Lie Bracket and Lie Derivative

Two properties of functions

- Given $f, g \in C^\infty(M)$ we can form $fg \in C^\infty(M)$
- Given $f \in C^\infty(M)$ and a v.f. X we can compute $X(f)$ the directional derivative of f.

A vector field $X : M \longrightarrow TM$ is a generalization of a function. $\left(X(x) = (x, V(x)) \right.$ ← vector valued function $\left. \right)$

Q1) Can we multiply two vector fields X, Y. ?

Q2) Can we take directional derivative of Y w.r.t. X?

A1) yes $[X, Y]$ Lie Bracket

A2) Yes $\mathcal{L}_X Y$ Lie derivative.

Bonus $[X, Y] = \mathcal{L}_X Y$!

Q1 How do we multiply X, Y to get new v.f. ?

A vector field X is a linear map

$$C^\infty(M) \longrightarrow C^\infty(M)$$
$$f \longmapsto X(f)$$

which satisfies the Liebnitz rule $X(fg)(p) = X(f)(p)\, g(p) + f(p)\, X(g)(p)$

we can compose the maps for X and Y

$$XY : C^\infty(M) \longrightarrow C^\infty(M)$$
$$f \longmapsto X(Y(f))$$

<u>Problem</u> XY does not satisfy Liebnitz rule and so is not a vector field!

$$\begin{aligned} XY(fg) &= X(Y(fg)) \\ &= X((Yf)g + f(Yg)) \\ &= X(Yf)g + (Yf)(Xg) + (Xf)(Yg) + f(X(Yg)) \\ &= (XY(f))g + f(XY(g)) + \underbrace{(Yf)(Xg) + (Xf)(Yg)}_{\text{error terms}} \end{aligned}$$

These error terms are symmetric in X, Y.

__Prop__ $XY - YX$ is a vector field

$XY - YX$ is called the Lie Bracket of X and Y and is denoted by $[X,Y]$.

Properties of $[X,Y]$.

1) bilinear

2) $[X,Y] = -[Y,X]$

3) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

Jacobi Identity

4) $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$

In local coordinates $X = \sum X_i \partial/\partial x^i \quad Y = \sum_i Y_i \partial/\partial x^i$

$$[X,Y] = \sum_j \left(\sum_i \left(X_i \frac{\partial Y_j}{\partial x^i} - Y_i \frac{\partial X_j}{\partial x^i} \right) \right) \partial/\partial x^j$$

Q2) How can we take directional derivative of Y w.r.t. X?

- Given $X(p) \in T_pM$ and $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = X(p)$

$$X(p)\, f = \frac{d}{dt}\Big|_{t=0} f(\gamma(t))$$

- Let ϕ_t^X be the flow of X

$$\phi_0^X(p) = p \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0} \left(\phi_t^X(p)\right) = X(p)$$

Hence $$X(p)\, f = \frac{d}{dt}\Big|_{t=0} f(\phi_t^X(p))$$

or $$X(p)\, f = \lim_{t \to 0} \frac{f(\phi_t^X(p)) - f(p)}{t}$$

What about derivative of v.f Y at p in direction $X(p)$?

$$\mathcal{L}_X Y(p) = \lim_{t \to 0} \frac{Y(\phi_t^X(p)) - Y(p)}{t} \quad ?$$

<u>Problem</u> $Y(\phi_t^X(p)) \in T_{\phi_t^X(p)} M \neq T_p M \ni Y(p)$.

We can fix this using isomorphism from $T_{\phi^X_t(p)}M$ to T_pM.

$$(\phi^X_t)_* : T_pM \longrightarrow T_{\phi^X_t(p)}M$$

$$(\phi^X_t)_*^{-1} : T_{\phi^X_t(p)}M \longrightarrow T_pM$$

$$\phi^X_{-t} \circ \phi^X_t = Id_M \Rightarrow (\phi^X_{-t})_* \circ (\phi^X_t)_* = Id_{T_pM}$$

$$\Rightarrow (\phi^X_t)_*^{-1} = (\phi^X_{-t})_*.$$

<u>Def</u>$^{\underline{n}}$
$$\mathcal{L}_X Y(p) = \lim_{t\to 0} \frac{(\phi^X_{-t})_*(Y(\phi^X_t(p))) - Y(p)}{t}$$

$$= \underbrace{\dot{(\phi^X_{-t})_*(Y(\phi^X_t(p)))}}_{\text{curve in } T_pM}(0)$$

<u>Ex</u> $X(x) = \frac{\partial}{\partial x^1}\Big|_x \quad Y(x) = x^1 \frac{\partial}{\partial x^2}\Big|_x$

$$\mathcal{L}_X Y(x) = \lim_{t\to 0} \frac{1}{t}\left((\phi^X_{-t})_*(Y(\phi^X_t(x))) - Y(x)\right)$$

$$\phi^X_t(x^1, x^2) = (x^1 + t, x^2)$$

$$\left[(\phi^X_{-t})_*\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $(\phi^X_{-t})_* \left(\frac{\partial}{\partial x^i}\Big|_{\phi^X_t(x)} \right) = \frac{\partial}{\partial x^i}\Big|_x$

$$\mathcal{L}_X Y(x) = \lim_{t \to 0} \frac{1}{t} \left((\phi^X_{-t})_* \left(Y(x^1+t, x^2) \right) - x^1 \frac{\partial}{\partial x^2}\Big|_x \right)$$

$$= \lim_{t \to 0} \frac{1}{t} \left((\phi^X_{-t})_* \left((x^1+t) \frac{\partial}{\partial x^2}\Big|_{\phi^X_t(x)} \right) - x^1 \frac{\partial}{\partial x^2}\Big|_x \right)$$

$$= \lim_{t \to 0} \frac{1}{t} \left((x^1+t) \frac{\partial}{\partial x^2}\Big|_x - x^1 \frac{\partial}{\partial x^2}\Big|_x \right)$$

$$= \lim_{t \to 0} \frac{1}{t} \left(t \frac{\partial}{\partial x^2}\Big|_x \right)$$

$$= \frac{\partial}{\partial x^2}\Big|_x$$

<u>Thm</u> $\mathcal{L}_X Y = [X, Y]$.

Proof Next Time.