Q. Are M and N diffeomorphic?

$$d \circ d = 0$$

$$\Lambda^k(M) \supseteq Z^k(M) = \{ \alpha \in \Lambda^k(M) \mid d\alpha = 0 \} \supseteq B^k(M) = \{ \alpha \in \Lambda^k(M) \mid \alpha = d\beta \}$$

$$H^k(M) = Z^k / B^k(M)$$

$$b^k(M) = \dim(H^k(M))$$

$$M \simeq N \Rightarrow b^k(M) = b^k(N) \quad \text{for all } k = 1, \dots, n$$

<u>Example</u> $M = S^1$

$$H^0(M) = \{\text{constant functions}\} \simeq \mathbb{R} \quad \text{so} \quad b^0(S^1) = 1$$

$$\Lambda^1(M) = \{ h \, d\theta \mid h \in C^\infty(M) \}$$

$$Z^1(M) = \Lambda^1(M) \qquad (d(h\, d\theta) = dh \wedge d\theta = 0)$$

$$B^1(M) = \{ df \mid f \in C^\infty(M) \}$$

$$\subseteq \left\{ g\, d\theta \;\middle|\; \int_{S^1} g\, d\theta = 0 \right\} \qquad \left(\int_{S^1} df = \int_{\partial S^1} f = 0 \right)$$

In fact $B^1(M) = \left\{ g\, d\theta \;\middle|\; \int_{S^1} g\, d\theta = 0 \right\}$

$$f(\theta) = \int_0^\theta g(\tau)\, d\tau \quad \text{and FTC} \Rightarrow df = g(\theta)\, d\theta$$

is periodic since $f(0) = f(2\pi) = 0$

Now

$$\underset{\substack{\cap \\ Z^1(S^1)}}{h\, d\theta} = \underset{\substack{\cap \\ B^1(M)}}{\left(h - \frac{1}{2\pi}\int_0^{2\pi} h\, d\theta \right) d\theta} + \underset{\substack{\cap \\ \mathbb{R}\, d\theta}}{\left(\frac{1}{2\pi}\int_0^{2\pi} h\, d\theta \right) d\theta}$$

So $[h\,d\theta] = \Big[\underbrace{\left(\frac{1}{2\pi}\int_0^{2\pi} h\,d\theta\right)}_{\in\, \mathbb{R}\, d\theta} d\theta\Big]$ and $H^1(S^1) \simeq \mathbb{R}$.

So $b^0(S^1) = 1$

<u>Lemma</u> M connected $\Rightarrow$ $b^0(M) = 1$

<u>Lemma</u> M compact orientable $\Rightarrow$ $b^n(M) = 1$

The real information is in b^k for $k = 2, \ldots, n-1$.

<u>Fact</u> $b^1(\Sigma_g) = 2g$

So b^1 classifies orientable surfaces.

Some of the information in the middle is redundant.

<u>Thm</u> (Poincaré Duality) For M compact orientable.

$$b^k(M) = b^{n-k}(M) \quad \text{for all } k = 0, \ldots, n$$

Given M let $\underset{\substack{\uparrow \\ \text{Poincaré Polynomial}}}{P_M(x)} = b^0(M) + b^1(M)x + \cdots + b^n(M)x^n$

<u>Thm</u> (Künneth)

$$P_{M \times N}(x) = P_M(x)\, P_N(x)$$

ex $\quad \Sigma_1 = S^1 \times S^1$

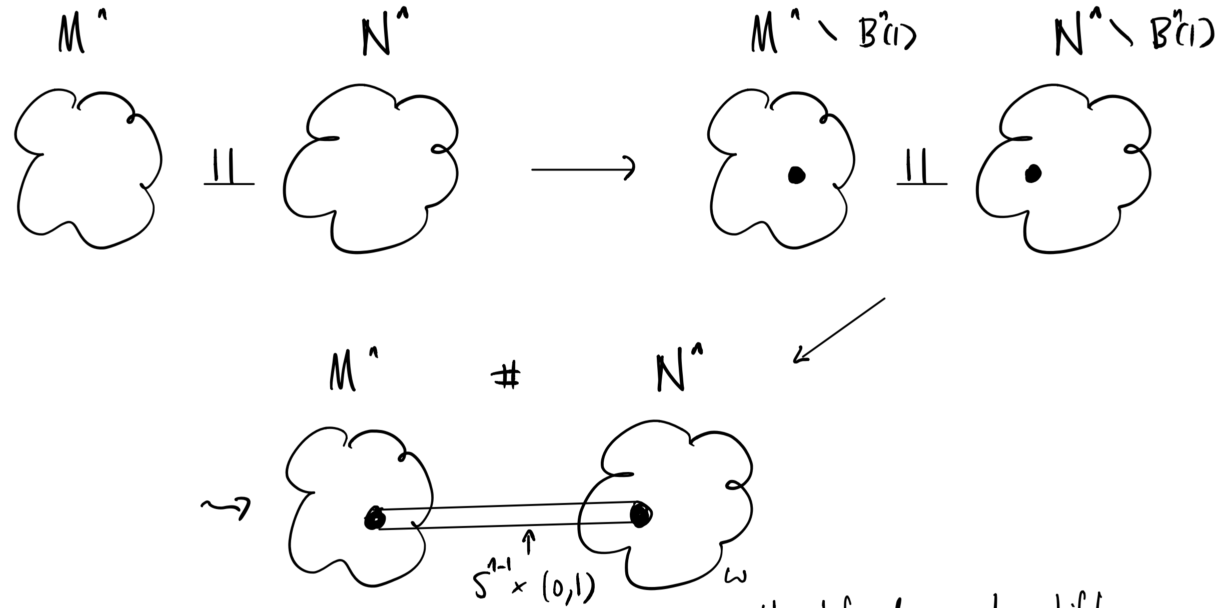
$$\begin{aligned} P_{\Sigma_1}(x) &= (P_{S^1}(x))^2 \\ &= (1+x)^2 \\ &= 1 + 2x + x^2 \end{aligned}$$

So $\quad b_1(\Sigma_1) = 2(1)$ as claimed above.

Example $\quad b^k(M) = b^k(N) \not\Rightarrow M \approx N$

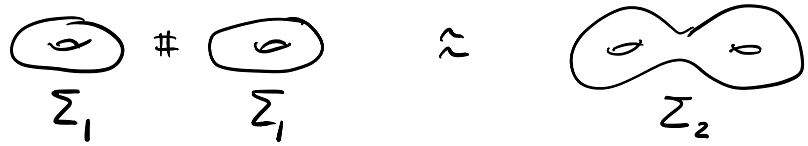
- replace $\mathbb{R}$ with $\mathbb{C}$ in definition of $\mathbb{RP}^n$ and we get $\mathbb{CP}^n = \{ [z_1, \ldots, z_{n+1}] \}$ where $[z_1, \ldots, z_{n+1}] = [\lambda z_1, \ldots, \lambda z_{n+1}]$ for all $\lambda \in \mathbb{C} \setminus 0$

- Given M^n and N^n can form $(M \# N)^n$ (well defined up to diffeomorphism)



well defined up to diffeo.

ex



<u>Fact</u> $\quad b^k(S^2 \times S^2) = b^k(\mathbb{C}P^2 \# \mathbb{C}P^2) \quad k = 1, 2, 3, 4$

but $\quad S^2 \times S^2 \not\cong \mathbb{C}P^2 \# \mathbb{C}P^2$

Can still use the $H^k(M)$ to tell them apart

Set $\quad H^*(M) = \bigoplus_{k=0}^{n} H^k(M)$.

This is not just a vector space. It is a ring!

(Think $n \times n$ matrics with usual product)

The product $[\alpha] \cdot [\beta] = [\alpha \wedge \beta]$.

$d\alpha = 0$, $d\beta = 0$ $\Rightarrow$ $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta = 0$

$$(\alpha + d\eta) \wedge \beta = \alpha \wedge \beta + d\eta \wedge \beta$$
$$= \alpha \wedge \beta + d(\eta \wedge \beta)$$

Recall $F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$

If F is a diffeomorphism then

$F^* : H^*(N) \longrightarrow H^*(M)$.

is a ring isomorphism. (multiplication tables get relabelled)

$H^*(S^2 \times S^2)$ and $H^*(\mathbb{C}P^2 \# \mathbb{C}P^2)$ are not isomorphic as rings.

Topics for Exam 1

(r,s) - tensor fields

- $T : M \to \mathcal{T}_{r,s}(M)$ • local

  $\pi \circ T = Id_M$ $\qquad T(x) = \sum T^{k_1 \ldots k_s}_{h_1 \ldots h_r} \frac{\partial}{\partial x^{k_1}} \otimes \cdots$

- Riemannian metric g

$\|V\|_g$, $L_g(\gamma)$, $d_g(p,q)$.

- k - forms

local basis, $\wedge$ (Alt), d

- F^* , orientations , partition of unity.

$\rightsquigarrow \int_M \alpha$

- Manifolds w/ boundary , inherited orientation

$$\int_M d\alpha = \int_{\partial M} \alpha$$

- $\nexists$ $F: M \to \partial M$ s.t. $F|_{\partial M} = Id$

- Brouwer Fixed pt Thm

- Z^k , B^k , H^k , b^k.