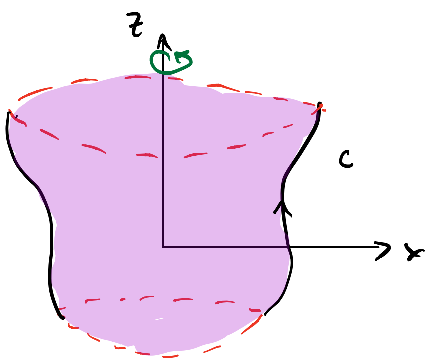
Surfaces of Revolution

Consider
$$c: (a,b) \longrightarrow \mathbb{R}^3$$
$$t \longmapsto (F(t), 0, G(t))$$

Assume that $F(t) > 0$ and $G'(t) > 0$

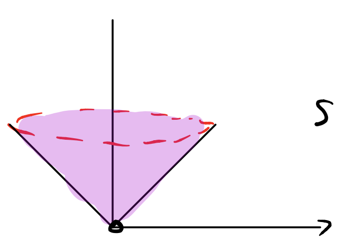


Rotating c around the z-axis we get a surface

$$S = \{ (F(x^2)\cos x^1,\ F(x^2)\sin x^1,\ G(x^2)) \mid x^1 \in [0, 2\pi),\ x^2 \in (a,b) \}$$

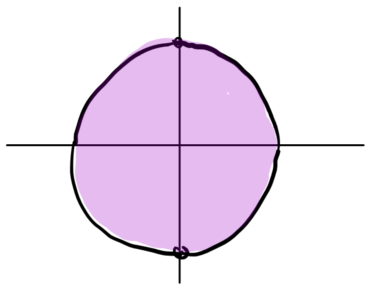
$$S \simeq S^1 \times (0,1)$$

<u>Ex</u> $c(t) = (t, 0, t) \qquad t \in (0,1)$



S is an open cone

$\underline{\text{Ex}}$ $c(t) = (\sqrt{1-t^2}, 0, t) \quad t \in (-1,1)$



$S = S^2 \setminus \{\text{poles}\}$

S inherits a metric g from standard metric g_0 on $\mathbb{R}^3$

$T_p S = \text{Span}\{\partial/\partial x^1, \partial/\partial x^2\} \subset \text{Span}\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$

$$\frac{\partial}{\partial x^1} = \frac{\partial x}{\partial x^1}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x^1}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x^1}\frac{\partial}{\partial z}$$

$$= -F(x^2)\sin x^1 \, \partial/\partial x + F(x^2)\cos x^1 \, \partial/\partial y$$

$$\frac{\partial}{\partial x^2} = F'(x^2)\cos x^1 \, \partial/\partial x + F'(x^2)\sin x^1 \, \partial/\partial y + G'(x^2)\, \partial/\partial z.$$

$$g(x^1,x^2) = \sum g_{ij}(x^1,x^2)\, dx^i \otimes dx^j$$

$$g_{ij}(x^1,x^2) = g_0(x^1,x^2)\left(\partial/\partial x^i, \partial/\partial x^j\right)$$

$$g_{11}(x^1,x^2) = (F(x^2))^2$$

$$g_{22}(x^1,x^2) = (F'(x^2))^2 + (G'(x^2))^2$$

$$g_{12} = g_{21} = 0$$

With this we can compute the Christoffel symbols of ∇.

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{F'(x^2)}{F(x^2)} \qquad \Gamma^1_{11} = \Gamma^1_{22} = 0$$

$$\Gamma^2_{11} = -\frac{FF'}{(F')^2+(G')^2}, \quad \Gamma^2_{22} = \frac{F'F''+G'G''}{(F')^2+(G')^2}, \quad \Gamma^2_{12} = \Gamma^2_{21} = 0.$$

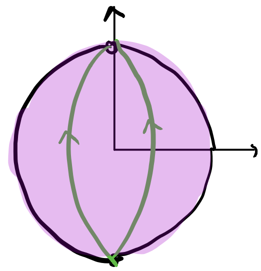
$$\ddot{\gamma}^1 + 2\frac{F'(\gamma^2)}{F(\gamma^2)}\dot{\gamma}^1\dot{\gamma}^2 = 0 \qquad ①$$

$$\ddot{\gamma}^2 - \frac{FF'}{(F')^2+(G')^2}(\dot{\gamma}^1)^2 + \frac{F'F''+G'G''}{(F')^2+(G')^2}(\dot{\gamma}^2)^2 = 0 \qquad ②$$

Rmk The corresponding geodesic on S is

$$\left(F(\gamma^2)\cos\gamma^1,\ F(\gamma^2)\sin\gamma^1,\ G(\gamma^2)\right)$$

Thm 1 $(k, \gamma^2(t))$ is geodesic iff $\|\dot\gamma\|_g$ is constant.

ex  meridians are geodesics

Pf γ satisfies ① since $\dot\gamma^1 = 0$

$$\|\dot\gamma\|_g^2 = g_{11}(\dot\gamma^1)^2 + g_{22}(\dot\gamma_2)^2$$

$$= \left((F'(\gamma^2))^2 + (G'(\gamma^2))^2\right)(\dot\gamma^2)^2$$

$$\|\dot\gamma\|_g^2 = c \iff (\dot\gamma^2)^2\left((F')^2 + (G')^2\right) = C$$

$$\iff 2(\dot\gamma^2)(\ddot\gamma^2)\left((F')^2 + (G')^2\right) + (\gamma^2)^2\, 2\,(F'F'' + G'G'') = 0$$

$$\iff \ddot\gamma^2 + \frac{F'F'' + G'G''}{(F')^2 + (G')^2} = 0$$

which is ②

<u>Thm 2</u> $(\gamma^1(t), k)$ is a geodesic iff $\|\dot{\gamma}\|_g = \text{const}$ and $F'(k) = 0$.

$$\|\dot{\gamma}\|_g^2 = \text{const} \Rightarrow F^2(k)\,(\dot{\gamma}^1)^2 = \text{const}$$

$$\Rightarrow \dot{\gamma}^1 = \pm \frac{\text{const}}{F'(k)}$$

$$\Rightarrow \gamma^1(t) = \pm \frac{\text{const}}{F'(k)}\, t + \gamma^1(0)$$

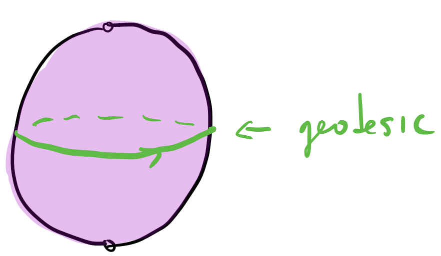
Check

① $$\underset{0}{\underset{\uparrow}{\ddot{\gamma}^1}} + 2\dot{\gamma}^1 \underset{0}{\underset{\uparrow}{\dot{\gamma}^2}} \frac{F'(\gamma^2)}{F(\gamma^2)} = 0 \quad \checkmark$$

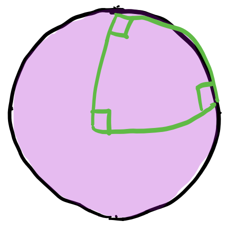
② $$\underset{0}{\underset{\uparrow}{\ddot{\gamma}_2}} - (\dot{\gamma}^1)^2 \frac{FF'}{(F')^2 + (G')^2} + \underset{0}{\underset{\uparrow}{(\dot{\gamma}^2)^2}} \frac{F'F'' + G'G''}{(F')^2 + (G')^2}$$

$= 0$ iff $F'(k) = 0$.

<u>Ex</u>



Ex Consider the following geodesic triangle on S^2



- The sum of interior angles is $3(\pi/2)$.
- Compare this to π for triangles in $(\mathbb{R}^2, g_0)$ and $< \pi$ for triangles in $(\mathbb{H}, g)$ (See Homework 10, 2b)
- The theorem which explains this phenomenon is the Gauss-Bonnet Thm. It involves Curvature