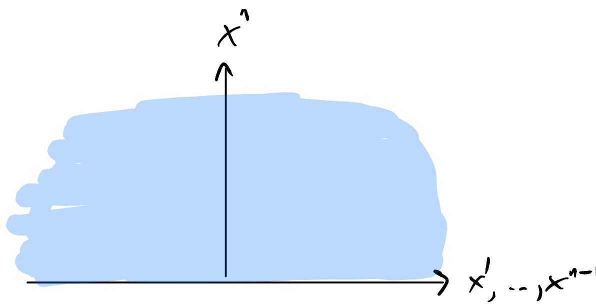
M compact oriented with boundary, ∂M

<u>Stokes' Thm</u> For any $\alpha \in \Lambda^{n-1}(M)$ we have

$$\int_M d\alpha = \int_{\partial M} \alpha$$

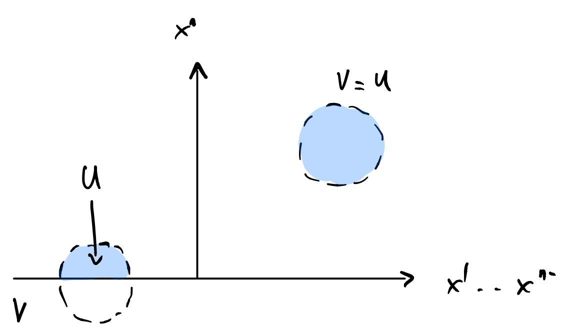
<u>Example 0</u>

$$\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n \mid x^n \geq 0 \}$$



is a manifold with boundary $\partial \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n_+ \mid x^n = 0 \}$

<u>Def^n</u> $U \subset \mathbb{R}^n_+$ is open if $U = V \cap \mathbb{R}^n_+$ for some $V \subset \mathbb{R}^n$ which is open.



<u>Example 1</u> $U \underset{\text{open}}{\subset} \mathbb{R}^n_+$ is an n-dimensional mfld with boundary $\partial U = U \cap \partial \mathbb{R}^n_+$

This is our local model.

<u>Def^n</u> M is an n-dimensional manifold w/ boundary if it admits an atlas whose charts (U_α, ϕ_α) satisfy .

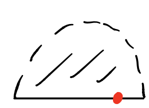
i) $\phi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n_+$ or $\mathbb{R}^n$

ii) ϕ_α is 1-1

iii) $\phi_\alpha(U_\alpha)$ is open (in $\mathbb{R}^n_+$ or $\mathbb{R}^n$)

Q Where is ∂M ?

<u>Def^n</u> $p \in M$ is a boundary pt if for any chart (U, ϕ) with $p \in U$, $\phi : U_\alpha \longrightarrow \mathbb{R}^n_+$ and $\phi(p) \in \partial \mathbb{R}^n_+$



$\partial M = \{ q \in M \mid q \text{ is a boundary point} \}$

The interior of M is $\mathrm{int}(M) = M \setminus \partial M$.

<u>Exercise</u> $\mathrm{int}(M)$ is a manifold w/out boundary

Example $M = [0,1]$

$$\mathcal{A} = \{ ([0,1), Id), ((0,1], \phi(x) = 1-x) \}$$

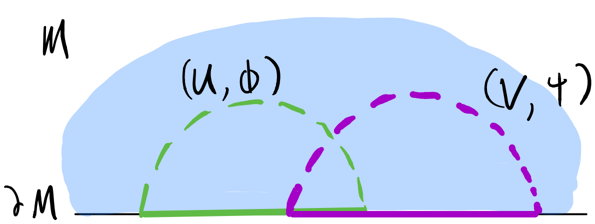
$\phi((0,1]) = [0,1) =$  open in $\mathbb{R}^1_+$

$\partial M = \{0, 1\}$ $\quad\quad$ $\mathrm{int}(M) = (0,1)$.

<u>Def^n</u> M is orientable if $\mathrm{int}(M)$ is orientable.

Facts about M^n and ∂M.

1) ∂M inherits from M the structure of a manifold of dimension n.



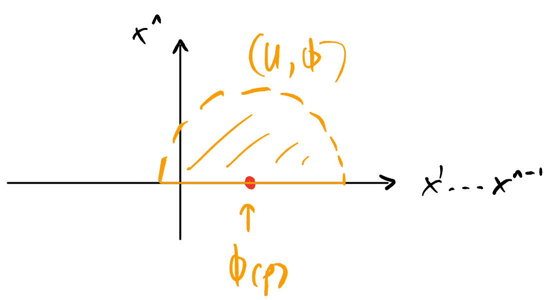
$(U \cap \partial M, \phi|_{U\cap \partial M})$, $(V \cap \partial M, \psi|_{V \cap \partial M})$

Note $\partial(\partial M) = 0$ since no boundary charts for ∂M

2) If M is compact, so is ∂M.

3) Suppose $p \in \partial M$. $T_p(\partial M)$ can be defined as a subspace of T_pM as follows.

Let (U, ϕ) be chart around p.



$\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is basis for T_pM

$\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^{n-1}}\Big|_p \right\}$ is a basis for $T_p\,\partial M$.

4) If $\mathrm{int}(M)$ is oriented, then ∂M inherits an orientation.

<u>Aside</u>: Alternative Def$^{\underline{n}}$ of Orientation

Orientation $= [\mathcal{A}]$ where $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is orienting.

Note $p \in U_\alpha \cap U_\beta$

$$\left\{ \frac{\partial}{\partial x_\beta^1}\bigg|_p, \dots, \frac{\partial}{\partial x_\beta^n}\bigg|_p \right\} = \left[\left(\phi_\beta \circ \phi_\alpha^{-1} \right)_* \right] \left\{ \frac{\partial}{\partial x_\alpha^1}\bigg|_p, \dots, \frac{\partial}{\partial x_\alpha^n}\bigg|_p \right\}$$

Since $\mathcal{A}$ is orienting, the bases are equivalent

Orientation $\Longrightarrow$ Rule for specifying orientation for each T_pM:

basis $\{V_1, \dots, V_n\}$ of T_pM is positive if

$\{V_1, \dots, V_n\} \sim \{ \partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p \}$ for

any chart.

In fact $\Longleftrightarrow$.

Back to 4).

- Let $\mathring{\mathcal{A}}$ be orienting atlas for $\text{int}(M)$

  Need rule to orient each $T_p(\partial M)$

- Extend $\mathring{\mathcal{A}}$ to atlas $\mathcal{A}$ of M by adding only boundary charts (V, ψ) s.t. $V \cap \partial M \neq \emptyset$.

- <u>Def^n</u> (V, ψ) is positive if $\det\left[\left(\phi_\alpha \circ \psi^{-1} \right)_* \right] > 0$

  for all $(U_\alpha, \phi_\alpha) \in \mathring{\mathcal{A}}$.

- Let $p \in \partial M$ and choose a boundary chart $(V, \psi) \in \mathcal{A}$

  We may assume (V, ψ) is positive.

<u>Rule</u> A basis $\{ V_1, \dots, V_{n-1} \}$ of $T_p(\partial M)$ is positive

if $\left\{ -\frac{\partial}{\partial x^n}\Big|_p , V_1, \dots, V_{n-1} \right\}$ is equivalent to

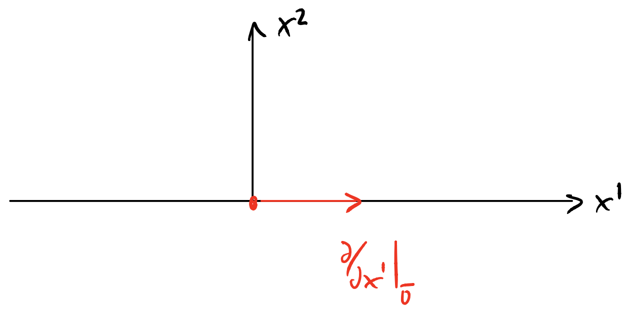
$\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$. as a basis for $T_p M$.

<u>Example</u> $M = \mathbb{R}^2_+$ $\quad \partial M = \partial \mathbb{R}^2_+$

$\text{int}(M) = \mathbb{H}$ is oriented by $\mathring{\mathcal{A}} = \{ (\mathbb{H}, Id) \}$

<u>Q</u> Is $\partial/\partial x^1 \big|_{\bar{0}}$ a positive basis of $T_{\bar{0}}(\partial \mathbb{R}^2_+)$

w.r.t inherited orientation on $\partial \mathbb{R}^2_+$.



- Add to $\mathring{\mathcal{A}}$ the boundary chart $(\mathbb{R}^2_+, Id)$ to get

  $\mathcal{A} = \{ (\mathbb{H}, Id), (\mathbb{R}^2_+, Id) \}$

- $(\mathbb{R}^2_+, \mathbb{I}d)$ is a positive boundary chart, $\det(I_n) = 1 > 0$.

- $\left\{ -\frac{\partial}{\partial x^2}\Big|_{\bar{0}}, \frac{\partial}{\partial x^1}\Big|_{\bar{0}} \right\} \sim \left\{ \frac{\partial}{\partial x^1}\Big|_{\bar{0}}, \frac{\partial}{\partial x^2}\Big|_{\bar{0}} \right\}$

since $\det\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 > 0$

so $\frac{\partial}{\partial x^1}\Big|_{\bar{0}}$ is positive

<u>FACT</u> $\left\{ \frac{\partial}{\partial x^1}\Big|_{\bar{0}}, \dots, \frac{\partial}{\partial x^{n-1}}\Big|_{\bar{0}} \right\}$ is positive basis of

$T_{\bar{0}}(\partial\mathbb{R}^n_+)$ w.r.t inherited orientation on $\partial\mathbb{R}^n_+$

iff n is even.