A Connection on M is a bilinear map

$$\{\text{vector fields}\} \times \{\text{vector fields}\} \longrightarrow \{\text{vector fields}\}$$
$$(X, Y) \longrightarrow \nabla_X Y$$

s.t. $\nabla_{fX} Y = f(\nabla_X Y)$

$$\nabla_X fY = f(\nabla_X Y) + (Xf)Y$$

Locally ∇ is determined by

$$\nabla_{\partial/\partial x^i} \partial/\partial x^j = \sum_k \Gamma^k_{ij} \, \partial/\partial x^k$$ Christoffel Symbols of ∇.

<u>Ex 0²</u> The standard ($\Gamma^k_{ij} = 0$) connection on $\mathbb{R}^n$

$$\nabla_{\partial/\partial x^i} Y = \sum_j \frac{\partial Y^j}{\partial x^i} \, \partial/\partial x^j$$

<u>Rmk</u> ANY collection of n^3 functions Γ^k_{ij} determines a connection on $\mathbb{R}^n$.

Recall: We also have lots of metrics on M.

These two choices can be made in concert.

<u>Thm</u> For every metric g on M $\exists!$ connection ∇ with the following additional properties

a) $\nabla_X Y - \nabla_Y X = [X, Y]$ (Torsion Free)

b) $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

∇ is compatible with g.

(∇ encodes g-Leibnitz Rule)

This unique ∇ corresponding to g is called its Riemannian connection.

In local coordinates $g(x) = \sum_{i,j} g_{ij}(x)\, dx^i \otimes dx^j$

and $$\Gamma^k_{ij} = \frac{1}{2} \sum_{\ell} g^{k\ell} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right)$$

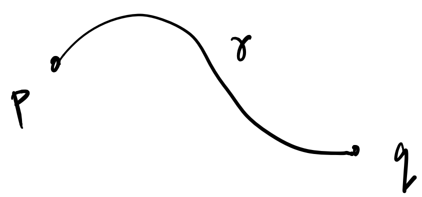
where $(g_{ij})^{-1} = (g^{ij})$

OK, what can we do with ∇ and why is it called a connection?

Parallel Translation

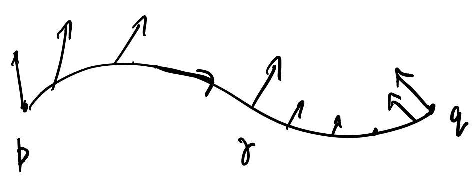
Given $p, q \in M$, want a isomorphism from T_pM to T_qM

Let $\gamma : [0,1] \longrightarrow M$ be smooth s.t $\gamma(0) = p$, $\gamma(1) = q$.



A vector field along γ is a map $V : [0,1] \longrightarrow TM$.

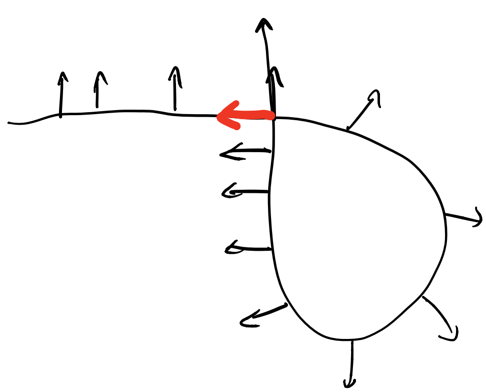
s.t. $V(t) \in T_{\gamma(t)}M$



<u>ex1</u> $\dot{\gamma}(0)$ is a v.f along γ.

<u>ex2</u> A vector field X on M determines one along γ, $X(\gamma(t))$.

<u>ex3</u>



Not every v.f. along γ is a restriction as in ex 2 !

A connection ∇ on M defines a unique map.

$$\{\text{Vector fields along } \gamma\} \longrightarrow \{\text{Vector fields along } \gamma\}$$
$$V \longmapsto \nabla_{\dot\gamma} V$$

s.t. a) $\nabla_{\dot\gamma}(V+W) = \nabla_{\dot\gamma} V + \nabla_{\dot\gamma} W$

b) $\nabla_{\dot\gamma}(fV) = f\,\nabla_{\dot\gamma} V + \frac{df}{dt} V$

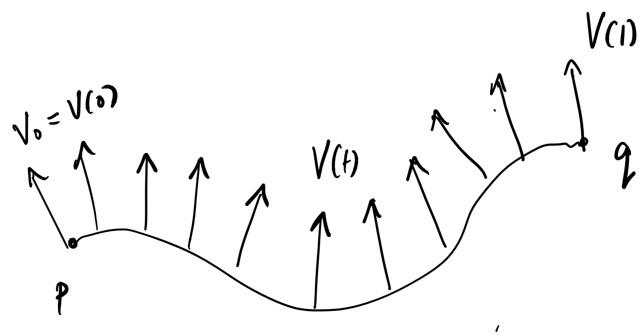
c) If $V = X(\gamma(t))$ then $\nabla_{\dot\gamma} V = \nabla_{\dot\gamma} X$.

In local coordinates $V(t) = v^1(t)\,\partial/\partial x^1 + \dots + v^n(t)\,\partial/\partial x^n$

and $$(\nabla_{\dot\gamma} V)(t) = \sum_k \left(\dot v^k(t) + \sum_{i,j} \Gamma^k_{ij}\, \dot\gamma^i(t)\, v^j(t) \right) \partial/\partial x^k$$

<u>Def^n</u> A v.f. V along γ is parallel if $\nabla_{\dot\gamma} V = 0$.

Thm Given $\gamma : [0,1] \longrightarrow M$ smooth at $V_0 \in \gamma(0)$ there is a unique parallel v.f V along γ s.t. $V(0) = V_0$



<u>Pf</u> Apply Existence and Uniqueness Thm to

$$\dot{V}^k(t) + \sum_{i,j} \Gamma^k_{ij}(\gamma(t))\, \dot{\gamma}^i(t)\, V^j(t) = 0$$

This are linear equation first order ODE's.

Thm The map

$$T_p M \longrightarrow T_q M \quad \text{is an isomorphism!}$$
$$V_0 \longmapsto V(1)$$

(It depends on ∇).

Hence " connection "

Fix a metric g on M. Let ∇ be the unique Riemannian connection for g.

<u>Definition</u> $\gamma : (a,b) \longrightarrow M$ is a geodesic for g if $\nabla_{\dot\gamma} \dot\gamma = 0$ ("acceleration" $= 0$)

In local coordinates

$$\nabla_{\dot\gamma} \dot\gamma = 0$$

$$\Leftrightarrow \sum_k \left(\ddot\gamma^k + \sum_{i,j} \Gamma^k_{ji} \dot\gamma^i \dot\gamma^j \right) \partial / \partial x^k = 0$$

$$\Leftrightarrow \quad \ddot\gamma^k + \sum_{i,j} \Gamma^k_{ji} \dot\gamma^i \dot\gamma^j = 0$$

- n 2^{nd} order ODE's
- nonlinear ($\dot\gamma^i \dot\gamma^j$)
- $\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$

<u>Example 0</u> $\quad M = \mathbb{R}^n \quad g_0 = \sum_i dx^i \otimes dx^i$

$$(g_{ij}) = I_n = (g^{ij})$$

$$\Gamma^k_{ij} = 0 \quad \forall\ i, j, k.$$

The Riemannian connection for g_0 is

$$\nabla_{\partial/\partial x^i} Y = \sum_j \frac{\partial Y^j}{\partial x^i} \, \partial/\partial x^j$$

The geodesic equations for g_0 are.

$$\ddot{\gamma}^k = 0$$

$$\Rightarrow \quad \gamma^k(t) = \gamma^k(0) + \dot{\gamma}^k(0)\, t$$

$$\Rightarrow \quad \gamma(t) = \gamma(0) + \dot{\gamma}(0)\, t$$ a straight line.