Introduction to Fukaya Categories

Lecture 3: Examples of Fukaya categories

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Hausdorff Institute, 2020-10-12 Mon

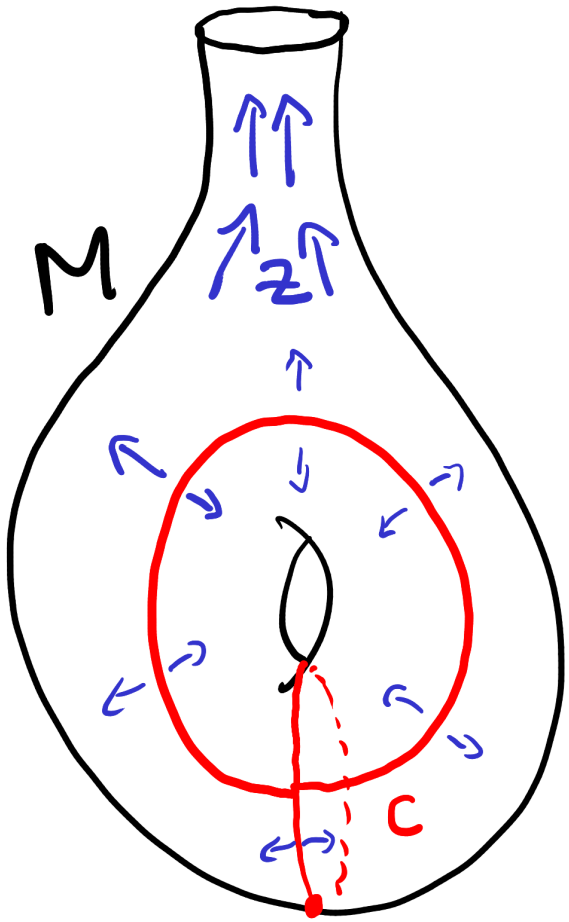
Outline

Liouville vector field

- Let (M, ω) be an exact symplectic manifold. A *Liouville form* is a one form θ such that $\omega = d\theta$.
- A choice of Liouville form θ induces a *Liouville vector field* Z on M, by the relation $\theta = \omega(Z, \cdot) = \iota_Z \omega$.
- By Cartan's magic formula, $\mathcal{L}_Z \omega = d\iota_Z \omega + \iota_Z d\omega = d\theta = \omega$.
- $\exp(tZ)_* \omega = e^t \omega$. So $(M, \omega) \cong (M, \lambda\omega)$ for $\lambda > 0$, provided Z is complete.
- In general the dynamics of Z may be complicated.
- Consider the limit set $C = \{\lim_{t \to -\infty} \exp(tZ)x \mid x \in M\}$, called the *core*.

Weinstein manifolds

- *Weinstein manifold*: Assume there is a proper bounded below function $\phi : M \to [0, \infty)$ such that Z is gradient-like for ϕ. Dynamics of Z is "tamed" by the function ϕ.
- The core is a stratifed isotropic subset.
- The program is to try to compress all of the symplectic geometry of M "onto the core C." (Roughly like taking $\lambda \to 0$ in $(M, \lambda\omega)$.)
- Leads to microlocal sheaves on the core/skeleton. (Nadler, Zaslow, many others.)



Wrapping Hamiltonians

- When considering Weinstein manifolds, we often include noncompact, properly embedded exact Lagrangians L (with tameness condition at ∞).
- M has a "wrapping Hamiltonian H" (grows quadratically at ∞; analogue of geodesic flow on T^*Q).
- The *wrapped Fukaya category* is defined as a type of localization of the Fukaya category with respect the flow of H. The *wrapped Floer cohomology* is

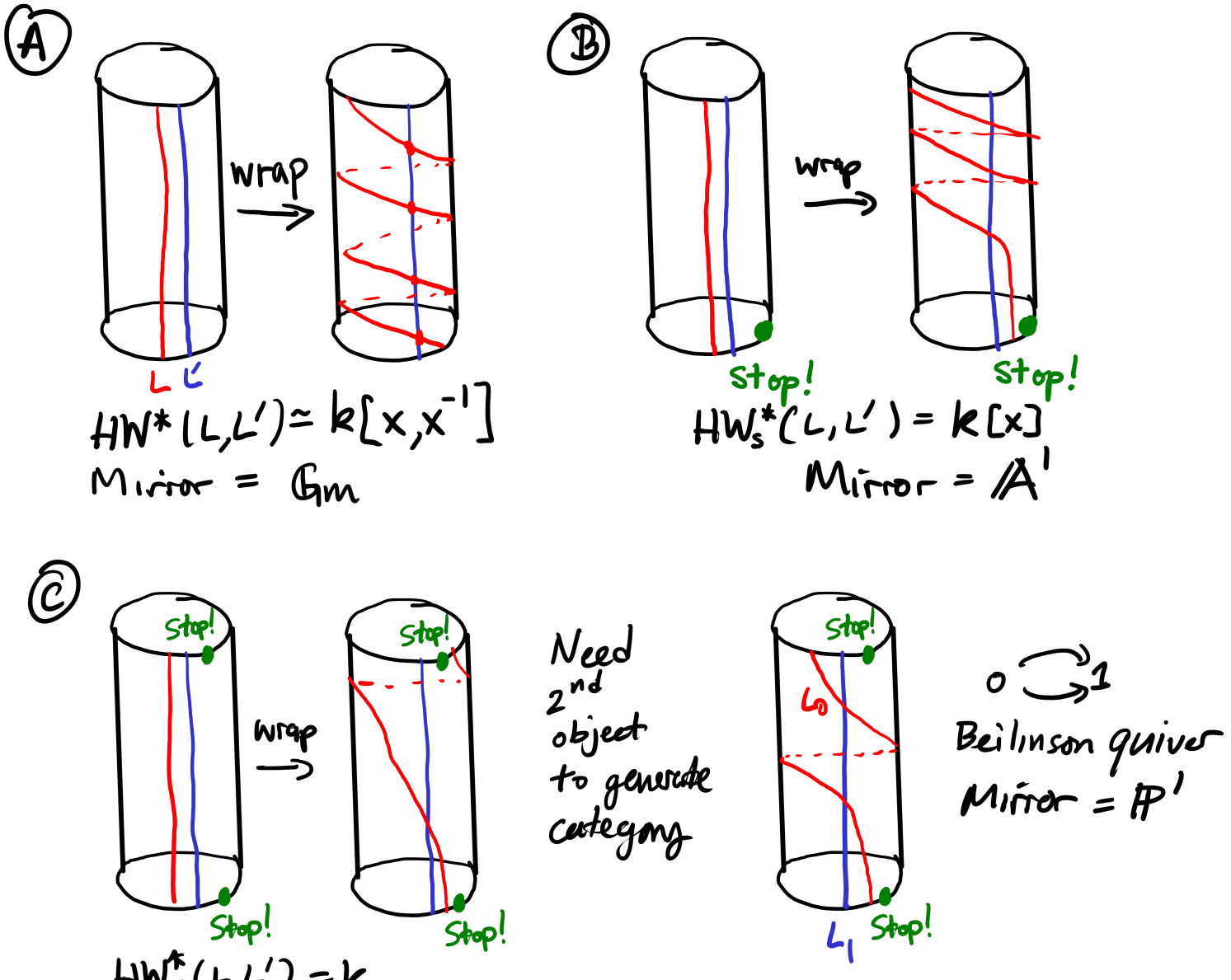
$$HW(L_0, L_1) = \lim_{t \to \infty} HF(\phi_H^t L_0, L_1)$$

- We can also introduce *stops*, markings that prescribe when to break off the limiting process.

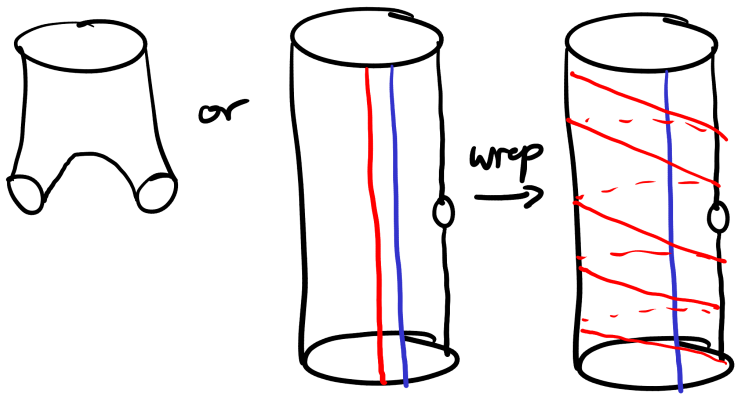
Case of cotangent bundle

- Let Q be a closed Pin manifold. Abouzaid shows:
- The wrapped Fukaya category is generated by a cotangent fiber $T^*_q Q$.
- The endomorphism algebra of $T^*_q Q$ is A_∞ equivalent to chains on the based loop space $C_{-*}(\Omega_q Q)$.
- Thus we have a complete description of the wrapped Fukaya category in terms of classical topology:
- Wrapped Fukaya category of $T^*Q \cong C_{-*}(\Omega_q Q) - \mathrm{mod}$.

Cylinder $= T^*S^1$



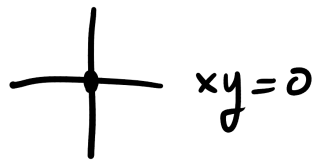
Pair of pants

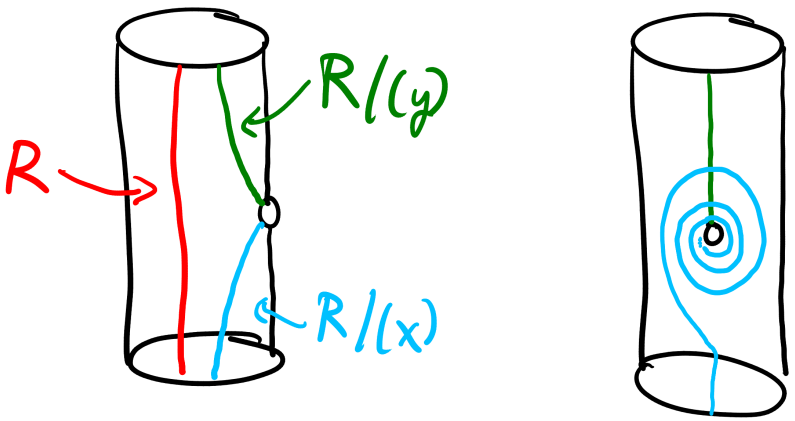


Now $HW^*(L,L') = k[x,y]/(xy)$
(product is different)

{We use the grading
Where $\deg(x) = \deg(y) = 0$}

$\mathrm{Spec}(R := k[x,y]/(xy))$ is a mirror :





$$\mathrm{Ext}^i_R(R/(x), R/(y)) = \begin{cases} k & \text{for } i \text{ odd} \\ 0 & \text{for } i \text{ even} \end{cases}$$

Weinstein handle attachment

- We can modify a Weinstein manifold by attaching a handle (corresponds to cell attachment in topology).
- "Critical" handle is attached along an Legendrian sphere.
- Legendrian submanifolds have a Floer theory that uses similar ingredients as the Fukaya category in a different way.
- Bourgeois-Ekholm-Eliashberg show how to get the wrapped Fukaya category out of it.

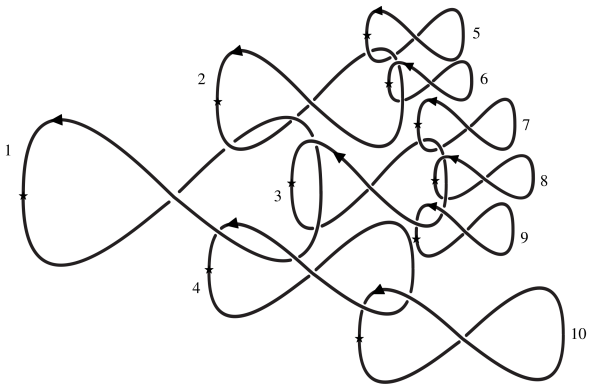
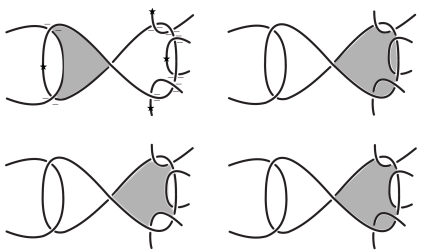


Figure 4: Lagrangian projection of Λ decorated with orientations and basepoints



(Figures from Lekili-Etgü arXiv:1502.07922)

Conclusion

- I hope to have convinced you that Fukaya categories are a natural thing to study, if somewhat technically complicated.
- If you are an algebraist, you may not want to worry so much about the foundations of Fukaya categories.
- In some cases, particularly the Weinstein case, theoretical advances have allowed us to access the information contained in the Fukaya category more easily.

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Mermin-Lekili slogan

Shut up and calculate . . . Fukaya categories.