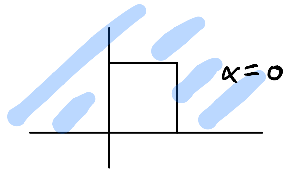
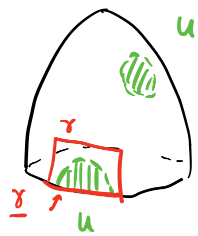
FTC $\Rightarrow$ Stokes'

0) $M = \mathbb{R}^n_+$



1) M oriented compact,



$\alpha = 0$ on $M \setminus U$

<u>General Case</u> M oriented compact, $\alpha \in \Lambda^{n-1}(M)$

- $\{V_i\}$ $\quad V_i \subset \gamma_i([0,1]^n)$, $\quad V_i \cap \partial M = \underline{\gamma}_i([0,1]^{n-1}) \cap \partial M$
- $\{f_i\}$

$$\int_M d\alpha = \sum_{i=1}^N \int_{\gamma_i} f_i(d\alpha)$$

$$= \sum_{i=1}^N \int_{\gamma_i} \left(d(f_i\alpha) - df_i \wedge \alpha \right)$$

$$= \sum_{i=1}^N \left(\int_{\gamma_i} d(f_i\alpha) - \sum_{i=1}^N \int_{\gamma_i} df_i \wedge \alpha \right)$$

$$= \sum_{i=1}^N \int_M d(f_i\alpha) - \sum_{i=1}^N \int_M df_i \wedge \alpha$$

since f_i and $df_i = 0$ outside $\gamma_i([0,1]^n)$

$$= \sum_{i=1}^{N} \int_{\partial M} f_i \alpha - \sum_{i=1}^{N} \int_{M} df_i \wedge \alpha \qquad \text{by Step (1)}$$

$$= \int_{\partial M} \left(\sum_{i=1}^{N} f_i\right)\alpha - \int_{M} \left(\sum_{i=1}^{N} df_i\right)\wedge \alpha \qquad \text{(by linearity)}$$

But $\sum_{i=1}^{N} f_i = 1$ and $\sum_{i=1}^{N} df_i = 0 \in \Lambda^1(M)$

So $$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$

Now we have completed Calculus on Manifolds.

Immediate Applications

1) $\partial M = \emptyset \Rightarrow \int_M d\alpha = 0 \quad \forall\, \alpha \in \Lambda^{n-1}(M)$

2) $d\alpha = 0 \Rightarrow \int_{\partial M} \alpha = 0 \quad \forall\, \alpha \in \Lambda^{n-1}(M)$

$$\int_N \alpha = 0 \quad \text{if} \quad N = \partial M \quad \text{and} \quad \alpha \text{ extends to } M \text{ as } \eta$$

such that $d\eta = 0$.

Deeper Applications

<u>Thm</u> Let M be compact oriented with boundary ∂M. There does not exist a smooth map $F: M \longrightarrow \partial M$ s.t. $F|_{\partial M} = Id_M$.

To prove this we need to reimagine orientations again.

M cpt orientable $\iff \exists\, \omega \in \Lambda^n(M)$ s.t. $\omega(p) \neq 0 \quad \forall\, p \in M$

"$\Rightarrow$" Let $\mathcal{A} = \{(U_i, \phi_i)\}_{i=1}^N$ be orienting atlas

Let $\{f_i\}$ be partition of unity subordinate to $\{U_i\}$.

Set $\omega = \sum_{i=1}^N f_i(x)\, dx_i^1 \wedge \dots \wedge dx_i^n$

Claim ω is nonvanishing.

Exercise $dx_i^1 \wedge \dots \wedge dx_i^n = \det\left(\left[(\phi_j \circ \phi_i^{-1})_*\right]\right) dx_j^1 \wedge \dots \wedge dx_j^n$

Rmk $\int_M \omega > 0$ (ω called volume form)

Pf of Thm

By contradiction.

Assume $F: M \to \partial M$ smooth and $F|_{\partial M} = Id_{\partial M}$.

Since M oriented so is ∂M.

Choose a nonvanishing $(n-1)$-form ω on ∂M.

Set $\alpha = F^*\omega$

Note $d\alpha = d(F^*\omega) = F^*(d\omega) = 0$ since $\omega \in \Lambda^{n-1}(\partial M^{n-1})$

So

$$0 = \int_M d\alpha \overset{\text{Stokes'}}{=} \int_{\partial M} \alpha = \int_{\partial M} F^*\omega \overset{F|_{\partial M} = Id_M}{=} \int_{\partial M} \omega \neq 0.$$

This is the desired contradiction. ↯

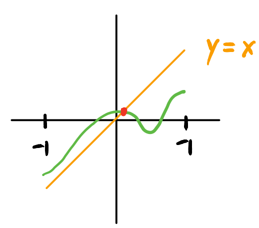
Brouwer's Fixed Point Thm (Smooth case)

Let $D^n = \{ x \in \mathbb{R}^n \mid \|x\|^2 \leq 1 \}$

This is a manifold with boundary, $\partial D^n = S^{n-1}$

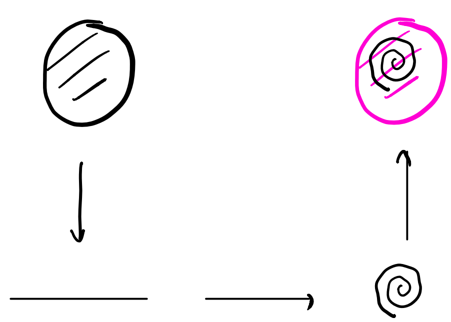
For every smooth map $F: D^n \longrightarrow D^n$ there is a point $p \in D^n$ s.t. $F(p) = p$.

Ex 1 $\quad F : [-1,1] \longrightarrow [-1,1]$



$$F(x) = x$$
$$\Updownarrow$$
$$(x, F(x)) = (x, x)$$

Ex 2 $\quad F : D^2 \longrightarrow D^2$



Is it clear there is a fixed point?

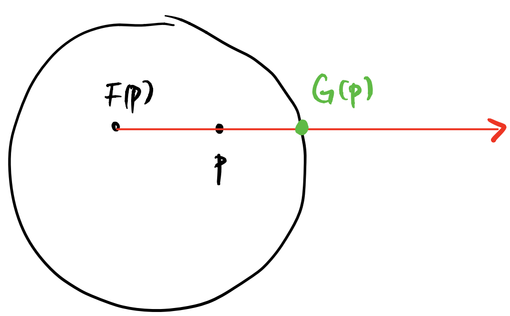
Pf Again by Contradiction using previous Thm

Assume $\exists$ $F : D^n \to D^n$ smooth w/ no fixed points

$F(p) \neq p \quad \forall \; p \in D^n$

- We use this to build a smooth map $G : D^n \longrightarrow \partial D^n = S^{n-1}$ such that $G|_{\partial D^n} = Id_{\partial D^n}$, thus contradicting Thm above.

- $F(p)$ and p are distinct points in D^n. Consider the ray from $F(p)$ to p.



- Let $G(p)$ be the unique point of S^{n-1} on this ray

- Note $G : D^n \longrightarrow \partial D^n$ and $G|_{\partial D^n} = Id|_{\partial D^n}$