Last time we saw Stokes' $\Rightarrow$ FTC

$$\int_M d\alpha = \int_{\partial M} \alpha$$

$$\Rightarrow \int_{[0,1]} df = \int_{\{0^-, 1^+\}} f = -f(0) + f(1)$$

(Here $\text{int}([0,1]) = (0,1)$ oriented by $\mathring{a} = \{((0,1), Id)\}$.)

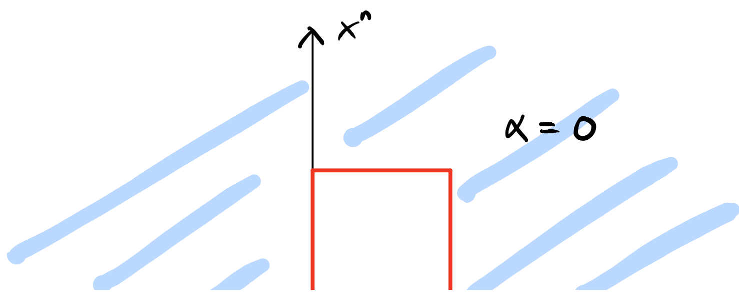
Today FTC $\Rightarrow$ Stokes'

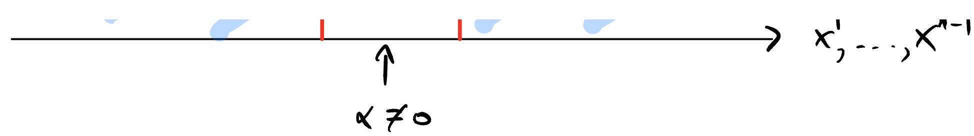
<u>Case 0</u> $M = \mathbb{R}^n_+$ oriented by $a^\circ = \{(\text{int}(\mathbb{R}^n_+), Id)\}$

Assume $\alpha \in \Lambda^{n-1}(\mathbb{R}^n_+)$ satisfies

i) for $i = 1, \dots, n-1$ $\alpha(x) = 0$ when $x_i \leq 0$ or $x_i \geq 1$

ii) $\alpha(x) = 0$ for $x^n \geq 1$





$$\alpha(x) = \sum_{i=1}^{n} a_i(x)\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

$$d\alpha\,(x) = \sum_{i=1}^{n} da_i \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

$$= \sum_{i=1}^{n} \left(\sum_j \frac{\partial a_i}{\partial x^j} dx^j \right) \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial a_i}{\partial x^i}\, dx^1 \wedge \dots \wedge dx^n$$

$$\int_M d\alpha = \int_{[0,1]^n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial a_i}{\partial x^i}\, dx^1 \wedge \dots \wedge dx^n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \int_0^1 \cdots \int_0^1 \frac{\partial a_i}{\partial x^i}\, dx^1 \cdots dx^n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \int_0^1 \cdots \int_0^1 \frac{\partial a_i}{\partial x^i}\, dx^i \quad dx^1 \cdots \widehat{dx^i} \cdots dx^n$$

By FTC $\displaystyle \int_0^1 \frac{\partial a_i}{\partial x^i}\, dx^i = a_i(x^1, \dots, x^i, \dots, x^n) \Big|_0^1$

$$= \begin{cases} 0 & \text{if} \quad i = 1, \dots, n-1 \\ -a_n(x^1, \dots, x^{n-1}, 0) & i = n \end{cases}$$

So $\int_M d\alpha = (-1)^{n-1} \int_0^1 \cdots \int_0^1 (-a_n(x^1, \dots, x^{n-1}, 0))\, dx^1 \cdots dx^{n-1}$

$$= (-1)^n \int_0^1 \cdots \int_0^1 a_n(x^1, \dots, x^{n-1}, 0)\, dx^1 \cdots dx^{n-1}$$

Next $\int_{\partial M} \alpha$

Here $\alpha = \alpha|_{\partial \mathbb{R}^n_+}$

$x^n|_{\partial \mathbb{R}^n_+} = 0$ and $d\left(x^n|_{\partial \mathbb{R}^n_+}\right) = 0$

So $\alpha|_{\partial \mathbb{R}^n_+}(x) = \sum_i \alpha(x)\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$

$$= \alpha_n(x^1, \dots, x^{n-1}, 0)\, dx^1 \wedge \dots \wedge dx^{n-1}$$

With respect to the inherited orientation on $\partial \mathbb{R}^n_+$

$$\int_{\partial \mathbb{R}^n_+} \alpha|_{\partial \mathbb{R}^n_+} = (-1)^n \int_{\mathbb{R}^{n-1}} \alpha|_{\mathbb{R}^{n-1}} =$$

$$= (-1)^n \int_{[0,1]^{n-1}} \alpha |_{\mathbb{R}^{n-1}}$$

$$= (-1)^n \int_0^1 \cdots \int_0^1 \alpha_n (x^1, \dots, x^{n-1}, 0)\, dx^1 \dots dx^{n-1}$$

$$= \int_{\mathbb{R}^n_+} d\alpha \qquad \text{from above.}$$

<u>Case 1</u> M oriented with boundary

$U \subset$ open

$\alpha \in \Lambda^{n-1}(M)$ s.t. $\alpha = 0$ on $M \setminus U$

(So $d\alpha = 0$ on $M \setminus U$ as well.)

Choose $\gamma : [0,1]^n \longrightarrow M$ 1-1 orientation preserving

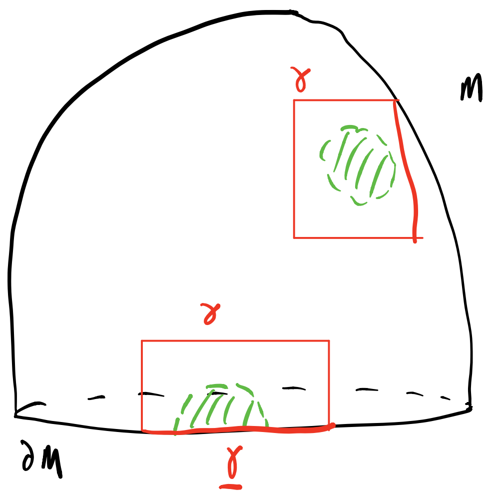
such that $\gamma([0,1]^n) \supset U$.

Define $\underline{\gamma} : [0,1]^{n-1} \longrightarrow M$

$$(t_1, \dots, t_{n-1}) \longmapsto \gamma(t_1, \dots, t_{n-1}, 0)$$

Assume also that $U \cap \partial M = U \cap \underline{\gamma}([0,1]^{n-1})$

(this is empty condition if $U \cap \partial M = \emptyset$)



$$\int_M d\alpha = \int_\gamma d\alpha$$

$$= \int_{[0,1]^n} \gamma^*(d\alpha)$$

$$= \int_{[0,1]^n} d(\gamma^*\alpha)$$

$$= \int_{\mathbb{R}^n_+} d(\gamma^*\alpha)$$

$$= \int_{\partial\mathbb{R}^n_+} \gamma^*\alpha \qquad \text{by Case 0}$$

$$= \int_{[0,1]^{n-1}} \underline{\gamma}^*\alpha$$

$$= \int_{\underline{\gamma}} \alpha$$

$$= \int_{\partial M} \alpha$$

General Case $\quad M$ oriented with boundary

$\alpha \in \Lambda^{n-1}(M)$

- Choose $\{V_i\}_{i=1}^{N}$ s.t. $V_i \subset \gamma_i([0,1]^n)$ with γ_i 1-1 and orientation preserving and

$$V_i \cap \partial M = V_i \cap \underline{\gamma}_i([0,1]^{n-1})$$

- Let $\{f_i\}$ be partition of unity subordinate to $\{V_i\}$.

$$\int_M d\alpha = \sum_{i=1}^{N} \int_{\gamma_i} f_i(d\alpha)$$

$$= \sum_{i=1}^{N} \int_{\gamma_i} \left(d(f_i\alpha) - df_i \wedge \alpha\right)$$

$$= \sum_{i=1}^{N} \int_{\gamma_i} d(f_i\alpha) - \sum_{i=1}^{N} \int_{\gamma_i} df_i \wedge \alpha$$

$$= \sum_{i=1}^{N} \int_{M} d(f_i\alpha) - \sum_{i=1}^{N} \int_{M} df_i \wedge \alpha$$

$$= \sum_{i=1}^{N} \int_{\partial M} f_i\alpha - \sum_{i=1}^{N} \int_{M} df_i \wedge \alpha$$

$$= \int_{\partial M} \left(\sum_{i=1}^{N} f_i\right)\alpha - \int_{M} \left(\sum_{i=1}^{N} df_i\right) \wedge \alpha$$

But $\sum_{i=1}^{N} f_i = 1$ and $\sum_{i=1}^{N} df_i = 0 \in \Lambda^1(M)$

So $\int_{M} d\alpha = \int_{\partial M} \alpha$ ✓

Applications

1) $\partial M = \emptyset \Rightarrow \int_{M} d\alpha = 0 \quad \forall\, \alpha \in \Lambda^{n-1}(M)$

2) $d\alpha = 0 \Rightarrow \int_{\partial M} \alpha = 0 \quad \forall\, \alpha \in \Lambda^{n-1}(M)$

<u>Thm</u> Let M be compact oriented with boundary ∂M. There does not exist a map $F: M \longrightarrow \partial M$ s.t. $F|_{\partial M} = Id_M$.

<u>Aside</u> M compact oriented $\Rightarrow$ $\exists\, \omega \in \Lambda^n(M)$ nonvanishing

<u>Pf</u> Let $\mathcal{A} = \{(U_i, \phi_i)\}_{i=1}^N$ be orienting atlas

Let $\{f_i\}$ be partition of unity subordinate to $\{U_i\}$.

<u>Exercise</u> $\omega = \sum_{i=1}^n f_i(x_i)\, dx_i^1 \wedge \dots \wedge dx_i^n$ nonvanishing

<u>Exercise</u> $\int_M \omega \neq 0$

<u>Pf of Thm</u>

Assume $F: M \to \partial M$ smooth and $F|_{\partial M} = Id_{\partial M}$.

Since M oriented so is ∂M.

Choose a nonvanishing $(n-1)$-form ω on ∂M

Set $\alpha = F^*\omega$

Note $d\alpha = d(F^*\omega) = F^*(d\omega) = 0$

$$0 = \int_M d\alpha = \int_{\partial M} \alpha = \int_{\partial M} F^*\omega = \int_{\partial M} \omega \neq 0.$$

So assumption must be false.