M oriented (by $[\underbrace{a}_{\text{orienting}}]$) and compact.

$\alpha \in \Lambda^n(M)$

- $\{V_i\}_{i=1}^N$ open cover s.t. $V_i \subset \gamma_i([0,1]^n)$ positively oriented
- $\{f_i\}_{i=1}^N$ partition of unity subordinate to $\{V_i\}$.

$$\int_M \alpha = \sum_{i=1}^N \int_{[0,1]^n} \gamma_i^*(f_i \alpha)$$

- Independent of choice of $\{V_i\}$, $\{\gamma_i\}$ and $\{f_i\}$
- If $M \underset{\text{open}}{\subset} [0,1]^n$, then $\{V_i\} = \{M\}$, $\{f_i\} = \{\mathbb{1}_M\}$

and $\displaystyle\int_M \alpha = \int_{[0,1]^n} f(x)\, dx^1 \wedge \dots \wedge dx^n = \int_0^1 \cdots \int_0^1 f(x)\, dx^1 \dots dx^n.$

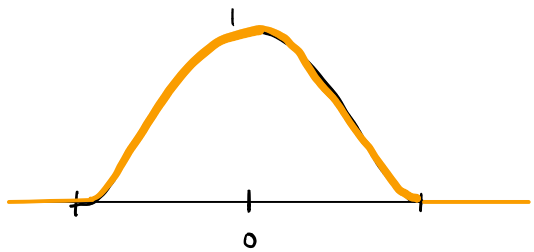
<u>Problem</u> This definition is difficult to use in practice.

<u>Reason 1</u> The $f_i : M \to \mathbb{R}$ are complicated to integrate.

<u>Ex</u> A basic model of a function used to construct partitions of unity is

$$f : \mathbb{R} \longrightarrow [0,1)$$

$$x \longmapsto \begin{cases} e^{\frac{1}{x^2-1}+1} & x \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$$



<u>Exercise</u> check this is smooth at $x = \pm 1$.

We can avoid the $\{f_i\}$ if we can avoid overlaps among the $\gamma_i([0,1]^n)$.

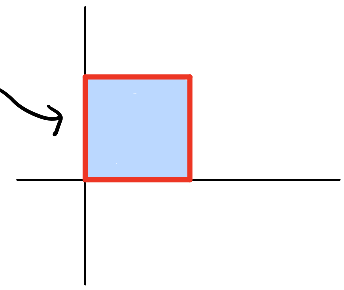
<u>Fact</u> Suppose M is compact and oriented. Suppose that $\gamma_1, \dots, \gamma_N : [0,1]^n \to M$ are 1-1 on $(0,1)^n$, positively oriented and satisfy

- $\displaystyle\bigcup_{i=1}^{N} \gamma_i([0,1])^n = M$

<u>AND</u>

- $p \in \gamma_i([0,1]^n) \cap \gamma_j([0,1]^n)$

$$\Longleftrightarrow \quad \gamma_i^{-1}(p),\ \gamma_j^{-1}(p) \in \partial([0,1]^n) = \left\{ x \in [0,1]^n \;\middle|\; x_k = 0 \text{ or } 1 \text{ for some } k \right\}$$

<u>ex</u> $\partial([0,1]^2)$ 

THEN

$$\int_M \alpha = \sum_{i=1}^{N} \int_{[0,1]^n} \gamma_i^* \alpha. \qquad (\text{no } f_j\text{'s or } V_j\text{'s})$$

Since only the contributions of $\gamma_i(\partial[0,1]^n))$ are redundant, we just need to show they are all zero.

$$\int_{[0,1]^n} \gamma_i^* \alpha = \int_0^1 \cdots \int_0^1 f_i(x)\, dx^1 \ldots dx^n$$

The contribution of the part of the boundary $x_j = 0$ is

$$\int_0^1 \cdots \underbrace{\int_0^0 \cdots \int_0^1 f_i(x)\, dx^1 \ldots dx^j}_{\int_0^0 \tilde{f}(x^j)\, dx^j = 0} \ldots dx^n$$

Example $M = S^1$

$$\gamma_1 : [0,1] \longrightarrow S^1 \subset \mathbb{R}^2$$
$$t \longmapsto (\cos 2\pi t, \sin \pi t)$$

- 1-1 on $(0,1)$
- positively oriented for one orientation

  replace by $\tilde{\gamma}_1(t) = (\cos 2\pi t, -\sin \pi t)$ if need be.
- $\gamma_1([0,1]) = S^1$
- $\gamma_1([0,1]) \cap \gamma_1([0,1]) = \{\gamma_1(0)\}$

$$\gamma_1^{-1}(\gamma_1(0)) = 0 \in \partial([0,1]) \checkmark$$

We can now compute $\int_{S^1} \alpha$ for any $\alpha \in \Lambda^1(S^1)$

and choice of orientation!

ex $\quad i : S^1 \longrightarrow \mathbb{R}^2$ inclusion

$$\alpha = i^* (x^1 dx^1 + 2x^1 dx^2)$$

Suppose S^1 is oriented so that γ_1 is positively oriented.

$$\int_{S^1} \alpha = \int_0^1 \gamma_1^* \, i^* (x^1 dx^1 + 2x^1 dx^2) \, dt$$

$$= \int_0^1 (i \circ \gamma_1)^* (x^1 dx^1 + 2x^1 dx^2) \, dt.$$

$$= \int_0^1 \left(\cos 2\pi t \; d(\cos 2\pi t) + 2\cos 2\pi t \; d(\sin 2\pi t) \right) dt$$

$$= \int_0^1 \left(2\pi \cos 2\pi t \sin 2\pi t + 4\pi \cos^2 2\pi t \right) dt$$

$$= 2\pi \int_0^1 (1 + \cos 4\pi t) \, dt$$

$$= 2\pi$$

Example

$$\gamma_1 : [0,1]^2 \longrightarrow S^2 \subset \mathbb{R}^3$$

$$(s,t) \longmapsto (\cos 2\pi t \sin \pi s, \sin 2\pi t \sin \pi s, \cos \pi s)$$

Allows one to compute $\int_{S^2} \alpha$ for any $\alpha \in \Lambda^2(S^2)$

Now we have derivatives $(F_*, d\omega, \dots)$ and integrals $(\int_\gamma \alpha, \int_M \alpha \dots)$ and so we need a version of the Fundamental Thm of Calculus

$$\int_a^b \frac{df}{dx}(x)\, dx = f(b) - f(a)$$

$\uparrow$

determined by f restricted to $\partial([a,b]) = \{a, b\}$

Stokes' Thm

$$\int_M d\omega = \int_{\partial M} \omega$$

- M is compact oriented manifold of dimension n
- $d\omega \in \Lambda^n(M)$ so $\omega \in \Lambda^{n-1}(M)$.
- ∂M is a manifold of M of dimension $n-1$. $\partial M \subset M$
- ∂M is orientable and inherits an orientation from the one on M.

The missing definition is that of a smooth manifold M with boundary ∂M.