$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

$$R(x) = R_{ijk}^{\ \ \ \ l}(x)\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

$$K(X,Y) = \frac{g(R(X,Y)X, Y)}{\|X\|_g^2 \|Y\|_g^2 - (g(X,Y))^2}$$

sectional curvature of $\operatorname{Span}\{X,Y\}$.

<u>Fact</u> $\{K(X,Y)\}$ determines R

If M is 2-dimensional we get a function

$$\begin{aligned} K : M &\longrightarrow \mathbb{R} \\ p &\longmapsto K(X,Y) \quad \text{for any } X, Y \in T_pM \text{ lin indep.} \end{aligned}$$

<u>Example</u> S determined by $c(t) = (F(t), 0, G(t))$

$$g_{ij} \to \Gamma_{ij}^k \to R_{ijl}^{\ \ \ k}$$

$$K(x^1, x^2) = \frac{(F'G'' - F''G')G'}{((F')^2 + (G')^2)^2 F}(x^2)$$

<u>Ex 1</u> S = cylinder $F(t) = 1$, $G(t) = t$.

Last time we saw that g was flat.

$$K = 0$$

<u>Ex 2</u> $S = S^2 \setminus \{poles\}$ $F(t) = \sqrt{1-t^2}$, $G(t) = t$.

$$F' = \frac{-t}{\sqrt{1-t^2}} \qquad F'' = \frac{-1}{\sqrt{1-t^2}} - \frac{t^2}{(1-t^2)^{3/2}} = \frac{-1}{(1-t^2)^{3/2}}$$

$$G'(t) = 1 \qquad G''(t) = 0$$

$$K = \frac{-F''}{((F')^2 + 1)^2 F} = \frac{\frac{1}{(1-t^2)^{3/2}}}{\sqrt{1-t^2}\left(\frac{t^2}{1-t^2} + 1\right)^2}$$

$$= \frac{1}{(1-t^2)^2} \frac{(1-t^2)^2}{1}$$

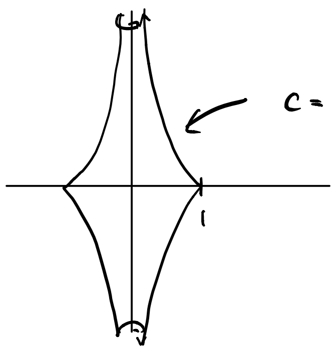
$$= 1$$

Ex 2R The curvature of $S^2(R)$ is $\frac{1}{R^2}$.

Ex 3 $(\mathbb{H}, g)$ have $g_{11} = \frac{1}{(x^2)^2} = g_{22}$, $g_{12} = g_{21} = 0$

Exercise $K\left(\partial/\partial x^1, \partial/\partial x^2\right) = -1$

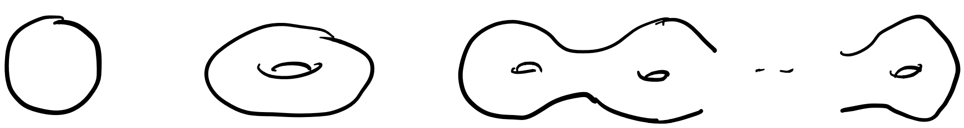
Ex 4 Consider S for $F(t) = t$, $G(t) = \int_t^1 \frac{\sqrt{1-x^2}}{x}\,dx$



This surface is called the pseudo sphere.

It has $K = -1$

Uniformization Theorem Every orientable connected 2-dimensional manifold M admits a metric g with $K = -1$, 0 or 1.



$K=1$ $\quad$ $K=0$ $\quad$ $\Sigma_{g\geq 2}$ $\quad$ $K=-1$

Back to geodesic triangles

<u>Aside</u> If M is oriented, each g determines a unique nonvanishing n-form $\mathrm{Vol}(g)$.

$\mathrm{Vol}(g)$ is the unique n-form such that $\forall p \in M$.

$$\mathrm{Vol}(g)(p)(V_1, \dots, V_n) = 1 \quad \text{if} \quad g(p)(V_i, V_j) = \delta_{ij}$$

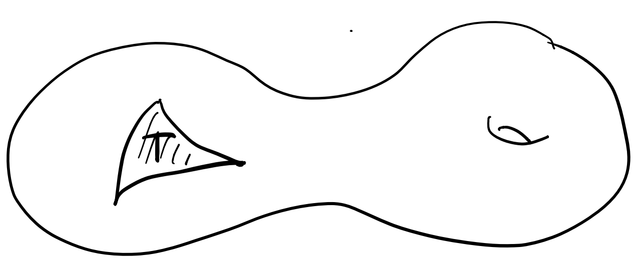
and $\{V_1, \dots, V_n\}$ is positive.

In local (positive) coordinates we have

$$\mathrm{Vol}(g)(x) = \sqrt{\det(g_{ij})}\, dx^1 \wedge \dots \wedge dx^n.$$

<u>Thm</u> (Gauss-Bonnet for geodesic triangles)

Let $T : \Delta \subset \mathbb{R}^2 \to (M, g)$ be a smooth 1-1 map s.t. such that $T|_{\Delta}$ is positive and $T(\Delta)$ is a geodesic triangle



If the interior angles of $T(\Delta)$ are $\theta_1, \theta_2, \theta_3$ then

$$\theta_1 + \theta_2 + \theta_3 = \pi + \int_T K \, \mathrm{Vol}(g)$$

This explains the triangle phenomenon!

$$\begin{array}{ll} < \pi & K < 0 \\ = \pi & K = 0 \\ > \pi & K > 0 \end{array}$$

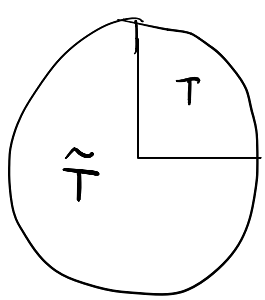
<u>Experiment</u> Computing radius of Earth in back yard.

Draw geodesic triangle.

Measure $\theta_1, \theta_2, \theta_2$ and Area.

$$\theta_1 + \theta_2 + \theta_3 - \pi = \frac{1}{R^2} \text{Area}$$

<u>Ex</u> S^2



$$\int \mathrm{Vol}\, g = \int_T \mathrm{Vol}\, g + \int_{\tilde{T}} \mathrm{Vol}(g)$$

$$= \frac{3\pi}{2} - \pi + \left(\frac{3\pi}{2} + \frac{3\pi}{2} + \frac{3\pi}{2} \right) - \pi$$

$= 4\pi$ Does not depend on g!

<u>Thm</u> (Gauss - Bonnet) Let (M, g) be compact and oriented.

$$\underbrace{\int_M K \, \mathrm{Vol}(g)}_{\text{Geometry}} = \underbrace{2\pi(2 - b^1)}_{\text{Topology}}$$