- ∇ , Γ^k_{ij}

- Given g $\exists!$ torsion free ∇ compatible with g

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell} g^{k\ell} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right)$$

- Given a smooth curve $\gamma : [0,1] \to M$, any ∇ determines a unique map

$$\{\text{v.f.'s along } \gamma\} \longrightarrow \{\text{v.f.'s along } \gamma\}$$

$$V \longmapsto \nabla_{\dot\gamma} V$$

$$V = \sum v^i(t) \frac{\partial}{\partial x^i} \qquad \nabla_{\dot\gamma} V = \sum_k \left(\dot v^k(t) + \sum_{i,j} \Gamma^k_{ij} \dot\gamma^i(t) v^j(t) \right) \partial/\partial x^k$$

<u>Def$^{\underline{n}}$</u> V is parallel if $\nabla_{\dot\gamma} V = 0$

- If ∇ is the Riemannian connection for g, $\gamma : [0,1] \to M$ is a geodesic if $\nabla_{\dot\gamma} \dot\gamma = 0$.

$$\ddot\gamma^k(t) + \sum_{i,j} \Gamma^k_{ij} \dot\gamma^i(t) \dot\gamma^j(t) = 0 \qquad k = 1, \ldots, n$$

Example 1 $\quad M = \mathbb{H} = \{(x^1,x^2) \in \mathbb{R}^2 \mid x^2 > 0\}$

$$g = \frac{1}{(x^2)^2}\left(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \right)$$

The Christoffel symbols of the Riemannian connection are

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\Gamma^2_{11} = -\frac{1}{x^2}$$

The other 4 are zero!

The geodesic equations are

$$\begin{cases} \ddot{\gamma}_1 - 2\dfrac{1}{\gamma_2}\dot{\gamma}_1\dot{\gamma}_2 = 0 \\ \ddot{\gamma}_2 + \dfrac{1}{\gamma_2}\left((\dot{\gamma}_1)^2 - (\dot{\gamma}_2)^2 \right) = 0 \end{cases}$$

Need to solve using tricks.

Assume $\dot{\gamma}_1 \neq 0$ $\quad$ Equations $\Rightarrow$ $\left(\dfrac{\gamma_2\dot{\gamma}_2}{\dot{\gamma}_1} + \gamma_1 \right)^{\cdot} = 0$

$$\Rightarrow \frac{\gamma_2\dot{\gamma}_2}{\dot{\gamma}_1} + \gamma_1 = c$$

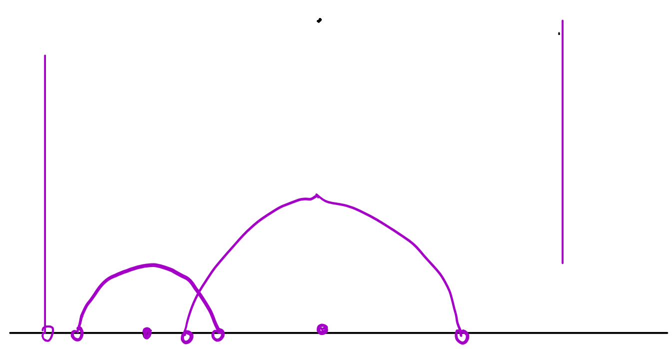
$$\Rightarrow \quad \gamma_1 \dot{\gamma}_1 + \gamma_2 \dot{\gamma}_2 = c\,\dot{\gamma}_1$$

$$\Rightarrow \quad \gamma_1 \dot{\gamma}_1 - c\,\dot{\gamma}_1 + \gamma_2 \dot{\gamma}_2 = 0$$

$$\Rightarrow \quad \left(\gamma_1 - \tfrac{c}{2}\right)^2 + \gamma_2^2 = b$$

$\Rightarrow$ (γ_1, γ_2) lies on circle with centre $(\frac{c}{2}, 0)$.

If $\dot{\gamma}_1 = 0$ then (γ_1, γ_2) lies on vertical line.



What about the actual functions $\gamma_1(t), \gamma_2(t)$?

<u>Thm</u> If γ is a geodesic of g then it's "speed" $\|\dot{\gamma}(t)\|_g$ is constant.

Pf $\quad \frac{d}{dt}\left(\|\dot\gamma(t)\|^2\right) = \frac{d}{dt}\Big|_{t=0}\Big(g(\dot\gamma(t), \dot\gamma(t)) \Big)$

$$= \dot\gamma(0)\,\big(g(\dot\gamma, \dot\gamma)\big)$$

$$= g(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) + g(\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)$$

by compatibility

$$= 0.$$

Exercise If $\gamma(t)$ is a geodesic then so is $\gamma(ct)$ for any $c \in \mathbb{R}$. Moreover

$$\|\dot\gamma(t)\| = c\|\dot\gamma(ct)\|$$

Back to Example 1.

- Any geodesic γ whose image is a vertical line is of the form

$$\gamma(t) = \left(\gamma(0),\ e^{\|\dot\gamma(0)\|_g t}\,\gamma_2(0)\right)$$

(not the standard parametrization $\gamma(0) + t\,\dot\gamma(0)$)

Note the x^1-axis is at $t = -\infty$.

- Check $\|\dot\gamma(t)\|_g$ = constant.

How are the circles parametrized?

View $\mathbb{H}$ as $\{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$.

Each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(A) = 1$ defines a map.

$$\mathbb{H} \longmapsto \mathbb{H}.$$

$$z \longmapsto \frac{az+b}{cz+d} = A \cdot z$$

Each geodesic with unit speed is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \cdot i$$

Euclid's Geometry

Basic Structures e.g points, lines, angles, ...

\+ Postulates eg "One can draw a straight line from p to q."

\+ Logic

= Geometry

One of Euclid's Postulates was controversial.

Parallel Postulate

Given line L and point p not on on L there is a unique line L^* through p which never intersects L

Q. Does this follow from the other postulates.

$(\mathbb{H}, g)$ settles this .... NO!

If "lines" are interpretted as geodesics all other postulates are true, however ... L^* is not unique.