Integrating differential forms

<u>Step 0</u> $\quad U = [0,1]^n \subset \mathbb{R}^n$

$$\alpha = f(x)\, dx^1 \wedge \dots \wedge dx^n$$

$$\int_U \alpha = \int_0^1 \int_0^1 \cdots \int_0^1 f(x^1, \dots, x^n)\, dx^1 \dots dx^n .$$

<u>Step 1</u> $\quad \gamma : [0,1]^k \longrightarrow M \quad , \quad \alpha \in \Lambda^k(M)$

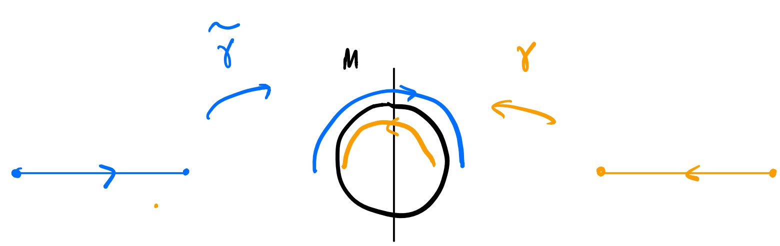
$$\int_\gamma \alpha = \int_{[0,1]^k} \gamma^* \alpha$$

Given $\gamma : [0,1]^n \longrightarrow M$ 1-1 and $\alpha \in \Lambda^n(M)$ one would hope that

$$\int_{\gamma([0,1]^n)} \alpha = \int_\gamma \alpha$$

is a good definition

It is not. There is a sign problem.



$$\int_{\bar{\gamma}} \alpha = - \int_{\gamma} \alpha$$

Orientation

- Two bases α, β of V are equivalent if $\det [Id]_{\alpha}^{\beta} > 0$.
- A map $T : (V, [\alpha]) \longrightarrow (W, [\beta])$ is orientation preserving if $\det [T]_{\alpha}^{\beta} > 0$.
- Two charts (U, ϕ) and (V, ψ) of M are orientation compatible if

$$\det \left[(\psi \circ \phi^{-1})_* \right] > 0 \qquad (\text{for all } x \in \phi(U \cap V))$$

Ex $\quad M = S^1 \qquad (U_1^+, \phi_1^+) \qquad (U_2^+, \phi_2^+)$

$$\phi_2^+ \circ (\phi_1^+)^{-1} (x^1) = \phi_2^+ \left(\sqrt{1-(x^1)^2}, x^1 \right)$$

$$= \sqrt{1-(x^1)^2}$$

$$\left[(\phi_2^+ \circ (\phi_1^+)^{-1})_* \right] = \frac{\partial}{\partial x^1} \left(\sqrt{1-(x^1)^2} \right)$$

$$= \frac{-x^1}{\sqrt{1-(x^1)^2}} < 0$$

$\forall\, x^1 \in \phi_1^+(U_+^1 \cap U_+^2) = (0,1)$

So not orientation compatible.

<u>Ex</u> Change ϕ_2^+ to $\tilde{\phi}_2^+(x^1,x^2) = -x^1$.

Then (U_1^+, ϕ_1^+) and $(U_2^+, \hat{\phi}_2^+)$ are orientation compatible.

<u>Defn</u> M is <u>orientable</u> if it admits an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$ such that any two charts in $\mathcal{A}$ are orientation compatible.

Such an atlas is said to be orienting.

<u>Rmk</u> Not all manifolds are orientable!

ex $\mathbb{RP}^2$ is not orientable

<u>Defn</u> Let $\mathcal{A}$ and $\mathcal{B}$ be orienting atlases for M.

$\mathcal{A} \sim \mathcal{B}$ if every pair of charts $(U,\phi) \in \mathcal{A}$ and $(V,\psi) \in \mathcal{B}$ are orientation compatible.

<u>Fact</u> If M is orientable it has exactly two equivalence classes of orienting atlases.

<u>Def^{n}</u> An orientation on an orientable manifold is a choice of one of these equivalence classes.

An oriented manifold is a pair $(M, [\mathcal{A}])$.

<u>Ex</u> $M = \mathbb{R}^n$

- $\mathcal{A} = \{(\mathbb{R}^n, Id_{\mathbb{R}^n})\}$ is an orienting atlas

  $\mathcal{B} = \{(\mathbb{R}^n, -Id_{\mathbb{R}^n})\}$ is an orienting atlas

  $\mathcal{A} \sim \mathcal{B} \iff n$ is even

i.e $\det\left[(-Id_{\mathbb{R}^n}) \cdot (Id_{\mathbb{R}^n})^{-1}_{*}\right] = \det(-\mathbb{I}_n) = (-1)^n$.

<u>Def^{n}</u> A chart (U, ϕ) of $(M, [\mathcal{A}])$ is positively oriented if it belongs to an oriented atlas in the chosen equivalence class.

Def$^{\underline{n}}$ $\gamma : [0,1]^n \to (M, [\alpha])$ is positively oriented if $\det[(\phi \circ \gamma)_*] > 0$ for any positively oriented chart on (U, ϕ) on M (and any $x \in [0,1]^n$)

Step 2

Let M be oriented. Suppose $\gamma : [0,1]^n \to M$ is 1-1 and positively oriented. Then for $U = \gamma([0,1]^n)$ we define

$$\int_U \alpha = \int_\gamma \alpha \qquad \text{for any } \alpha \in \Lambda^n(M).$$

Rmk $\int_\gamma \alpha$ makes sense for $\gamma : [0,1]^2 \to \mathbb{R}P^2$

$\int_{\gamma([0,1]^2)} \alpha$ does not.

Assume M is oriented and $\alpha \in \Lambda^n(M)$

We want to define $\int_M \alpha$ by dividing M up into pieces as in Step 2

Assume M is compact.

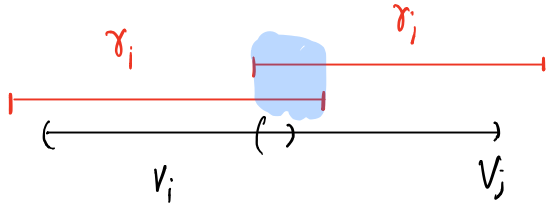
Then M admits a finite open cover $\{V_i\}_{i=1}^{N}$.

- each $V_i \subset M$ is open.
- $\bigcup_{i=1}^{N} V_i = M$.

<u>Prop</u> M (compact) admits a finite open cover such that $V_i \subset \gamma_i([0,1]^n)$ for some $\gamma_i : [0,1]^n \to M$ 1-1 and orientation preserving.

What about $$\int_M \alpha = \sum_{i=1}^{N} \int_{\gamma_i([0,1]^n)} \alpha \quad ?$$

Problem: overlapping contributions



<u>Def^{n}</u> A partition of unity subordinate to $\{V_i\}_{i=1}^{N}$ is

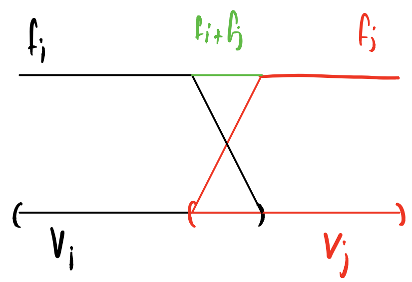
a collection of smooth functions $\{f_i : M \to \mathbb{R}\}_{i=1}^{N}$

s.t. i) $f_i(p) \geq 0 \quad \forall\ i, p$

ii) $f_i(p) = 0 \quad \forall\ p \notin V_i$

iii) $\sum_{i=1}^{N} f_i(p) = 1 \quad \forall\ p$

i.e.



Thm There is a partition of unity subordinate to any finite (in fact countable) open cover of M.

Given M oriented and $\alpha \in \Lambda^n(M)$

- choose $\{V_i\}_{i=1}^{N}$ such that $V_i \subset \phi_i([0,1]^n)$ as in Prop.
- choose a partition of unity subordinate to $\{V_i\}$ as in Thm.

Def^n $\displaystyle\int_M \alpha = \int_M 1\,\alpha$

$$= \int_M \left(\sum_{i=1}^N f_i\right)\alpha$$

$$= \sum_{i=1}^N \int_M f_i\,\alpha$$

$$= \sum_{i=1}^N \int_{\gamma_i([0,1])^n} f_i\,\alpha \qquad \text{since } f_i = 0 \text{ outside } V_i \subset \gamma_i([0,1])^n.$$

$$= \sum_{i=1}^N \int_{[0,1]^n} \gamma_i^*(f_i\,\alpha) \qquad \text{by Step 2.}$$