Brouwer's Fixed Point Thm

For every smooth $F: D^n \longrightarrow D^n$ there is point $p \in D^n$ s.t. $F(p) = p$.

<u>Rmk</u> The generalization F continuous follows from

<u>Thm</u> (Wierstrauss Approximation)

Given $F: D^n \to D^n$ continuous, for each $\varepsilon > 0$ there is a polynomial (hence smooth) map $P: D^n \longrightarrow D^n$ s.t. $\| F(p) - P(p) \| < \varepsilon$ for all $p \in D^n$.

<u>Exercise</u> (triangle inequality)

F has no fixed pt $\Rightarrow$ P has no fixed pt for ε small enough

This would contradict smooth Brouwer Thm.

Hard Problem:

Let M and N be compact connected manifolds of dimension n.

<u>ex</u> $M = F^{-1}(q)$ for some $F: S^{12} \longrightarrow O(3)$

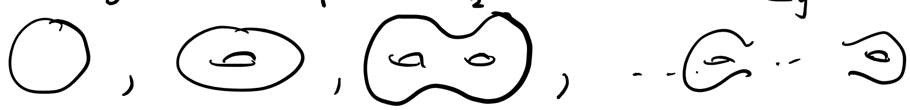
N has atlas $\mathcal{A} = \{ (U_\alpha, \phi_\alpha) \}$

Determine if M and N are diffeomorphic.

<u>Dimension 1</u> $M^1 \approx S^1$

<u>Dimension 2</u>

$S^2 = \Sigma_0$ $\quad S^1 \times S^1 = \Sigma_1$ $\quad \Sigma_2$ $\quad \Sigma_g$

M^2 orientable $\approx$ 

$[M]$ is determined by 1-number g (genus).

<u>Dimension ≥ 3</u> ???

A tool (which includes genus).

FACT 1. If $F: M \longrightarrow N$ is a diffeomorphism then $F^*: \Lambda^k(N) \longrightarrow \Lambda^k(M)$ is a vector space isomorphism for all $k = 0, \dots, n$. $\left((F^*)^{-1} = (F^{-1})^* \right)$

More importantly, the calculus of k-forms is the same for diffeomorphic manifolds.

<u>Prop</u> Let $F: M \to N$ be a diffeomorphism.
Let $\alpha \in \Lambda^k(N)$

i) $d\alpha = 0 \iff d(F^*\alpha) = 0$

α is closed $\qquad$ $F^*\alpha$ is closed

ii) $\alpha = d\beta \iff F^*\alpha = d(\eta)$ for some $\eta \in \Lambda^{k-1}(M)$

α exact $\qquad$ $F^*\alpha$ exact

Note α exact $\Rightarrow$ α closed

Since $\alpha = d\beta \Rightarrow d\alpha = d(d\beta) = 0$

The converse is not true !

Ex $\alpha = \dfrac{-x^2}{(x^1)^2 + (x^2)^2}\, dx^1 + \dfrac{x^1}{(x^1)^2 + (x^2)^2}\, dx^2 \in \Lambda^1(\mathbb{R}^2 \setminus 0)$

Exercise : $d\alpha = 0$

Claim $\alpha \neq df$ for any $f \in C^\infty(\mathbb{R}^2 \setminus 0)$

Pf Assume $\alpha = df$

Consider $i : S^1 \longrightarrow \mathbb{R}^2 \setminus 0$ inclusion

Now $i^*\alpha = i^* df = d(i^* f) = d(f \circ i)$

So $\int_{S^1} i^*\alpha = \int_{S^1} d(f\circ i) = \int_{\partial S^1} f\circ i = 0$ since $\partial S^1 = \emptyset$

But $\int_{S^1} i^*\alpha = \int_{\gamma_1} -x^2\,dx^1 + x^1\,dx^2$ for $\gamma_1 : [0,1] \to S^1$
$t \mapsto (\cos 2\pi t, \sin 2\pi t)$

$$= \int_0^1 -\sin 2\pi t \; d(\cos 2\pi t) + \cos 2\pi t \; d(\sin 2\pi t)$$

$$= \int_0^1 2\pi$$

$$= 2\pi \neq 0$$

This contradicts assumption

We can define a number which measures the difference (if any) b/w closed and exact k-forms.

Given M and k set

$$Z^k(M) = \{ \alpha \in \Lambda^k(M) \mid d\alpha = 0 \}$$

$$B^k(M) = \{ \alpha \in \Lambda^k(M) \mid \alpha = d\beta \text{ for } \beta \in \Lambda^{k-1}(M) \}$$

$Z^k(M)$ is a subspace of $\Lambda^k(M)$

$B^k(M)$ is a subspace of $\Lambda^k(M)$

$\alpha = d\beta \qquad \tilde{\alpha} = d\tilde{\beta} \quad \Rightarrow \quad \alpha + c\tilde{\alpha} = d(\beta + c\tilde{\beta})$

In fact : $B^k(M)$ is a subspace of $Z^k(M)$

Set $\quad H^k(M) = Z^k(M) / B^k(M)$

$= Z^k(M)/\sim$ where

$\alpha \sim \beta \iff \alpha - \beta \in B^k(M)$

i.e. $\alpha - \beta = d\eta$

<u>FACT</u> $H^k(M)$ is a finite dimensional vector space.

<u>Def^n</u> $b^k(M) = \dim(H^k(M))$ (k^{th} Betti number)

<u>Ex</u> $M = S^1$

$\Lambda^0(S^1) = C^\infty(S^1) = \{ f(\theta) \mid f(\theta + 2\pi) = f(\theta) \}$

$B^0(S^1) = \emptyset$ since $\Lambda^{-1}(S^1) = \emptyset$.

$Z^0(S^1) = \{ f \in C^\infty(S^1) \mid df = 0 \}$

$= \{$ constant functions on $S^1 \}$

$\simeq \mathbb{R}$.

$H^0(S^1) \simeq \mathbb{R} \qquad b^0(S^1) = 1$

$$\Lambda^1(S^1) = \{ h\,d\theta \mid h \in C^\infty(S^1) \}$$

$$B^1(S^1) = \{ df \mid f \in C^\infty(S^1) \}$$

$$\subseteq \{ g\,d\theta \mid \int_{S^1} g\,d\theta = 0 \}$$

In fact $B^1(S^1) = \{ g\,d\theta \mid \int_{S^1} g\,d\theta = 0 \}$

$g\,d\theta = df$ for $f = \int_0^\theta g(\theta)\,d\theta$

$$Z^1(S^1) = \Lambda^1(S^1)$$

$$h = h - \frac{1}{2\pi}\int_{S^1} h\,d\theta + \frac{1}{2\pi}\int_{S^1} h\,d\theta$$

$$\underbrace{h\,d\theta}_{\in Z^1(S^1)} = \underbrace{\left(h - \frac{1}{2\pi}\int_{S^1} h\,d\theta\right) d\theta}_{\in B^1(S^1)} + \left(\frac{1}{2\pi}\int_{S^1} h\,d\theta\right) d\theta$$

$$Z^1(S^1)/B^1(S^1) = \{ c\,d\theta \mid c \in \mathbb{R} \} \simeq \mathbb{R}$$

So $b^1(S^1) = \mathbb{R}$.

<u>Lemma</u> If M and N are diffeomorphic then

$b^k(M) = b^k(N)$ for $k = 1, \ldots, n$.

Pf $\quad F: M \to N \quad \rightsquigarrow \quad F_* : \Lambda^k(N) \xrightarrow{\simeq} \Lambda^k(M)$

$$F_* : Z^k(N) \xrightarrow{\simeq} Z^k(M)$$

$$F_* : B^k(N) \xrightarrow{\simeq} B^k(M)$$

So $\quad F_* : Z^k(N)/B^k(N) \xrightarrow{\simeq} Z^k(M)/B^k(M)$

$$[\alpha] \longmapsto [F^*\alpha]$$

Lemma If M is connected then $b^0(M) = 1$

Lemma If M is orientable then $b^n(M) = 1$

Thm (Poincaré Duality) If M is compact and orientable $b^k(M) = b^{n-k}(M)$ for $k = 0, \dots, n$.

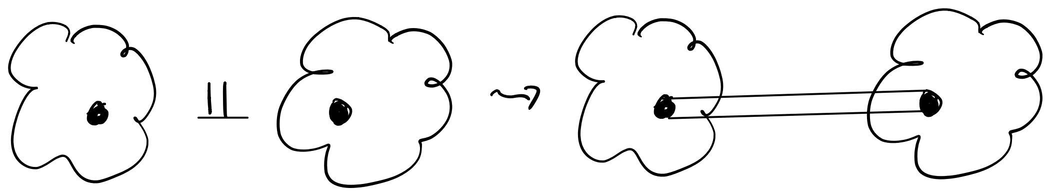
Lemma $b^1(\Sigma_g) = 2g$

$\{b^k\}$ is far from a perfect set of invariants

$$b^k(M) = b^k(N) \not\Longrightarrow M \approx N.$$

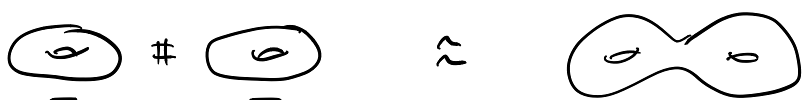
<u>Ex</u> • $\mathbb{R}P^n \overset{\mathbb{R}\to\mathbb{C}}{\rightsquigarrow} \mathbb{C}P^n$ dimension $2n$

• M^n $\quad$ N^n $\quad$ $M^n \# N^n$



well defined up to diffeo.

ex



$\Sigma_1 \# \Sigma_1 \approx \Sigma_2$

<u>Fact</u> $b^k(S^2 \times S^2) = b^k(\mathbb{C}P^2 \# \mathbb{C}P^2)$ $\quad k = 1, 2, 3, 4$

but $S^2 \times S^2 \not\approx \mathbb{C}P^2 \# \mathbb{C}P^2$