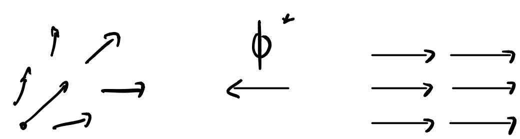
**Thm 1** $\mathcal{L}_X Y = [X,Y]$

$$\mathcal{L}_X Y = \lim_{t\to 0} \frac{1}{t}\left( (\phi^X_{-t})_* (Y(\phi^X_t)) - Y \right) \qquad [X,Y] = XY - YX$$

**Pf** It suffices to prove that $\mathcal{L}_X Y = [X,Y]$ near general $p \in M$.

**Thm 2** (Flow box theorem)

If $X(p) \neq 0$ then there is a chart $(U,\phi)$ near $p$ such that $X(x) = \frac{\partial}{\partial x^1}\Big|_x$

i.e. 

Idea $X(x) = f(x) \frac{\partial}{\partial x}$ $\quad f(x) > 0$

$y = \int_0^x \frac{1}{f(s)}\, ds \qquad X(y) = f(x) \frac{dy}{dx}\frac{d}{dy} = f(x) \frac{1}{f(x)} \frac{d}{dy} = \frac{d}{dy}.$

Back to Thm 1.

- If $X(p) = 0$ then $[X,Y](p) = 0$ and

$$\mathcal{L}_X Y(p) = \lim_{t\to 0} \frac{1}{t}\left( Y(p) - (\phi^X_{-t})^*(Y \circ \phi^X_t(p)) \right)$$

$$= \lim_{t\to 0} \frac{1}{t}\left( Y(p) - Y(p) \right)$$

$= 0$

- If $X(p) \neq 0$ then in flow box coordinates

$$X(x) = \partial/\partial x^1 \quad , \quad Y(x) = \sum Y^i \, \partial/\partial x^i$$

$$[X, Y] = \sum_j \left( \sum_i X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \partial/\partial x^j$$

$$= \sum_j \frac{\partial Y^j}{\partial x^1}$$

$$\phi_t^X (x^1, \dots, x^n) = (x^1 + t, x^2, \dots, x^n)$$

$$[(\phi_{-t}^X)_*] = I_n$$

$$(\phi_{-t}^X)_* Y(\phi_t^X(x)) = (\phi_{-t}^X)_* \left( \sum_i Y^i (x^1+t, x^2, \dots, x^n) \, \partial/\partial x^i \right)$$

$$= \sum_i Y^i (x^1 + t, x^2, \dots, x^n)$$

$$\mathcal{L}_X Y(x) = \lim_{t \to 0} \frac{1}{t} \left( \sum_i \left( Y^i(x^1+t, x^2, \dots, x^n) - Y^i(x^1, \dots, x^n) \right) \partial/\partial x^i \right.$$

$$= \sum_i \frac{\partial Y^i}{\partial x^i} \quad \checkmark$$

For a k-form $\omega \in \Lambda^k(M)$ we can also define

$$\mathcal{L}_X \omega(p) = \frac{d}{dt}\Big|_{t=0} \left( \left( (\phi_t^X)^* \omega \right)(p) \right)$$

<u>Thm 3</u> Cartan's Magic formula.

$$\mathcal{L}_X \omega = d(\omega(X,\cdot)) + d\omega(X,\cdot)$$

<u>Exercise</u>: Prove this using idea above.

Connections on $M$.

Unlike $Xf = \sum X^i \frac{\partial f}{\partial x^i}$, $\mathcal{L}_X Y = \sum \left( X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \partial/\partial x^j$

depends on the values of $X$ in a nbhd of $p$.

Q. Can we define a directional derivative of $Y$ w.r.t $X$ at $p$ which only depends on $X(p)$?

A connection at $p \in M$ is a bilinear map

$$T_pM \times \{\text{vector fields}\} \longrightarrow T_pM$$

$$(X(p), Y) \longmapsto \nabla_{X(p)} Y$$

s.t. $\nabla_{X(p)} fY = f(\nabla_{X(p)} Y) + X(f)\, Y.$

A connection on $M$ is a map which assigns to each $p$ a connection $\nabla(p)$ at $p$ such that

1) For vector fields $X, Y$ on $M$

$\nabla_X Y(p) = \nabla(p)_{X(p)} Y$ is a smooth vector field.

2) $(X, Y) \longrightarrow \nabla_X Y$ is bilinear

3) $\nabla_{fX} Y = f\, \nabla_X Y$

4) $\nabla_X (fY) = f(\nabla_X Y) + X(f)\, Y$

Let's look at such a $\nabla$ must look like in local coordinates

$$\nabla_X Y = \nabla_{\sum X^i \partial/\partial x^i} \sum Y^j \partial/\partial x^j$$

$$= \sum_{i,j} X^i \left( Y^j \nabla_{\partial/\partial x^i} \partial/\partial y^j + \frac{\partial Y^j}{\partial x^i} \partial/\partial y^j \right)$$

$\nabla$ is determined by $\nabla_{\partial/\partial x^i} \partial/\partial x^j = \sum_k \Gamma_{ij}^k \, \partial/\partial x_k$

$\nabla$ is determined by $n^3$ functions $\Gamma_{ij}^k$ (Christoffel symbols)

There are LOTS of connections on $M$.

Ex $\Gamma_{ij}^k = 0 \Rightarrow \nabla_{\partial/\partial x^i} Y = \nabla_{\partial/\partial x^i} \left( \sum Y^j \, \partial/\partial x^j \right) = \sum_j \frac{\partial Y^j}{\partial x^i} \, \partial/\partial x^j$