Lecture 36 Addendum

$$\mathbb{H} = \{ (x^1, x^2) \in \mathbb{R}^2 \mid x^2 > 0 \}$$

$$g = \frac{1}{(x^2)^2} (dx^1 \otimes dx^1 + dx^2 \otimes dx^2) \qquad g_{11} = g_{22} = \frac{1}{(x^2)^2}$$

$$g_{12} = g_{21} = 0$$

Christoffel symbols:

$$\Gamma_{ij}^{\ k} = \frac{1}{2} \sum_{\ell} g^{k\ell} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right)$$

Introduce notation

$$[ij, \ell] = \frac{1}{2} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right)$$

(maybe also called a Christoffel symbol)

So $\quad \Gamma_{ij}^{\ k} = \sum_{\ell} g^{k\ell} [ij, \ell]$

Now, $(g^{k\ell}) = (g_{ij})^{-1}$ so $\quad g^{11} = g^{22} = (x^2)^2 \quad g^{12} = g^{21} = 0$

There are 8 symbols $[ij, \ell]$

$$[11,1] = \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right) = 0$$

$$[11,2] = \frac{1}{2} \left(\frac{\partial g_{21}}{\partial x^1} + \frac{\partial g_{21}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2} \right) = -\frac{1}{2} \frac{\partial}{\partial x^2} \left((x^2)^{-2} \right) = (x^2)^{-3}$$

$$[12,1] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial}{\partial x^2}(x^2)^{-2} = -(x^2)^{-3}$$

$$[12,2] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2}\right) = 0$$

$$[21,1] = [12,1] = -(x^2)^{-3}$$

$$[21,2] = [12,2] = 0$$

$$[22,1] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) = 0$$

$$[22,2] = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial}{\partial x^2}\left((x^2)^{-2}\right) = -(x^2)^{-3}$$

So

$$\Gamma^1_{11} = g^{11}[11,1] + g^{12}[11,2] = 0$$

$$\Gamma^2_{11} = g^{21}[11,1] + g^{22}[11,2] = (x^2)^2(x^2)^{-3} = \frac{1}{x^2}$$

$$\Gamma^1_{12} = g^{11}[12,1] + g^{12}[12,2] = (x^2)^2(-(x^2)^{-3}) = \frac{-1}{x^2}$$

$$\Gamma^2_{12} = g^{21}[12,1] + g^{22}[12,2] = 0$$

$$\Gamma^1_{21} = \Gamma^1_{12} = \frac{-1}{x^2}$$

$$\Gamma^2_{21} = \Gamma^2_{12} = 0$$

$$\Gamma^1_{22} = g^{11}[22,1] + g^{12}[22,2] = 0$$

$$\Gamma^2_{22} = g^{21}[22,1] + g^{22}[22,2] = (x^2)^2(-(x^2)^{-3}) = -1/x^2$$

Geodesic equations for $\gamma = (\gamma^1, \gamma^2) : \mathbb{R} \to \mathbb{H}$

$$0 = \ddot{\gamma}^1 + \sum_{i,j} \Gamma^1_{ij} \dot{\gamma}^i \dot{\gamma}^j = \ddot{\gamma}^1 + \Gamma^1_{12} \dot{\gamma}^1 \dot{\gamma}^2 + \Gamma^1_{21} \dot{\gamma}^2 \dot{\gamma}^1$$

$$= \ddot{\gamma}^1 + \frac{-1}{\gamma^2} \dot{\gamma}^1 \dot{\gamma}^2 + \frac{-1}{\gamma^2} \dot{\gamma}^2 \dot{\gamma}^1$$

$$= \ddot{\gamma}^1 - \frac{2 \dot{\gamma}^1 \dot{\gamma}^2}{\gamma^2}$$

$$0 = \ddot{\gamma}^2 + \sum_{i,j} \Gamma^2_{ij} \dot{\gamma}^i \dot{\gamma}^j = \ddot{\gamma}^2 + \Gamma^2_{11} \dot{\gamma}^1 \dot{\gamma}^1 + \Gamma^2_{22} \dot{\gamma}^2 \dot{\gamma}^2$$

$$= \ddot{\gamma}^2 + \frac{1}{\gamma^2} (\dot{\gamma}^1)^2 - \frac{1}{\gamma^2} (\dot{\gamma}^2)^2$$

$$= \ddot{\gamma}^2 + \frac{1}{\gamma^2} \left((\dot{\gamma}^1)^2 - (\dot{\gamma}^2)^2 \right)$$

So $\left\{ \begin{array}{l} \ddot{\gamma}^1 - \dfrac{2 \dot{\gamma}^1 \dot{\gamma}^2}{\gamma^2} = 0 \\ \ddot{\gamma}^2 + \dfrac{1}{\gamma^2} \left((\dot{\gamma}^1)^2 - (\dot{\gamma}^2)^2 \right) = 0 \end{array} \right\}$ are the geodesic equations. (Note $\gamma^2 > 0$ required.)

To solve, we look for conserved quatities, constant along a solution.

Always know that $\|\dot{\gamma}\|_g^2 = g(\dot{\gamma}, \dot{\gamma})$ is conserved

$$g(\dot{\gamma}, \dot{\gamma}) = \frac{1}{(\gamma^2)^2} \left((\dot{\gamma}^1)^2 + (\dot{\gamma}^2)^2 \right)$$

So $\dfrac{(\dot{\gamma}^1)^2 + (\dot{\gamma}^2)^2}{(\gamma^2)^2} = b$, some constant.

Next look at first equation, and suppose $\dot{\gamma}^1 \neq 0$, so we can divide by it.

$$0 = \frac{\ddot{\gamma}^1}{\dot{\gamma}^1} - \frac{2\dot{\gamma}^2}{\gamma^2} = \frac{d}{dt}\ln(\dot{\gamma}^1) - 2\frac{d}{dt}\ln(\gamma^2)$$

$$= \frac{d}{dt}\left[\ln(\dot{\gamma}^1) - 2\ln(\gamma^2)\right] = \frac{d}{dt}\ln\left(\dot{\gamma}^1/(\gamma^2)^2\right)$$

So $\ln\left(\dot{\gamma}^1/(\gamma^2)^2\right)$ is conserved, so $\dot{\gamma}^1/(\gamma^2)^2$ is conserved.

Thus $\dot{\gamma}^1/(\gamma^2)^2 = c$, a constant.

Now we know $\begin{cases} \dot{\gamma}^1 = c(\gamma^2)^2 \\ (\dot{\gamma}^1)^2 + (\dot{\gamma}^2)^2 = b(\gamma^2)^2 \end{cases}$

Two types of solutions:
- $c = 0$, in which case $\dot{\gamma}^1 = 0$ so $\gamma^1 =$ constant $\Rightarrow \gamma$ is a vertical strait line
- $c \neq 0$, in which case $\dot{\gamma}^1$ is never 0, so we can divide by it

$$1 + \frac{(\dot{\gamma}^2)^2}{(\dot{\gamma}^1)^2} = \frac{b(\gamma^2)^2}{(\dot{\gamma}^1)^2} = \frac{b(\gamma^2)^2}{c^2(\gamma^2)^4} = \frac{b}{c^2(\gamma^2)^2}$$

Now $\dfrac{\dot{\gamma}^2}{\dot{\gamma}^1} = \dfrac{d\gamma^2}{d\gamma^1}$ (using γ^1 itself as a parameter, ok since $\dot{\gamma}^1$ never 0)

so $\left(\dfrac{d\gamma^2}{d\gamma^1}\right)^2 = \dfrac{b}{c^2(\gamma^2)^2} - 1$

$$\frac{dx^2}{dx^1} = \sqrt{\frac{b}{c^2(x^2)^2} - 1} = \frac{\sqrt{b - c^2(x^2)^2}}{c\,x^2}$$

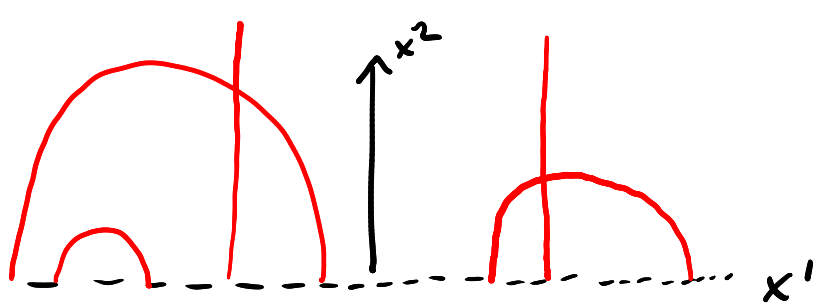
Separate variables:

$$\frac{c\,x^2\,dx^2}{\sqrt{b - c^2(x^2)^2}} = dx^1$$

$$\int \frac{c\,x^2\,dx^2}{\sqrt{b - c^2(x^2)^2}} = \int dx^1$$

$$\frac{1}{c}\sqrt{b - c^2(x^2)^2} = x^1 - a \qquad (a = \text{constant})$$

$$\frac{1}{c^2}\left(b - c^2(x^2)^2\right) = (x^1 - a)^2$$

$$(x^1 - a)^2 + (x^2)^2 = b/c^2 \qquad (x^2 > 0)$$

A semicircle with center $(a, 0)$ and radius $\sqrt{b}/c$!