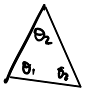
There is a difference in the geometry of .

$\mathbb{R}^2, g_0$ ← standard. Triangle phenomenon



$\theta_1 + \theta_2 + \theta_3$

$\mathbb{R}^2, g_0$: $= \pi$

S^2, g ← inherited from g_0: $> \pi$

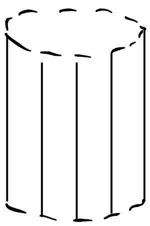
H, g ← hyperbolic. $< \pi$

We want to understand this difference.

Def$^{\underline{n}}$ A metric g on M is <u>flat</u> if $\forall\, p \in M$ there is a chart (U, ϕ) s.t. $g(\phi^{-1}(x)) = g_{st}$

Example $M = S = \{ (\cos x^1, \sin x^1, x^2) \}$ $(F=1,\ G=t)$

g = inherited metric



$$\phi^{-1}(x^1, x^2) = (\cos x^1, \sin x^1, x^2)$$

$$g(x^1, x^2) = (F^2)\, dx^1 \otimes dx^1 + ((F')^2 + (G')^2)\, dx^2 \otimes dx^2$$
$$= g_{st}$$

The most general tool for studying this idea is a . . .
tensor field

Def$^{\underline{n}}$ The curvature of (M, g) is the map

$$R : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$$

$$(X, Y, Z) \longmapsto \underbrace{-\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z}_{R(X,Y)Z}.$$

where $\mathcal{V} = \{\text{vector fields on } M\}$ and ∇ is <u>the</u> Riemannian connection for g.

<u>Lemma</u> $R(X,Y)Z(p)$ only depends on $X(p), Y(p), Z(p)$ and the map

$$T_pM \times T_pM \times T_pM \longrightarrow T_pM$$
$$(X, Y, Z) \longmapsto R(X,Y)Z$$

is trilinear. In other words R is a tensor field.

<u>Pf</u> It suffices to prove

$$R(fX,Y)Z = R(X,fY)Z = R(X,Y)fZ = f(R(X,Y)Z) \quad \forall f \in C^\infty(M)$$

$$R(fX,Y)Z = -\nabla_{fX}\nabla_Y Z + \nabla_Y \nabla_{fX} Z + \nabla_{[fX,Y]} Z.$$
$$= -f(\nabla_X \nabla_Y Z) + f\nabla_Y \nabla_X Z + Y(f)\nabla_X Z$$
$$+ \nabla_{f[X,Y] - Y(f)X} Z$$
$$= f(R(X,Y)Z)$$

R is a $(3,1)$-tensor field.

ie $R : V \times V \times V \longrightarrow V$

$$\longleftrightarrow \quad T : V \times V \times V \times V^* \longrightarrow \mathbb{R}$$

$$(X, Y, Z, \alpha) \longmapsto \alpha(R(X,Y)Z)$$

So $\quad R(x) = R^{l}_{ijk}\, dx^i \otimes dx^j \otimes dx^k \otimes \partial/\partial x^l$

Let's compute the R^{l}_{ijk}.

$$R^{l}_{ijk} = dx^l\left(R\left(\partial/\partial x^i, \partial/\partial x^j, \partial/\partial x^l\right)\right)$$

$$= dx^l\left(-\nabla_{\partial_i}\nabla_{\partial_j}\partial_k + \nabla_{\partial_j}\cdot\nabla_{\partial_i}\partial_k + \nabla_{[\partial_i,\partial_j]}\partial_k\right)$$

$$= dx^l\left(-\nabla_{\partial_i}\left(\Gamma^m_{jk}\partial_m\right) + \nabla_{\partial_j}\left(\Gamma^n_{ik}\partial_n\right)\right)$$

$$= dx^l\Big(-\Gamma^m_{jk}\Gamma^p_{im}\partial_p - \left(\frac{\partial}{\partial x^i}\Gamma^m_{jk}\right)\partial_m + \Gamma^n_{ik}\Gamma^q_{jn}\partial_q$$

$$+ \left(\partial/\partial x^j\,\Gamma^n_{ik}\right)\partial_n\Big)$$

$$= \Gamma^n_{ik}\Gamma^l_{jn} - \Gamma^m_{jk}\Gamma^l_{im} + \frac{\partial}{\partial x^j}\Gamma^l_{ik} - \frac{\partial}{\partial x^i}\Gamma^l_{jk}$$

$$R^l_{ijk} = \sum_m\left(\Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}\right) + \frac{\partial\Gamma^l_{ik}}{\partial x^j} - \frac{\partial}{\partial x^i}\Gamma^l_{jk}$$

Note for g_0 we have $R^l_{ikj} = 0$

" R measures distance from standard metric."

Let's add a bit more to that interpretation

1) Consider $u: [0,1] \times [0,1] \longrightarrow M$

$(s,t) \longmapsto u(s,t)$

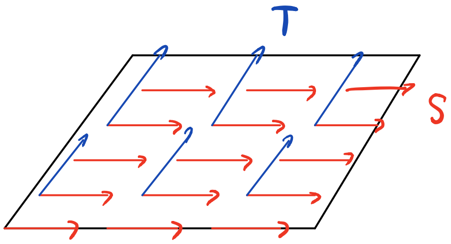
ex 0 $\quad u_0 : [0,1] \times [0,1] \longrightarrow \mathbb{R}^n$

$u_0(s,t) = (s,t,0,\dots,0)$ (flat)

There are two vector fields along $u([0,1]^2)$

$S = \frac{\partial u}{\partial s}$ and $T = \frac{\partial u}{\partial t}$

ex 0



Give X on M we can define $\nabla_T \nabla_S X$

and $\nabla_S \nabla_T X$

Lemma $\quad \nabla_T \nabla_S X - \nabla_S \nabla_T X = R(T,S)X.$

2) Let's contract the information contained in R.

Given $X, Y \in T_pM$ linearly indepent, set:

$$K(X,Y) = \frac{g(R(X,Y)Y, X)}{\|X\|_g^2 \|Y\|_g^2 - (g(X,Y))^2}$$

<u>Lemma</u> $K(X,Y)$ depends only on $\operatorname{Span}\{X,Y\}$.

<u>Note</u> $K(bX,Y) = K(X,cY) = K(X,Y)$

for any $b, c \in \mathbb{R}$ not zero.

K is called the sectional curvature of $\operatorname{Span}\{X,Y\} \subset T_pM$.

Suppose M is 2-dimensional.

Then we have
$$\begin{aligned} K : M &\longrightarrow \mathbb{R} \\ p &\longmapsto K(X,Y) \text{ for any } \{X,Y\} \subset T_pM \end{aligned}$$

K = Gaussian Curvature of (M,g).

This allows for a complete understanding of the triangle phenomenon.