Introduction to Fukaya Categories

Lecture 2: Floer cohomology and the Fukaya category

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Hausdorff Institute, 2020-10-09 Fri

Outline

Towards categorification

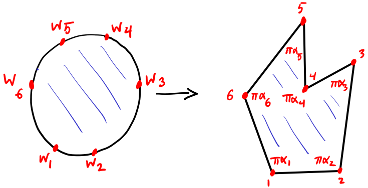
- Recognizing that a problem is a Lagrangian intersection problem reveals a path towards categorification: Fukaya categories!
- Given transversely intersecting oriented Lagrangians L_1, L_2, define $CF(L_1, L_2) = \bigoplus_{x \in L_1 \cap L_2} \mathbb{K} \cdot x$
- We need to understand the "processes" that can connect intersection points.

Holomorphic maps

- Equip (M, ω) with a compatible almost-complex structure J.
- Let (Σ, j) be a Riemann surface (2d complex manifold).
- If $u : \Sigma \to M$ is a map, we may formulate the Cauchy-Riemann equation $Du \circ j = J \circ Du$.
- If J is *integrable* ($\exists$ holomorphic coordinates on M), then this is a familiar notion.
- If Σ has boundary $\partial\Sigma$, we may require that $u : \partial\Sigma \to L$, where $L \subset M$ is a submanifold.

Polygons in the plane

- Let P be a polygon in $\mathbb{C}$ with angles $\pi\alpha_k$, $k = 1, \ldots, n$, $\alpha_k \in (0, 2]$.
- Riemann mapping theorem $\implies$ the interior of P is biholomorphic to $D^\circ = \{|w| < 1\}$.



Theorem (Schwarz-Christoffel Formula)

The function $z = F(w)$ that maps $\{|w| < 1\}$ biholomorphically onto P is of the form

$$F(w) = C \int_0^w \prod_{k=1}^{n} (w - w_k)^{\alpha_k - 1}\, dw + C'$$

where w_k are certain points on the unit circle and $C, C' \in \mathbb{C}$.

Boundary-punctured disks

- At a vertex where $\alpha_k \notin \mathbb{Z}$, the Schwarz-Christoffel formula has a branch point singularity. It makes sense to extend the domain of F to the closed disk with the points w_k removed. We call such a domain a *boundary-punctured disk*.
- $\mathrm{Aut}(D) \cong \mathrm{PSL}(2, \mathbb{R})$ acts on the set of boundary punctured disks. The quotient is the *moduli space* $\mathcal{R}^n$.
- $\dim \mathcal{R}^n = n - 3$. $\mathcal{R}^3 = \mathrm{pt}$. $\mathcal{R}^2 = \mathrm{pt}/\mathbb{R}$ (unstable situation).

Holomorphic polygons

- The "process" that connects intersection points are holomorphic polygons in M.
- Consider Lagrangian submanifolds $L_0, \ldots, L_n$ meeting transversally at intersection points $x_i \in L_{i-1} \cap L_i$, $x_0 \in L_n \cap L_0$. Let $(D, \{w_k\})$ be a disk with $n+1$ boundary punctures.

Definition

A *holomorphic polygon* in M with boundary data $\{L_i, x_i\}$ and domain $(D, \{w_k\})$ is a map $u : D \setminus \{w_k\} \to M$ such that

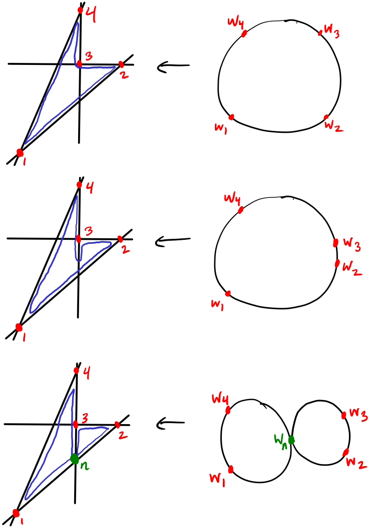
- u is holomorphic on D°.
- $\lim_{w \to w_k} u(w) = x_i$.
- u maps boundary between w_i and w_{i+1} to L_i.

What are we supposed to do with these polygons?

- We want to count them.
  - Are there finitely many? (finite-dimensional spaces of solutions; requires an analysis of the transversality of the moduli space; need to characterize situtations where it is finite.)
  - Need to count with signs? (Yes; need extra structure for this to work.)
- If we can count them, what structure does this collection of numbers have?
  - Relations come from degenerating the domain (Gromov compactification + gluing).
  - Most easily packaged as an A_∞ category.

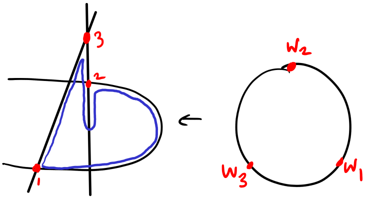
Example: 4-gon breaking into 3-gons

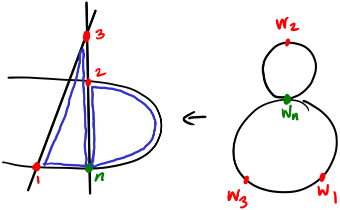
- Consider polygons with boundary on several straight lines.
- At non-convex corner, a slit may form.
- Eventually it breaks into smaller polygons.



Another degeneration: Floer breaking

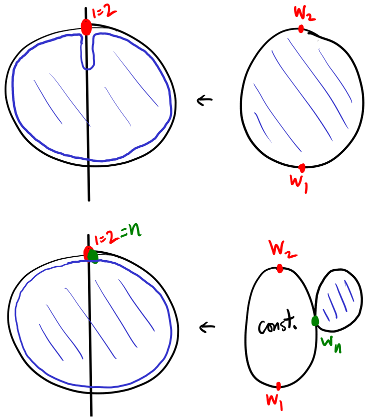
- Same phenomenon can occur when lines are not straight.
- Now a “bigon” or “strip” may break off.





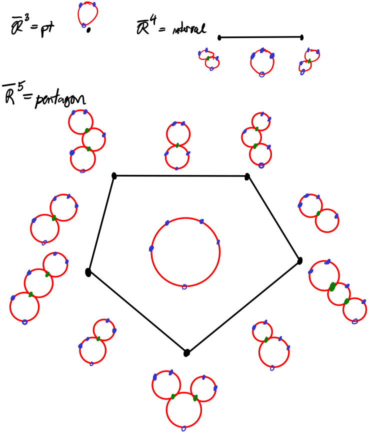
Yet another degeneration: disk breaking

- It is even possible that what breaks off is a disk (one boundary puncture).
- The broken configuration shown here contains a constant triangle.



Compactification of $\mathcal{R}^{d+1}$

- w_0 is the root.
- Degenerations correspond to groupings of other punctures $\{w_k \mid 1 \leq k \leq d\}$.
- Same as partial parenthesizations of d letters.
- $\overline{\mathcal{R}}^{d+1}$ = Associahedron K_d!



Compactification of the space of maps

- In general, given Lagrangians $L_0, \dots, L_d$ and intersections $x_0 \in L_d \cap L_0$ and $x_i \in L_{i-1} \cap L_i$ $(1 \le i \le d)$, we consider maps from polygons in $\mathcal{R}^{d+1}$ to M with these boundary data.
- As modulus of the domain varies, maps may degenerate, we compactify the space of maps "over the associahedron."
- We must also include Floer breaking and disk breaking in the compactification of the space of maps (even though domain does not degenerate).

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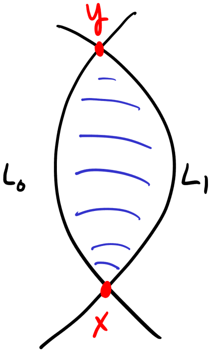
Gluing

For later arguments, it is necessary to know that every degenerate configuration actually appears in the boundary of the main stratum. This is among the most difficult analytical points.

Floer differential

- Recall $CF(L_0, L_1) = \sum_{x \in L_0 \cap L_1} \mathbb{K} \cdot x$.
- This carries a boundary map ∂ that counts holomorphic *strips*.
- Strips have an $\mathbb{R}$ action; consider orbits.
- Matrix element $n(x, y) =$ count "rigid modulo $\mathbb{R}$" strips.

$$\partial x = \sum_y n(x, y) y$$



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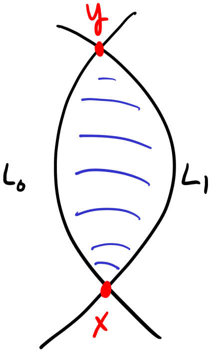
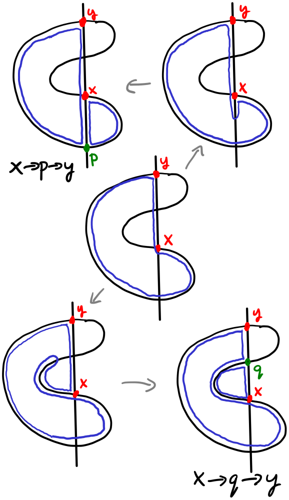


Potential issue

In this and later operations, it is possible that there are infinitely many terms. This can be handled by using formal variables (coefficients in Novikov field).

$\partial^2 = 0$

- ∂^2 counts pairs of rigid (modulo $\mathbb{R}$) strips.
- This set is the boundary of the one-dimensional (modulo $\mathbb{R}$) familes of strips.
- This cobordism pairs up the terms in ∂^2. With appropriate system of signs, they all cancel out.



$\partial^2 = 0$ ... unless we have disks with boundary on L_i.

This does not happen in the exact situation.

A holomorphic disk u has $\int_u \omega > 0$. If $\omega = d\theta$ and $\theta|_L = dF$ for $F : L \to \mathbb{R}$, then

$$\int_u \omega = \int_u d\theta = \int_{\partial u} \theta = \int_{\partial u} dF = \int_{\partial\partial u} F = 0.$$

Otherwise

We actually have to deal with the disk count. This leads to what is called a curved A_∞ structure; it includes an operation m_0 with zero inputs. We assume that there are no disks for now.

Floer cohomology

- The cohomology of the complex $(CF(L_0, L_1), \partial)$ is the *Floer cohomology* $HF(L_0, L_1)$ of the pair L_0, L_1.
- It is invariant under Hamiltonian deformations of L_i.
- If L_1 and L_2 are not transverse we use Hamiltonian deformation to perturb them.

Example

For $L =$ zero section in T^*Q, we have $HF(L, L) = H^*(Q)$.

m_d operation

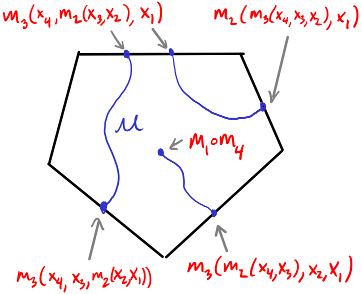
- Given Lagrangians $L_0, L_1, \ldots, L_d$, we aim to define a map

$$m_d : CF(L_{d-1}, L_d) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_d)$$

  by counting rigid holomorphic $(d+1)$ gons with boundary on L_i and punctures mapping to intersection points.
- m_1 is essentially ∂.
- By considering the one-dimensional moduli spaces of $(d+1)$ gons, we get relations involving m_k for $1 \leq k \leq d$.

Terms in A_∞ equations

- Visualize a one-dimensional component $\mathcal{M}$ by mapping to $\overline{\mathcal{R}}^{d+1}$.
- Where $\partial\mathcal{M}$ hits $\partial\overline{\mathcal{R}}^{d+1}$, we get a combination of m_k's with $k > 1$.
- Terms involving m_1 and m_d appear as points of $\partial\mathcal{M}$ that map to interior.
- Get all possible terms in the A_∞ equations.



Gradings and orientations

- We need to fill in some gaps in the story told so far.

Dimensions

The moduli spaces of maps break up into components according to the homotopy class of the map. How can we tell the dimension of each component?

Grading

How to put $\mathbb{Z}$ grading on $CF(L_0, L_1)$? Connected to previous question.

Orientations

Need to orient the moduli spaces to make the cobordism argument work (unless characteristic $= 2$).

Lagrangian Grassmannians

- From polygon example, we see dimension has to do with "non-convex" corners.
- What is the higher-dimensional generalization of this "angle" phenomenon?

Lagrangian Grassmannian

$\mathrm{LGr}(n) = \mathrm{U}(n)/\mathrm{O}(n)$; it has fundamental group $\mathbb{Z}$, witnessed by map $\det^2 : \mathrm{U}(n)/\mathrm{O}(n) \to \mathrm{U}(1)$.

Lagrangian Grassmannian bundle

For a symplectic manifold M, $\mathrm{LGr}(M) \to M$ is the bundle of Lagrangian Grassmannians of the tangent spaces.

Grading on M

- Need to "measure" fundamental group of fibers of $\mathrm{LGr}(M)$ consistently over M.
- What to choose a map $\alpha : \mathrm{LGr}(M) \to S^1$ that restricts to each fiber as $\det^2$.
- This is possible if and only if $2c_1(M) = 0$ in $H^2(M, \mathbb{Z})$. We now assume this.

Moral

The condition $2c_1(M) = 0$ is a "weak Calabi-Yau" condition that is necessary for the Fukaya category to be $\mathbb{Z}$ graded.

Single Lagrangian case

- Suppose that M is equipped with a grading α. Given a Lagrangian L, we lift to $L \to \tilde{L} \subset \mathrm{LGr}(M)$.
- Pullback of α to L defines map $\alpha_L : L \to S^1$, or class $\mu_L \in H^1(L, \mathbb{Z})$, called the *Maslov class* of L.
- $\mathcal{M}$ = moduli space of holomorphic maps $(\Sigma, \partial\Sigma) \to (M, L)$ in some fixed homotopy class $\beta \in \pi_2(M, L)$.

Dimension = index of linearized operator (Riemann-Roch)

$\dim \mathcal{M} = n\chi(\Sigma) + \mu_L(\partial\beta)$. (Involves μ_L.)

Orientation of $\mathcal{M}$

$w_1(\mathcal{M}) = T(w_2(L)) + (T(\mu_L) - 1)U(\mu_L)$. (Involves $w_2(L)$ and μ_L.)

Polygon case

- The case of polygons with boundary on $L_0, \ldots, L_d$ is more complicated since we have boundary punctures.
- Still, the key quantities are μ_{L_i} and $w_2(L_i)$.
- The easiest solution is to assume that these classes vanish.

Graded Lagrangians

If $\mu_L = 0$, then $\alpha_L : L \to S^1$ admits a lift $\tilde{\alpha}_L : L \to \mathbb{R}$. Such a lift is called a *grading on L*. The shift functor [1] changes this choice.

(S)pin structure

If $w_2(L) = 0$, then L admits a Pin structure. Easier: if $w_1(L) = 0$ also, then L admits an orientation and a Spin structure.

Polygon case

- Our "Lagrangian branes" are Lagrangians equipped with chosen grading and Pin structure.
- To each we intersection point x of such we may assign an absolute index $i(x) \in \mathbb{Z}$. This gives $CF(L_i, L_j)$ a $\mathbb{Z}$ grading.
- The dimension of the space of polygons with inputs $x_1, \ldots, x_d$ and output x_0 is

$$i(x_0) - i(x_1) - \cdots - i(x_d) + d - 2$$

- Counting zero dimensional moduli spaces forces $i(x_0) = \sum i(x_i) + (2 - d)$. So m_d has degree $2 - d$.

Full A_∞ equations

- Taking signs into account requires a slight redefinition of $CF(L_i, L_j)$. I won't spell this out here.
- Finally we have the full A_∞ structure:

$$m_d : CF(L_{d-1}, L_d) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_d)[2-d]$$

$$\sum_{k,\ell} (-1)^\dagger m_{d-k+1}(x_d, \ldots, x_{k+\ell+1}, m_k(x_{k+\ell}, \ldots, x_{\ell+1}), x_\ell, \ldots, x_1) = 0$$

$$\dagger = i(x_1) + \cdots + i(x_\ell) - \ell \quad \text{(Convention in Seidel's book.)}$$

- This is an A_∞ category with a shift functor [1]. We can then apply any formal enlargement we want: triangulated envelope, idempotent completion, localization, Ind-completion, etc.