Integrating differential forms

<u>Step 0</u> $U = [0,1]^n \subset \mathbb{R}^n$

$$\alpha = f(x)\, dx^1 \wedge \dots \wedge dx^n$$

$$\int_U \alpha = \int_0^1 \int_0^1 \cdots \int_0^1 f(x^1, \dots, x^n)\, dx^1 \dots dx^n .$$

A) Pullback. $F : M \longrightarrow N$

$$\alpha \in \Lambda^k(N)$$

$$\Lambda^k(M) \ni (F^*\alpha)_{(p)}(V_1, \dots, V_k) = \alpha(F(p))(F_* V_1, \dots, F_* V_k)$$

<u>Ex</u> $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x^1, x^2) \longmapsto (x^1 - x^2,\ x^2 - x^1,\ (x^1)^2)$$

$$\alpha(y) = \sqrt{y^1}\, dy^1 \wedge dy^3 \in \Lambda^2(\mathbb{R}^3)$$

$$(F^*\alpha)(x) = \sqrt{x^1 - x^2}\, d(x^1 - x^2) \wedge d((x^1)^3)$$

$$= \sqrt{x^1 - x^2}\, \left[(dx^1 - dx^2) \wedge (3(x^1)^2 dx^1 \right]$$

$$= -\sqrt{x^1 - x^2}\ 3(x^1)^2\, dx^2 \wedge dx^1$$

$$= \sqrt{x^1 - x^2}\; 3(x^1)^2 \; dx^1 \wedge dx^2 .$$

<u>Ex</u> $\gamma : (a,b) \longrightarrow \mathbb{R}^n$

$t \longmapsto (\gamma_1(t), \dots, \gamma_n(t))$

$$\alpha(x) = \sum a_i(x)\, dx^i$$

$$(\gamma^*\alpha)(t) = \sum_i a_i(\gamma(t))\; d(\gamma_i(t))$$

$$= \sum_i a_i(\gamma(t))\; \frac{d\gamma_i}{dt}(t)\; dt$$

$$= \left[\sum_i a_i(\gamma(t))\, dx^i\right]\left(\sum \frac{d\gamma_j}{dt}(t) \frac{\partial}{\partial x^j}\right) dt .$$

$$= \alpha(\gamma(t))\,(\dot\gamma(t))\; dt .$$

So

$$\int_\gamma \alpha = \int_a^b \alpha(\gamma(t))(\dot\gamma(t))\, dt = \int_a^b \gamma^*(\alpha)$$

<u>Step 1</u> Given $\omega \in \Lambda^k(M)$ and a smooth map

$\gamma : [0,1]^k \to M$ (singular k-cube)

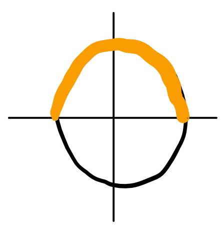
$$\int_\gamma \omega = \int_{[0,1]^k} \gamma^*\omega \quad \leftarrow \text{ as defined in Step 0.}$$

Now suppose $U \subset M$ and $U = \gamma([0,1]^n)$ for some $\gamma : [0,1]^n \to M$ which is smooth and 1-1.

Q. Does it make sense to define $\int_U \omega = \int_\gamma \omega$?

A. NO. There is a sign problem.

Example $M = S^1$ $U = \{ x \in S^1 \mid x^2 \geq 0 \}$



$$\gamma_1 : [0,1] \longrightarrow S^1$$
$$t \longmapsto (\cos(\pi t), \sin(\pi t))$$

$$\gamma_2 : [0,1] \longrightarrow S^1$$
$$s \longmapsto (\cos(\pi(1-s)), \sin(\pi(1-s)))$$

$$\gamma_1([0,1]) = U = \gamma_2([0,1])$$

Exercise: For any 1-form α on S^1 $\int_{\gamma_1} \alpha = -\int_{\gamma_2} \alpha$.

We need a way to consistently choose b/w signs.

B) Orientations

Let $\alpha = \{e_1, \dots, e_n\}$ and $\beta = \{f_1, \dots, f_n\}$ be basis for V.

$[Id_V]^\beta_\alpha$ is an invertible matrix. $\left(\left([Id_V]^\beta_\alpha \right)^{-1} = [Id_V]^\alpha_\beta \right)$.

<u>Defn</u> $\alpha \sim \beta$ iff $\det\left([Id_V]^\beta_\alpha \right) > 0$

<u>Ex</u> $V = \mathbb{R}^2$

$$\alpha = \{ (1,0), (0,1) \}$$

$$\beta = \{ (0,1), (1,0) \}$$

$$\gamma = \{ (1,1), (1,-1) \}$$

$$[Id]^\beta_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{so} \quad \alpha \nsim \beta.$$

$$[Id]^\gamma_\alpha = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{so} \quad \alpha \nsim \gamma$$

$$[Id]^\gamma_\beta = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{so} \quad \beta \sim \gamma$$

Fact For any V there are exactly 2 equivalence classes of bases. $\{+,-\}$

Def^n An orientation of V is a choice of one equivalence class of bases $(+)$.

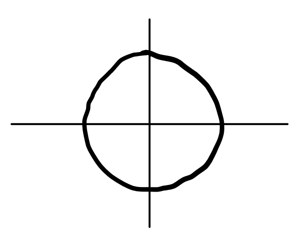
Let (U,ϕ) and (V,ψ) be charts on M.

Def^n (U,ϕ) and (V,ψ) are orientation compatible if either $U \cap V = \emptyset$ or if.

$$\det\left(\left[(\psi \circ \phi^{-1})_* \right]_{\{\partial/\partial x^i|_x\}}^{\{\partial/\partial y^i|_y\}} \right) > 0$$

for all $x \in \phi(U \cap V)$

Ex $M = S^1$ $\quad (U_1^+, \phi_1^+) \quad (U_2^+, \phi_2^+)$



$$\phi_2^+ \circ (\phi_1^+)^{-1}(x^1) = \phi_2^+\left(\sqrt{1-(x^1)^2}, x^1\right)$$

$$= \sqrt{1-(x^1)^2}$$

$$\left[\left(\phi_2^+ \circ (\phi_1^+)^{-1}\right)_*\right] = \frac{\partial}{\partial x^1}\left(\sqrt{1-(x^1)^2}\right)$$

$$= \frac{-x^1}{\sqrt{1-(x^1)^2}} < 0$$

So not orientation compatible.

Ex Change ϕ_2^+ to $\tilde{\phi}_2^+(x^1, x^2) = -x^1$.

Then (U_1^+, ϕ_1^+) and $(U_2^+, \hat{\phi}_2^+)$ are orientation compatible.

Def^n M is orientable if it admits an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ such that any two charts in $\mathcal{A}$ are orientation compatible.

Such an atlas is said to be orienting.

Rmk Not all manifolds are orientable!

ex $\mathbb{RP}^2$ is not orientable

Def$^{\underline{n}}$ Let $\mathcal{A}$ and $\mathcal{B}$ be orienting atlases for M.

$\mathcal{A} \sim \mathcal{B}$ if every pair of charts $(U, \phi) \in \mathcal{A}$

and $(V, \psi) \in \mathcal{B}$ are orientation compatible.

Fact If M is orientable it has exactly two equivalence classes of orienting atlases.

Def$^{\underline{n}}$ An orientation on an orientable manifold is a choice of one of these equivalence classes.

Def$^{\underline{n}}$ M oriented. (U, ϕ) is positively oriented if it belongs to an oriented atlas in the chosen equivalence class.

<u>Ex</u> $M = \mathbb{R}^n$

$\mathcal{A} = \{(\mathbb{R}^n, Id_{\mathbb{R}^n})\}$ is an orienting atlas.

$\mathcal{B} = \{(\mathbb{R}^n, -Id_{\mathbb{R}^n})\}$ is an orienting atlas

$\mathcal{A} \sim \mathcal{B} \iff$ n is even

i.e $\det\left[(-Id_{\mathbb{R}^n}) \cdot (Id_{\mathbb{R}^n})^{-1}{}_{*}\right] = \det(-\mathbb{I}_n) = (-1)^n.$

<u>Def$^{\underline{n}}$</u> Suppose M is oriented. $\gamma : [0,1]^n \to M$ is positively oriented if $\det[(\phi \circ \gamma)_*]$ for any positively oriented chart on (U, ϕ) on M (with $U \cap \gamma([0,1]^n) \neq \emptyset$).