$\Lambda^k(M)$

$\wedge : \Lambda^l(M) \times \Lambda^m(M) \longrightarrow \Lambda^{l+m}(M)$

ex $\quad dx^1 \wedge dx^2 = dx^1 \otimes dx^2 - dx^2 \wedge dx^1$

Thm For $k = 0, 1, \ldots$ there is a unique map

$$d : \Lambda^k(M) \longrightarrow \Lambda^{k+1}(M) \quad \text{s.t.}$$

i) $d(f) = df$

ii) $d(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^l \alpha \wedge d\omega . \qquad (\alpha \in \Lambda^l(M))$

iii) $d \circ d = 0$

$d \equiv$ the exterior derivative on M.

We define d in local coordinates

$$\alpha(x) = \sum a_{t_1 \cdots t_k}(x)\, dx^{t_1} \wedge \cdots \wedge dx^{t_k}.$$

$$(d\alpha)(x) = \sum d(a_{t_1 \cdots t_k}(x)) \wedge dx^{t_1} \wedge \cdots \wedge dx^{t_k}.$$

Ex $\quad \alpha = f(x^1,x^2,x^3)\, dx^1 \wedge dx^2 \in \Lambda^2(\mathbb{R}^3)$

$$d\alpha = df \wedge dx^1 \wedge dx^2$$

$$= \left(\frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \frac{\partial f}{\partial x^3} dx^3 \right) \wedge dx^1 \wedge dx^2$$

$$= \frac{\partial f}{\partial x^1} dx^1 \wedge dx^1 \wedge dx^2 + \frac{\partial f}{\partial x^2} dx^2 \wedge dx^1 \wedge dx^2 + \frac{\partial f}{\partial x^3} dx^3 \wedge dx^1 \wedge dx^2$$

$$= \frac{\partial f}{\partial x^3} dx^1 \wedge dx^2 \wedge dx^3$$

Let's verify properties (i), (ii) and (iii).

(i) $\quad d(f) = df \quad \checkmark$

ii) $\quad \alpha = a(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_\ell}$

$\quad \omega = b(x)\, dx^{j_1} \wedge \dots \wedge dx^{j_m}$

$$\alpha \wedge \omega = a(x)b(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_m}$$

$$d(\alpha \wedge \omega) = d(a(x)b(x)) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_m}$$

$$= (a\, db + b\, da) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_m}$$

$$= \quad da \wedge dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge b\, dx^{j_1} \wedge \dots \wedge dx^{j_m} \quad +$$

$$(-1)^\ell a\, dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge db \wedge dx^{j_1} \wedge \dots \wedge dx^{j_m}$$

$$= \quad d\alpha \wedge \omega + (-1)^\ell \alpha \wedge d\omega$$

(iii) $\quad \alpha = a(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$d\alpha = \sum_{j \notin \{i_1, \dots, i_k\}} \frac{\partial a}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$d(d\alpha) = \sum_{k \notin \{j, i_1, \dots, i_k\}} \frac{\partial^2 a}{\partial x^k \partial x^j} dx^k \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \sum_{k \neq j \notin \{i_1, \dots, i_k\}} \frac{\partial^2 a}{\partial x^k \partial x^j} dx^k \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \sum_{\substack{k<j \\ \notin \{i_1, \dots, i_k\}}} \left(\frac{\partial^2 a}{\partial x^k \partial x^j} - \frac{\partial^2 a}{\partial x^j \partial x^k} \right) dx^k \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$= 0$ since partial order derivatives commute!

Without coordinates we can define $d\alpha$ for $\alpha \in \Lambda^k(M)$ by

$$d\alpha(V_1, \ldots, V_{k+1}) = \sum_i (-1)^{i-1} V_i\left(\alpha(V_1, \ldots, \hat{V}_i, \ldots, V_{k+1})\right)$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha\left([V_i, V_j], V_1, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_{k+1}\right)$$

where $[V_i, V_j]$ is the vector field defined by

$$[V_i, V_j](f) = V_i(V_j(f)) - V_j(V_i(f))$$

<u>Ex</u> If α is a 1-form then

$$d\alpha(V_1, V_2) = V_1\,\alpha(V_2) - V_2\,\alpha(V_1) - \alpha([V_1, V_2])$$

Let's integrate

<u>Step 0</u> $M = \mathbb{R}^n$ $\omega = f(x)\, dx^1 \wedge \ldots \wedge dx^n \in \Lambda^n(\mathbb{R}^n)$

$$\mathbb{R}^n \supset U = [0,1]^n = [0,1] \times [0,1] \times \ldots \times [0,1]$$

Def^{n} $$\int_U \omega = \int_0^1 \cdots \int_0^1 f(x)\, dx^1\, dx^2 \cdots dx^n$$

Need

A) Pullbacks B) Orientations C) Partitions of Unity

A) Given $F: M \longrightarrow N$ and $\alpha \in \Lambda^k(N)$

define the pullback of α by F to

$$(F^*\alpha)(p)(\underset{\substack{\cap \\ T_pM}}{V_1}, \dots, V_k) = \alpha(F(p))(F_* V_1, \dots, F_* V_k)$$

Facts

1) $F^*(f) = f \circ F$

2) $F^*(d\alpha) = d(F^*\alpha)$

3) $F^*(\alpha \wedge \omega) = F^*(\alpha) \wedge F^*(\omega)$

In local coordinates, if $\alpha(y) = f(y)\, dy^{t_1} \wedge \dots \wedge dy^{t_k}$

then

$$(F^*\alpha)(x) = f(F(x))\, d(y^{j_1}(F(x))) \wedge \dots \wedge d(y^{j_k}(F(x)))$$