, Proposition 4.4.4], write $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ as a colimit of a simplicial diagram

$$\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)) \simeq \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Spec}(A^n)$$

where $A^0 \simeq \Bbbk$ and each A^n is discrete. This allows us to write

$$\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{ShvCat}(\mathrm{Spec}(A^n)) \simeq \lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Lin}_{A^n} \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}.$$

Thus, we can interpret a sheaf of categories $\mathscr{F}$ over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ as the datum of a presentably $\Bbbk$-linear category $\Gamma(\mathrm{Spec}(\Bbbk), \mathscr{F})$ together with a system of equivalences

$$\Gamma(\mathrm{Spec}(\Bbbk), \mathscr{F}) \otimes_{\mathrm{Mod}_{\Bbbk}} \mathrm{Mod}_{A^n} \simeq \Gamma(\mathrm{Spec}(A^n), \mathscr{F})$$

which has to be compatible with pullback along the maps forming the simplicial diagram $\mathrm{Spec}(A^\bullet) \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. In particular, taking the base change of the homotopy (3.32) along the maps $\mathrm{Spec}(A^n) \to \mathrm{Spec}(\Bbbk)$ lifts the homotopy $[\mathrm{aff}_X^*(\tau)] \simeq [\mathrm{Loc}_{X_{\mathrm{B}}}(\tilde{\tau})]$ to a homotopy $[\tau] \simeq [\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))}(\tilde{\tau})]$. □

We are ready to prove Theorem 3.22 when $n = 1$.

Proposition 3.33. *Let X be a pointed 2-Koszul space over a field $\Bbbk$ of characteristic 0 whose homotopy groups $\pi_q(X)$ are finitely generated for each $q \geqslant 0$. Then the affinization map $\mathrm{aff}_X : X_{\mathrm{B}} \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ induces an equivalence of 2-categories*

$$\mathrm{aff}_X^* : \mathbf{2ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \xrightarrow{\simeq} \mathbf{2LocSysCat}(X;\Bbbk).$$

Proof. The $\Bbbk$-cochains $\mathrm{C}^\bullet(\Omega_* X;\Bbbk)$ on the based loop space $\Omega_* X$ are equipped with the structure of a Hopf algebra because $\Omega_* X$ is a grouplike $\mathbb{E}_1$-monoid. Therefore, $\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk))$ is a group stack in virtue of Lemma 3.27, and Proposition 3.28 allows us to interpret $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ as the delooping of $\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk))$. Since $\Omega_* X$ is 1-Koszul in virtue of Lemma 3.26, we know that pulling back along affinization map $\mathrm{aff}_{\Omega_* X} : (\Omega_* X)_{\mathrm{B}} \to \mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk))$ induces a strongly monoidal equivalence

$$\mathrm{aff}_X^* : \mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk))) \xrightarrow{\simeq} \mathrm{LocSys}(\Omega_* X;\Bbbk).$$

In particular, $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk)))$ is fully dualizable and self-dual as an object of $\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ (because $\mathrm{LocSys}(\Omega_* X;\Bbbk)$ is self-dual). Moreover, the functor

$$\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk))} : \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right) \longrightarrow \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk)))$$

is fully faithful (Lemma 3.29), so we can apply the discussion in [Gai15, Section 10.2] and write

$$\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right),$$

where now $\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk)))$ is seen as a monoidal category via the convolution tensor product induced by the group structure on $\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk))$. Under the equivalence of Corollary 3.20, this monoidal structure corresponds to the Day convolution monoidal structure on $\mathrm{LocSys}(\Omega_* X;\Bbbk)$, hence we obtain a chain of equivalences

$$\begin{aligned}\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) &\xrightarrow{\simeq} \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)\\ &\xrightarrow{\simeq} \mathrm{Lin}_{\mathrm{LocSys}(\Omega_* X;\Bbbk)}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)\\ &\xrightarrow{\simeq} \mathrm{LocSysCat}(X;\Bbbk),\end{aligned}$$

where the second equivalence is obtained by base change along $\mathrm{aff}^*_{\Omega_* X}$ and the third equivalence is due to Remark 1.7. To check that this functor agrees with aff^*_X, we simply notice that the first equivalence

$$\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk)))}\left(\mathrm{Lin}_\Bbbk \mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)$$

sends a sheaf of categories $\mathscr{F}$ to the presentably $\Bbbk$-linear category $\Gamma(\mathrm{Spec}(\Bbbk), \mathscr{F})$ equipped with a $\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_* X;\Bbbk))$-module structure. Indeed, the inverse of the above equivalence factors as a chain of equivalences

$$\begin{aligned}\mathrm{Lin}_{\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))}\Big(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big) &\simeq \lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk)^{\times n})}\Big(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big)\\ &\simeq \lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))^{\times n})\\ &\simeq \mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))).\end{aligned}$$

The first equivalence is given by taking the dual $\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$-comodule structure on a categorical $\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))$-module $\mathscr{C}$ and producing the associated co-bar cosimplicial category $\mathrm{QCoh}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))^{\otimes n}\otimes_{\mathrm{Mod}_\Bbbk}\mathscr{C}$ ([Gai15, Corollary 10.1.5]). The second equivalence is given by taking the term-wise $\mathrm{Loc}_{\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))^{\times n}}$ functor ([Gai15, Proposition 10.1.3]). In both cosimplicial categories, the 0-th term is $\mathscr{C}$ itself. Under the third and last equivalence, this is precisely the category of local sections on $\mathrm{Spec}(\Bbbk)$ on the corresponding sheaf of categories over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$. Since the equivalence

$$\mathrm{Lin}_{\mathrm{QCoh}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\Big(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\Big)\simeq \mathrm{LocSysCat}(X;\Bbbk)$$

sends $\mathscr{C}$ to the categorical local system over X with stalk at the base point $\eta\colon\{*\}\to X$ equivalent to the underlying presentably $\Bbbk$-linear category of $\mathscr{C}$, it follows that the equivalence

$$\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))\simeq \mathrm{LocSysCat}(X;\Bbbk)$$

does not alter the local sections of a sheaf of categories over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$, so it is realized by the pullback along the affinization map.

We are left to promote such equivalence to a 2-categorical equivalence. In order to do this, we just need to check that the equivalence aff_X^* intertwines the coaugmentations from $\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}$ on both sides. This is clear since such coaugmentations are induced by pulling back along the terminal morphisms $X_{\mathrm{B}}\to\mathrm{Spec}(\Bbbk)$ and $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))\to\mathrm{Spec}(\Bbbk)$, and aff_X obviously commutes with them. □

Proposition 3.33 is the stepping stone for the inductive proof of Theorem 3.22. Before completing the proof, we observe the following easy fact concerning pushforward $(n+1)$-functors of quasi-coherent sheaves of n-categories for $n\geqslant 2$, which will be used in order to apply the inductive argument.

Remark 3.34. Let $f:\mathscr{X}\to\mathscr{Y}$ be a morphism of prestacks over a commutative ring spectrum $\Bbbk$, and let $n\geqslant 2$ be an integer. For a quasi-coherent sheaf of n-categories $n\mathscr{F}$ over $\mathscr{X}$, the $(n+1)$-functor f_* sends $n\mathscr{F}$ to a quasi-coherent sheaf of n-categories over $\mathscr{Y}$ whose local sections on an affine scheme $\mathrm{Spec}(R)$ over $\mathscr{Y}$ are described as

$$n\mathbf{\Gamma}(\mathrm{Spec}(R),f_*(n\mathscr{F}))\simeq \lim_{\mathrm{Spec}(S)\to\mathrm{Spec}(R)\times_{\mathscr{Y}}\mathscr{X}} n\Gamma(\mathrm{Spec}(S),n\mathscr{F}).$$

Such limit is computed along the pullback $(n+1)$-functors. Since the pushforward functor f_* is both right and left adjoint to the pullback functor f^*, the above limit can be equivalently computed as the colimit along the pushforward $(n+1)$-functors, i.e.,

$$n\Gamma(\mathrm{Spec}(R), f_*(n\mathcal{F})) \simeq \operatorname*{colim}_{\mathrm{Spec}(S)\to\mathrm{Spec}(R)\times_{\mathcal{Y}}\mathcal{X}} n\Gamma(\mathrm{Spec}(S), n\mathcal{F}).$$

Both the above limit and colimit are computed inside $(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}$.

Proof of Theorem 3.22. Proposition 3.33 proves the case for $n=1$. For a general $n \geqslant 2$: assume that we have proved Theorem 3.22 for all integers $1 \leqslant k \leqslant n-1$. Let

$$\mathrm{cSpec}(\eta)\colon \mathrm{Spec}(\Bbbk) \xrightarrow{\eta_{\mathrm{B}}} X_{\mathrm{B}} \xrightarrow{\mathrm{aff}_X^*} \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$$

be the pointing of $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ induced by the chosen base point $\eta\colon \{*\} \to X$. This produces a commutative diagram of categories

$$\begin{array}{ccc} \mathrm{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)) & \xrightarrow{\mathrm{aff}_X^*} & \mathrm{LocSysCat}^n(X;\Bbbk) \\ & {\scriptstyle \mathrm{cSpec}(\eta)^*} \searrow \quad \mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}. \quad \swarrow {\scriptstyle \eta_{\mathrm{B}}^*} & \end{array} \tag{3.35}$$

We will prove that the diagram (3.35) satisfies the hypotheses of [Lur17, Corollary 4.7.3.16]. This will allow us to apply the Barr–Beck–Lurie's monadicity theorem, and then conclude that the n-categorical equivalence holds as well thanks to Remark 1.19.

(1) The functor η_{B}^* is both monadic and comonadic: it is conservative, it commutes with all colimits, and is part of an ambidextrous adjunction. Its adjoint is computed as a left Kan extension along the pointing $\eta\colon \{*\} \to X$, which is the same as a right Kan extension in virtue of Remark 1.17. With our connectedness assumptions on X, this adjoint is extremely simple to describe: under the equivalence

$$\mathrm{LocSysCat}^n(X;\Bbbk) \simeq \mathrm{Lin}_{n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

the functor η_{B}^* corresponds to forgetting the $\mathrm{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)$-module structure, and the adjoint is given by

$$n\mathcal{C} \mapsto n\mathcal{C} \otimes_{n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk).$$

(2) The functor $\mathrm{cSpec}(\eta)^*$ is conservative. Indeed, suppose that a morphism of two quasi-coherent sheaves of n-categories $F\colon n\mathcal{F} \to n\mathcal{G}$ over $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ is an equivalence when considering local sections over $\mathrm{Spec}(\Bbbk)$: we want to prove that it is actually an equivalence on *all* local sections. Since $\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$ is a coaffine stack, we argue as in the proof of Lemma 3.29 and write

$$\mathrm{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))) \simeq \lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Lin}_{A^n}\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}$$

for some colimit simplicial diagram $\mathrm{Spec}(A^\bullet) \to \mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk))$, where $A^0 \simeq \Bbbk$ and each A^n is discrete. Then the claim is clear because for any stack $\mathcal{X}$ and any quasi-coherent sheaf of n-categories $n\boldsymbol{\mathcal{F}}$, for a morphism of affine schemes $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ living over $\mathcal{X}$ one has an equivalence of n-categories

$$n\boldsymbol{\Gamma}(\mathrm{Spec}(R), n\boldsymbol{\mathcal{F}}) \simeq n\boldsymbol{\Gamma}(\mathrm{Spec}(S), n\boldsymbol{\mathcal{F}}) \otimes_{n\mathbf{Lin}_S\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{Lin}_R\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}.$$

(3) The functor $\mathrm{cSpec}(\eta)^*$ commutes with all limits and colimits. Indeed, as observed in Remark 1.17, it admits a both left and right adjoint $\mathrm{cSpec}(\eta)_*$.

(4) For any $\Bbbk$-linear presentable n-category $n\boldsymbol{\mathcal{C}}$, the natural n-functor

$$\mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\boldsymbol{\mathcal{C}})\right) \longrightarrow \eta_{\mathrm{B},*}(n\boldsymbol{\mathcal{C}})$$

obtained via adjunction from the counit n-functor

$$\mathrm{cSpec}(\eta)^*(\mathrm{cSpec}(\eta)_*(n\boldsymbol{\mathcal{C}})) \simeq \eta_{\mathrm{B}}^*\left(\mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\boldsymbol{\mathcal{C}})\right)\right) \longrightarrow n\boldsymbol{\mathcal{C}}$$

is an equivalence. Since both $\mathrm{cSpec}(\eta)^*$ and η_{B}^* are conservative, we can reduce ourselves to check whether the n-functor at the level of local sections over $\mathrm{Spec}(\Bbbk)$

$$n\boldsymbol{\Gamma}\left(\mathrm{Spec}(\Bbbk), \mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\boldsymbol{\mathcal{C}})\right)\right) \longrightarrow n\boldsymbol{\Gamma}\left(\mathrm{Spec}(\Bbbk), \eta_{\mathrm{B},*}(n\boldsymbol{\mathcal{C}})\right) \tag{3.36}$$

is an equivalence. Under the equivalence

$$\mathrm{LocSysCat}^n(X;\Bbbk) \simeq \mathrm{LMod}_{n\mathbf{LocSysCat}^{n-1}(\Omega_*X;\Bbbk)}\left(\mathrm{Lin}_\Bbbk\mathrm{Pr}^{\mathrm{L}}_{(\infty,n)}\right),$$

the codomain of the functor (3.36) can be written as

$$n\boldsymbol{\Gamma}(\mathrm{Spec}(\Bbbk), \eta_{\mathrm{B},*}(n\boldsymbol{\mathcal{C}})) \simeq n\boldsymbol{\mathcal{C}} \otimes_{n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{LocSysCat}^{n-1}(\Omega_*X;\Bbbk).$$

The left hand side, using Remark 3.34 and Proposition 3.28, can be instead described as

$$n\boldsymbol{\Gamma}\left(\mathrm{Spec}(\Bbbk), \mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\boldsymbol{\mathcal{C}})\right)\right) \simeq \operatorname*{colim}_{\substack{\mathrm{Spec}(R)\to\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))\\ R\in\mathrm{CAlg}_\Bbbk^{\mathrm{disc}}}} n\boldsymbol{\mathcal{C}} \otimes_{n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{Lin}_R\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}.$$

Since the tensor product of presentable n-categories is compatible with colimits, we can swap the tensor product and the colimit and using once again Remark 1.17 we can write

$$\begin{aligned} n\boldsymbol{\Gamma}\left(\mathrm{Spec}(\Bbbk), \mathrm{aff}_X^*\left(\mathrm{cSpec}(\eta)_*(n\boldsymbol{\mathcal{C}})\right)\right) &\simeq n\boldsymbol{\mathcal{C}} \otimes_{n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} \left(\operatorname*{colim}_{\substack{\mathrm{Spec}(R)\to\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))\\ R\in\mathrm{CAlg}_\Bbbk^{\mathrm{disc}}}} n\mathbf{Lin}_R\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\right) \\ &\simeq n\boldsymbol{\mathcal{C}} \otimes_{n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}} n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(\Omega_*X;\Bbbk))). \end{aligned}$$

Therefore, the n-functor (3.36) can be interpreted as the tensor product over $n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}$ of the affinization $(n-1)$-functor

$$\mathrm{aff}^{*}_{\Omega_* X}\colon n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(\Omega_* X;\Bbbk))) \longrightarrow n\mathbf{LocSysCat}^{n-1}(\Omega_* X;\Bbbk)$$

with the identity functor of $n\mathscr{C}$. Since $\Omega_* X$ is $(n-1)$-Koszul (Lemma 3.26) and all its homotopy groups are once again finitely generated, the n-functor (3.36) is an equivalence because of the inductive hypothesis, as desired.

So, Barr–Beck–Lurie's monadicity theorem allows us to conclude. □

Remark 3.37. The reason why the above proof does not extend straightforwardly to the case when $n=1$, forcing us to tackle the latter in a somewhat more convoluted way, is that in this case the functor $\mathrm{cSpec}(\eta)^*$ can only be proved to be comonadic – i.e., it is not obvious that the map $\mathrm{aff}_X\colon X_{\mathrm{B}} \to \mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))$ is affine schematic, which is what guarantees that $\mathrm{cSpec}(\eta)_*$ is both a left and right adjoint to $\mathrm{cSpec}(\eta)^*$ ([Ste21, Corollary 14.2.10]). In particular, it is not obvious how to check that the natural functor

$$\mathscr{C}\otimes_{\mathrm{Mod}_{\Bbbk}}\left(\lim_{\substack{\mathrm{Spec}(R)\to\mathrm{cSpec}(\mathrm{C}^{\bullet}(\Omega_* X;\Bbbk))\\ R\in\mathrm{CAlg}^{\geqslant 0}_{\Bbbk}}}\mathrm{Mod}_R\right)\longrightarrow \lim_{\substack{\mathrm{Spec}(R)\to\mathrm{cSpec}(\mathrm{C}^{\bullet}(\Omega_* X;\Bbbk))\\ R\in\mathrm{CAlg}^{\geqslant 0}_{\Bbbk}}}\mathscr{C}\otimes_{\mathrm{Mod}_{\Bbbk}}\mathrm{Mod}_R$$

is an equivalence. This is true, *a posteriori*, because of Proposition 3.33.

We conclude this section with some explicit examples to which Theorem 3.22 applies.

Example 3.38. Let $X := \mathbf{B}\mathbb{CP}^{\infty}$. When $\Bbbk$ is a field of characteristic 0, its $\Bbbk$-cochain algebra $\mathrm{C}^{\bullet}(X;\Bbbk)$ is the symmetric $\Bbbk$-algebra on the $\Bbbk$-module $\Bbbk[-3]$. In particular, as a stack, $\mathrm{C}^{\bullet}(X;\Bbbk)\simeq \mathbf{B}^3(\mathbb{G}_{\mathrm{a},\Bbbk})$. The latter is known to be 1-affine ([Gai15, Theorem 2.5.7.(b)]). Notice that $\Omega_*\mathbf{B}^3\mathbb{G}_{\mathrm{a},\Bbbk}\simeq \mathbf{B}^2\mathbb{G}_{\mathrm{a},\Bbbk}\simeq \mathrm{cSpec}(\mathrm{C}^{\bullet}(\mathbb{CP}^{\infty};\Bbbk))$, which is again 1-affine ([Gai15, Theorem 2.5.7.(a)]). In particular, $\mathrm{Loc}_{\mathbf{B}^3\mathbb{G}_{\mathrm{a},\Bbbk}}$ and $\mathrm{Loc}_{\mathbf{B}^2\mathbb{G}_{\mathrm{a},\Bbbk}}$ are both trivially fully faithful.
So, we can argue as in Proposition 3.33 to describe $\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk)))$ as

$$\begin{aligned}\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(X;\Bbbk))) &\simeq \mathrm{Lin}_{\mathrm{QCoh}(\mathrm{C}^{\bullet}(\mathbb{CP}^{\infty};\Bbbk))}\left(\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)\\ &\simeq \mathrm{Lin}_{\mathrm{LocSys}(\mathbb{CP}^{\infty};\Bbbk)}\left(\mathrm{Lin}_{\Bbbk}\mathrm{Pr}^{\mathrm{L}}_{(\infty,1)}\right)\simeq \mathrm{LocSysCat}(X;\Bbbk).\end{aligned}$$

Applying the same strategy to $Y := \mathbf{B}X$ and to $\mathrm{cSpec}(\mathrm{C}^{\bullet}(Y;\Bbbk))$, we obtain an analogous equivalence

$$\mathrm{ShvCat}(\mathrm{cSpec}(\mathrm{C}^{\bullet}(\mathbf{B}^2\mathbb{CP}^{\infty};\Bbbk)))\xrightarrow{\simeq}\mathrm{LocSysCat}(\mathbf{B}^2\mathbb{CP}^{\infty};\Bbbk).$$

We now present some curious consequences of the above computation.

Corollary 3.39. *For all $n\geqslant 1$, the Betti stack $(\mathbf{B}^n\mathbb{CP}^{\infty})_{\mathrm{B}}$ is n-affine, while the Betti stack $\left(\mathbf{B}^{n+1}\mathbb{CP}^{\infty}\right)_{\mathrm{B}}$ is not n-affine.*

Proof. The case $n=1$ is obvious from Example 3.38, since $\mathbf{B}^3\mathbb{G}_{a,\Bbbk} \simeq \mathrm{C}^\bullet(\mathbf{B}\mathbb{CP}^\infty;\Bbbk)$ is 1-affine but $\mathbf{B}^4\mathbb{G}_{a,\Bbbk} \simeq \mathrm{C}^\bullet(\mathbf{B}^2\mathbb{CP}^\infty;\Bbbk)$ is not ([Gai15, Theorem 2.5.7.(c)]). Then, a simple inductive argument using Theorem 2.26 yields the result. □

Remark 3.40. For all $n \geqslant 1$, Corollary 3.39 offers an example of an n-affine Betti stack corresponding to a non-n-truncated space X. However, as predicted by Corollary 2.41, the $(n+1)$-th homotopy group of X is always trivial.

Corollary 3.41. *The Betti stack* $(\mathbb{CP}^\infty)_{\mathrm{B}}$ *is almost* 0*-affine.*

Proof. Just combine Corollary 3.39 (in the $n=1$ case) with Proposition 2.11. □

Remark 3.42. To our knowledge, Corollary 3.41 is a novel result. Via private communication, Y. Harpaz showed us that the global sections functor

$$\Gamma(\mathbb{CP}^\infty,-)\colon \mathrm{LocSys}(\mathbb{CP}^\infty;\Bbbk) \longrightarrow \mathrm{Mod}_\Bbbk$$

is indeed obtained by composing two monadic functors – namely, the monadic fully faithful Koszul duality functor $\mathrm{LMod}_{\mathrm{C}_\bullet(S^1;\Bbbk)} \subseteq \mathrm{Mod}_{\mathrm{C}^\bullet(\mathbb{CP}^\infty;\Bbbk)}$ and the forgetful functor $\mathrm{Mod}_{\mathrm{C}^\bullet(\mathbb{CP}^\infty;\Bbbk)} \to \mathrm{Mod}_\Bbbk$. In particular, even with standard (i.e., de-categorified) arguments it is obvious that the global sections functor must be conservative. However, it is not clear how to prove that it preserves colimits of $\Gamma(\mathbb{CP}^\infty,-)$-split simplicial objects.

Remark 3.43. Suppose that X is a pointed n-Koszul space over a field $\Bbbk$ of characteristic 0. Then, since X is in particular $(n-1)$-Koszul, one can expect to recover the $\mathbb{E}_{n-1}$-Koszul duality equivalence between $(n-2)$-categorical modules by "delooping" $\mathbb{E}_n$-Koszul duality between $(n-1)$-categorical modules. This is indeed the case: notice that the unit for the monoidal structure on $(n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ is the sheaf $n\underline{\mathbf{ShvCat}}^{n-1}(-)$ whose global sections are precisely $n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$. So we have an equivalence of mapping n-categories between

$$n\underline{\mathbf{Fun}}^{\mathrm{L}}_{(n+1)\mathbf{ShvCat}^n(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))}\big(n\underline{\mathbf{ShvCat}}^{n-1}(-),\, n\underline{\mathbf{ShvCat}}^{n-1}(-)\big)$$

and

$$n\underline{\mathbf{Fun}}^{\mathrm{L}}_{(n+1)\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}\Big(n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)},\, n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))\Big),$$

which is just $n\mathbf{ShvCat}^{n-1}(\mathrm{cSpec}(\mathrm{C}^\bullet(X;\Bbbk)))$ because $n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$ is the monoidal unit inside $(n+1)\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,1)}$.

Similarly, the monoidal unit for $(n+1)\mathbf{LocSysCat}^n(X;\Bbbk)$ is the trivial local system

$$n\underline{\mathbf{LocSysCat}}^{n-1}(-) := \mathrm{const}\Big(n\mathbf{Lin}_\Bbbk\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}\Big),$$

and so we have an equivalence of mapping n-categories between

$$n\underline{\mathbf{Fun}}^{\mathrm{L}}_{(n+1)\mathbf{LocSysCat}^n(X;\Bbbk)}\big(n\underline{\mathbf{LocSysCat}}^{n-1}(-),\, n\underline{\mathbf{LocSysCat}}^{n-1}(-)\big)$$

and

$$n\underline{\mathbf{Fun}}^{\mathrm{L}}_{(n+1)\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n)}}\left(n\mathbf{Lin}_{\Bbbk}\mathbf{Pr}^{\mathrm{L}}_{(\infty,n-1)}, n\mathbf{LocSysCat}^{n-1}(X;\Bbbk)\right) \simeq n\mathbf{LocSysCat}^{n-1}(X;\Bbbk).$$

The $(n+1)$-functor aff_X^* sends $n\underline{\mathbf{ShvCat}}^{n-1}(-)$ to $n\underline{\mathbf{LocSysCat}}^{n-1}(-)$, because it is strongly monoidal and hence preserves the monoidal unit. Since aff_X^* is also an equivalence of $(n+1)$-categories, it induces an equivalence at the level of mapping n-categories, and so it recovers the $\mathbb{E}_{n-1}$-Koszul duality equivalence for $(n-2)$-categorical modules. Applying iteratively this argument, we recover the $\mathbb{E}_k$-Koszul duality equivalence for modules for all $k \leqslant n$, up to the classical $\mathbb{E}_1$-Koszul duality for modules of Corollary 3.20.

References

[AF15] David Ayala and John Francis. "Factorization homology of topological manifolds". In: *J. Topol.* 8.4 (2015), pp. 1045–1084. ISSN: 1753-8416,1753-8424. DOI: 10.1112/jtopol/jtv028. URL: https://doi.org/10.1112/jtopol/jtv028.

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel. "Koszul duality patterns in representation theory". In: *J. Amer. Math. Soc.* 9.2 (1996), pp. 473–527. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-96-00192-0. URL: https://doi.org/10.1090/S0894-0347-96-00192-0.

[BFN12] David Ben-Zvi, John Francis, and David Nadler. *Morita equivalence for convolution categories: Appendix to arXiv:0805.0157*. 2012. arXiv: 1209.0193 [math.AG].

[BN12] David Ben-Zvi and David Nadler. "Loop spaces and connections". In: *J. Topol.* 5.2 (2012), pp. 377–430. ISSN: 1753-8416. DOI: 10.1112/jtopol/jts007. URL: https://doi.org/10.1112/jtopol/jts007.

[DGI06] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar. "Duality in algebra and topology". In: *Adv. Math.* 200.2 (2006), pp. 357–402. ISSN: 0001-8708. DOI: 10.1016/j.aim.2005.11.004. URL: https://doi-org.pros1.lib.unimi.it/10.1016/j.aim.2005.11.004.

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*. Vol. 205. Graduate Texts in Mathematics. Springer-Verlag, New York, 2001, pp. xxxiv+535. ISBN: 0-387-95068-0. DOI: 10.1007/978-1-4613-0105-9. URL: https://doi.org/10.1007/978-1-4613-0105-9.

[Gai15] Dennis Gaitsgory. "Sheaves of categories and the notion of 1-affineness". In: *Stacks and categories in geometry, topology, and algebra*. Vol. 643. Contemp. Math. Amer. Math. Soc., Providence, RI, 2015, pp. 127–225. DOI: 10.1090/conm/643/12899. URL: https://doi.org/10.1090/conm/643/12899.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality*. Vol. 221. Mathematical Surveys and Monographs.

American Mathematical Society, Providence, RI, 2017, xl+533pp. ISBN: 978-1-4704-3569-1. DOI: 10.1090/surv/221.1. URL: https://doi.org/10.1090/surv/221.1.

[GHM23] Benjamin Gammage, Justin Hilburn, and Aaron Mazel-Gee. *Perverse schobers and 3d mirror symmetry*. 2023. arXiv: 2202.06833 [math.RT].

[HPT23] Peter J. Haine, Mauro Porta, and Jean-Baptiste Teyssier. "The homotopy-invariance of constructible sheaves". In: *Homology Homotopy Appl.* 25.2 (2023), pp. 97–128. ISSN: 1532-0073,1532-0081. DOI: 10.4310/hha.2023.v25.n2.a6. URL: https://doi.org/10.4310/hha.2023.v25.n2.a6.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[Lur11a] Jacob Lurie. *Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems*. 2011. URL: http://people.math.harvard.edu/~lurie/papers/DAG-VIII.pdf.

[Lur11b] Jacob Lurie. *Derived Algebraic Geometry X: Formal Moduli Problems*. 2011. URL: http://people.math.harvard.edu/~lurie/papers/DAG-X.pdf.

[Lur11c] Jacob Lurie. *Derived Algebraic Geometry XIII: Rational and p-adic Homotopy Theory*. 2011. URL: http://people.math.harvard.edu/~lurie/papers/DAG-XIII.pdf.

[Lur17] Jacob Lurie. *Higher Algebra*. 2017. URL: http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[Lur18] Jacob Lurie. *Spectral Algebraic Geometry (Under Construction)*. 2018. URL: https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[MP12] J. P. May and K. Ponto. *More concise algebraic topology*. Chicago Lectures in Mathematics. Localization, completion, and model categories. University of Chicago Press, Chicago, IL, 2012, pp. xxviii+514. ISBN: 978-0-226-51178-8.

[PPS25] James Pascaleff, Emanuele Pavia, and Nicolò Sibilla. *Higher local systems and categorified monodromy equivalence*. 2025.

[PS02] César Polcino Milies and Sudarshan K. Sehgal. *An introduction to group rings*. Vol. 1. Algebra and Applications. Kluwer Academic Publishers, Dordrecht, 2002, pp. xii+371. ISBN: 1-4020-0238-6. DOI: 10.1007/978-94-010-0405-3. URL: https://doi.org/10.1007/978-94-010-0405-3.

[PS20] Mauro Porta and Francesco Sala. *Simpson's shapes of schemes and stacks*. 2020. URL: https://people.dm.unipi.it/sala/assets/pdf/porta_sala_shapes.pdf.

[Ste20] Germán Stefanich. *Presentable (∞, n)-categories*. 2020. arXiv: 2011.03035 [math.AT].

[Ste21] German Stefanich. *Higher Quasicoherent Sheaves*. Thesis (Ph.D.)–University of California, Berkeley. ProQuest LLC, Ann Arbor, MI, 2021, p. 347. ISBN: 979-8535-56467-8. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:28540389.

[Tel14] Constantin Teleman. "Gauge theory and mirror symmetry". In: *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*. Kyung Moon Sa, Seoul, 2014, pp. 1309–1332.

[Toë06] Bertrand Toën. "Champs affines". In: *Selecta Math. (N.S.)* 12.1 (2006), pp. 39–135. ISSN: 1022-1824,1420-9020. DOI: 10.1007/s00029-006-0019-z. URL: https://doi.org/10.1007/s00029-006-0019-z.