Introduction to Fukaya Categories

Lecture 1: Basics of symplectic geometry for Fukaya categories

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Outline

Symplectic structures

Let M be a manifold of dimension $2n$.

Definition

A *symplectic form* is a two-form $\omega \in \Omega^2(M)$ which is

- Nondegenerate: $X \mapsto \omega(X, \cdot)$ is an iso $TM \to T^*M$.
- Closed: $d\omega = 0$.

Example

Q a manifold; $M = T^*Q$ the (total space of the) cotangent bundle. With coordinates q^i on Q, dual coordinates p_i on cotangent fiber, the 1-form $\theta = \sum_i p_i \, dq^i$ is coordinate-independent. Take $\omega = d\theta$.

Example

$M \subset \mathbb{P}^N$ a quasi-projective variety, ω restriction of Fubini-Study form.

History: Classical Mechanics

T^*Q is the phase space

q^i is a "generalized coordinate", and p_i is the "canonically conjugate momentum."

Dynamics

Dynamics is generated by a function $H(q, p)$ (Hamiltonian = total energy) by the ODE $\{\dot{q}^i = \partial H/\partial p_i, \dot{p}_i = -\partial H/\partial q^i\}$. In modern terms this ODE is the flow of the vector field X_H satisfying $\omega(\cdot, X_H) = dH$.

Canonical Transformation = Symplectomorphism

Old: A *canonical transformation* preserves $\sum p_i \, dq^i$ up to a total differential. New: A *symplectomorphism* preserves ω.

Lagrangian submanifolds

(M, ω) a symplectic manifold of dimension $2n$.

Definition

A submanifold $L \subset M$ is *isotropic* if $\omega|L = 0$. It is *Lagrangian* if in addition $\dim L = n$.

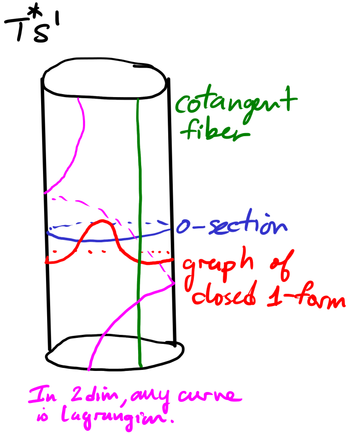
Examples in $M = T^*Q$

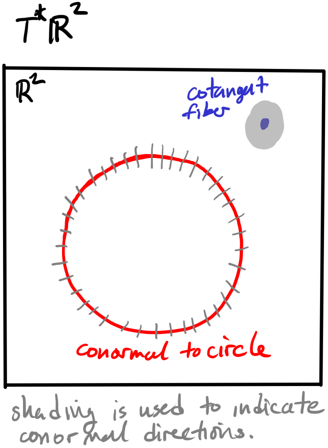
The zero section $L_0 = \{\text{all } p_i = 0\}$. For fixed $q \in Q$, the cotangent fiber $T^*_q Q$. For a smooth submanifold $N \subset Q$, the conormal bundle $T^*_N Q = \{(q, p) \mid q \in N \text{ and } (\sum p_i \, dq^i)|_{T_q N} = 0\}$.

Importance for us

Lagrangian submanifolds are the natural boundary conditions for processes that take place in a symplectic manifold.

Gallery


T^*S^1
cotangent fiber
0-section
graph of closed 1-form
In 2dim, any curve is Lagrangian.


$T^*\mathbb{R}^2$
$\mathbb{R}^2$
cotangent fiber
conormal to circle
shading is used to indicate conormal directions.

Infinitesimal (tangent space) symplectic geometry

- Consider $\mathbb{C}^n$ with standard Hermitian form $\langle z, w\rangle = \sum_i \bar{z}_i w_i$. Write as $\langle z, w\rangle = b(z, w) + \sqrt{-1}\omega(z, w)$, with b, ω being $\mathbb{R}$ valued, $\mathbb{R}$ linear forms; b is symmetric and ω is skew-symmetric.
- Then $(\mathbb{C}^n, \omega)$ is a model of the tangent space at any point of a symplectic manifold.
- $\mathrm{GL}(n, \mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n)$, $\mathrm{Sp}(2n) = \mathrm{Aut}_{\mathbb{R}}(\mathbb{C}^n, \omega)$, $\mathrm{U}(n) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n, b + i\omega)$.

Consequential observation

The groups $\mathrm{Sp}(2n)$, $\mathrm{U}(n)$, and $\mathrm{GL}(n, \mathbb{C})$ are mutually homotopy equivalent.

Consequence

Homotopy theory of symplectic vector bundles is the same as that of unitary or complex vector bundles. (Theory of Chern classes.)

Infinitesimal (tangent space) Lagrangian submanifolds

- A linear subspace $L \subset \mathbb{C}^n$ with $\dim_{\mathbb{R}} L = n$ is Lagrangian if $\omega|_L = 0$.
- Every Lagrangian subspace is equivalent under the action of $\mathrm{U}(n) \subset \mathrm{Sp}(2n)$ to the standard $\mathbb{R}^n \subset \mathbb{C}^n$. The stabilizer is $\mathrm{O}(n)$.
- Thus the set of Lagrangian subspaces, or *Lagrangian Grassmannian*, is $\mathrm{LGr}(n) \cong \mathrm{U}(n)/\mathrm{O}(n)$.

Fact

The map $\det^2 : \mathrm{U}(n)/\mathrm{O}(n) \to \mathrm{U}(1)$ is an isomorphism on π_1. Hence $H^1(\mathrm{LGr}(n), \mathbb{Z}) \cong \mathbb{Z}$. This leads to the theory of *Maslov classes and indices*.

Structures on the tangent bundle

(M, ω) a symplectic manifold. The form ω reduces the structure group of TM to $\mathrm{Sp}(2n)$. Hence structure group of TM can also be reduced to $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{U}(n)$.

Definition

An *almost-complex structure* (ACS) on M is $J : TM \to TM$ such that $J^2 = -Id$. An ACS J is *compatible* with ω if $g(X, Y) = \omega(X, JY)$ is a pos. def. symmetric form.

Theorem

The space of almost complex strutures compatible with a *given* symplectic form is contractible.

Local classification

Regard the ball $B^{2n}(r) \subset \mathbb{C}^n$ as a symplectic manifold with the standard ω.

Darboux Theorem

Any point in a symplectic manifold has a neighborhood symplectomorphic to $(B^{2n}(r), \omega)$.

Weinstein Theorem

A closed Lagrangian submanifold $L \subset M$ has a tubular neighborhood symplectomorphic to a tubuluar neighborhood of the zero section in T^*L.

Lesson

Local picture is always the same; interesting phenomena in the large.

Exactness

Since $d\omega = 0$, there is a class $[\omega] \in H^2_{dR}(M)$. Since ω is nondegenerate, ω^n is a volume form.

Definition

A symplectic manifold is *exact* if $[\omega] = 0$, i.e., $\omega = d\theta$.

Proposition

An exact symplectic manifold is never closed. Proof: If closed, $0 < \int \omega^n = \int d(\theta \wedge \omega^{n-1}) = 0$.

Examples

Cotangent bundle T^*Q with $\theta = \sum p_i \, dq^i$. Affine varieties. Many other constructions.

Exact Lagrangians

Suppose $(M, \omega = d\theta)$ is exact symplectic (specific θ chosen), and $L \subset M$ is Lagrangian. Then $d(\theta|_L) = \omega|_L = 0$, so there is a class $[\theta|_L] \in H^1_{dR}(L)$ (depends on choice of θ).

Definition

L is *exact* if $[\theta|_L] = 0$, that is, $\theta|_L = dF$ for some $F : L \to \mathbb{R}$.

Slogan

Exact Lagrangians in exact symplectic manifolds are easier to understand, particularly when stricter versions of exactness are imposed (Liouville manifolds, Weinstein manifolds).

Why easier?

For $\lambda > 0$, rescaling $\omega \mapsto \lambda\omega$ is a symmetry of the theory.

Let's intersect

(M, ω) oriented by $\omega^n > 0$. If L_1, L_2 are oriented Lagrangians, then since $\dim L_i = n = (1/2) \dim M$, we have an oriented intersection number $L_1 \cdot L_2$.

Example

$M = T^*Q$, $f : Q \to \mathbb{R}$ function. $L_1 =$ zero section, $L_2 = \Gamma(df) =$ graph of df. Then $L_1 \cap L_2 =$ critical points of f. With appropriate orientations $L_1 \cdot L_2 = \chi(Q)$.

Lagrange multipliers

Problem

To find the maximum/minimum of a function $f : \mathbb{R}^n \to \mathbb{R}$ on a simplex $\Delta = \{x \mid x_i \geq 0 \text{ and } \sum x_i \leq 1\}$.

Lagrange multipliers

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Undergraduate brain

Find critical points in interior, then apply Lagrange multiplier method to each stratum (# of multipliers = codim), testing vertices last.

Lagrange multipliers

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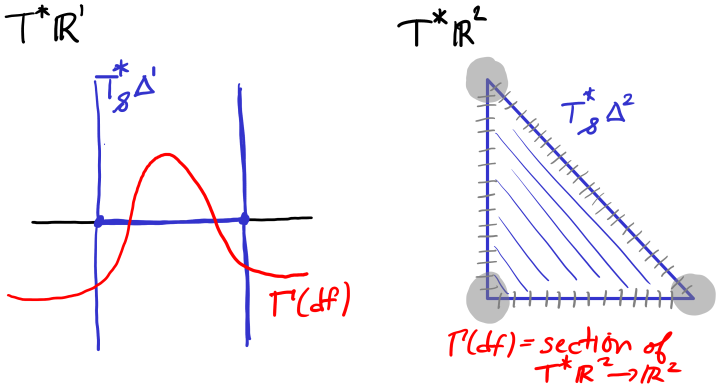
Galaxy brain

Let $\mathcal{S}$ = decomposition of Δ into strata, let $T^*_{\mathcal{S}}\mathbb{R}^n$ = union of the conormals to the strata. Take $T^*_{\mathcal{S}}\mathbb{R}^n \cap \Gamma(df)$.

Lagrange multipliers picture



Less obvious example (historically important)

- Let Y be an oriented integral homology 3-sphere. Consider irreps $\pi_1(Y) \to \mathrm{SU}(2)$ up to $\mathrm{SU}(2)$ conjugacy. The *Casson invariant* $\lambda(Y)$ is a signed count of the classes.
- Given a Heegaard splitting $Y = M_1 \cup_\Sigma M_2$, consider $\mathcal{R}(M_i) \subset \mathcal{R}(\Sigma)$, where $\mathcal{R}(\cdot)$ denotes the *variety* of conjugacy classes of irreps of $\pi_1 \to \mathrm{SU}(2)$.
- Then $\lambda(Y) = \frac{(-1)^g}{2}\mathcal{R}(M_1) \cdot \mathcal{R}(M_2)$.

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Atiyah-Bott

This is a Lagrangian intersection problem.

Let's categorify

- Recognizing that a problem is a Lagrangian intersection problem reveals a path towards categorification: Fukaya categories!
- Given transversely intersecting oriented Lagrangians L_1, L_2, define $CF(L_1, L_2) = \bigoplus_{x \in L_1 \cap L_2} \mathbb{K} \cdot x$

Issue 1: Grading

We have a $\mathbb{Z}/2$ grading by sign of intersections. Then trivially $\chi(CF(L_1, L_2)) = L_1 \cdot L_2$. We would rather have a $\mathbb{Z}$ grading if possible (relates to Maslov indices).

Issue 2: Invariance and categorical structure

Where is this going to come from?

Towards categorification

We need to understand the "processes" that can connect intersection points.

Algebraically

Certain maps $m_n : \bigotimes_{i=1}^{n} CF(L_{i-1}, L_i) \to CF(L_0, L_n)$.

Geometrically

Holomorphic maps from Riemann surfaces to (M, J), with boundary on various Lagrangians.

Recursive structure

Arises from geometric degenerations, implies relations between the m_n (A_∞ equations).