k-forms

$\Lambda^k(V) \subset \mathcal{T}_{k,0}(V)$ a subspace

$\{ \sigma^{t_1} \wedge \dots \wedge \sigma^{t_k} \mid 1 \le t_1 < t_2 \cdots < t_k \le n \}$ is a basis

$$\wedge : \Lambda^{l}(V) \times \Lambda^{m}(V) \longrightarrow \Lambda^{l+m}(V)$$
$$(\alpha, \omega) \longmapsto \frac{(l+m)!}{l!\, m!} \operatorname{Alt}(\alpha \otimes \omega)$$

Need the projection $\operatorname{Alt} : \mathcal{T}_{k,0}(V) \longrightarrow \Lambda^k(V)$.

$S_k \ni \psi$

There are $k(k-1)\cdots 2 \cdot 1 = k!$ elements in S_k.

- ψ can be written as composition of transpositions

ex

$$\begin{tikzcd}
 & \{2,1,3\} \arrow[dr, "(13)"] & \\
\{1,2,3\} \arrow[ur, "(12)"] \arrow[rr, "\psi"] \arrow[dr, "(13)"'] & & \{3,1,2\} \\
 & \{3,2,1\} \arrow[ur, "(23)"'] &
\end{tikzcd}$$

- # of transpositions used to express a given ψ is always even or odd. $\mathrm{sgn}(\psi) = (-1)^{\#}$ is well-defined

- $\tilde{\psi} \circ S_k = \{ \tilde{\psi} \circ \psi \mid \psi \in S_k \} = S_k$

- S_k "acts" on $\mathcal{T}_{k,0}(V)$ as follows

$$(\psi * W)(V_1, \dots, V_k) = W(V_{\psi(1)}, \dots, V_{\psi(k)})$$

ex $\quad (12)*(\sigma^1 \otimes \sigma^2) = \sigma^2 \otimes \sigma^1$

Note $\quad \tilde{\psi} * (\psi * W) = (\tilde{\psi} \circ \psi) * W$

Now $\quad W$ alternating

$\Leftrightarrow \quad (ij)*W = -W$ for every transposition

$\Leftrightarrow \quad \psi * W = \mathrm{sign}(\psi) W$ for every $\psi \in S_k$.

S_k identifies $\Lambda^k(V) \subset \mathcal{T}_{k,0}(V)$

Define

$$\mathrm{Alt} : \mathcal{T}_{k,0}(V) \longrightarrow \Lambda^k(V)$$

$$W \longmapsto \frac{1}{k!} \sum_{\psi} \mathrm{sgn}(\psi)\, \psi * W$$

Need $\tilde{\varphi}_*(\mathrm{Alt}(W)) = \mathrm{sgn}(\tilde{\varphi})\,\mathrm{Alt}(W)$

$$\tilde{\psi}_*(\mathrm{Alt}(W)) = \tilde{\psi}_*\left(\frac{1}{k!}\sum_{\psi} \mathrm{sgn}(\psi)\,(\psi * W)\right)$$

$$= \frac{1}{k!}\sum_{\psi} \mathrm{sgn}(\psi)\,(\tilde{\psi}\circ\psi * W)$$

$$= \frac{1}{k!}\sum_{\psi} \mathrm{sgn}(\tilde{\psi})\,\mathrm{sgn}(\tilde{\psi})\,\mathrm{sin}(\psi)\,(\tilde{\psi}\circ\psi * W)$$

$$= \mathrm{sgn}(\tilde{\psi})\,\frac{1}{k!}\sum_{\psi} \mathrm{sgn}(\tilde{\psi})\,\mathrm{sgn}(\psi)\,(\tilde{\psi}\circ\psi * W)$$

$$= \mathrm{sgn}(\tilde{\psi})\,\frac{1}{k!}\sum_{\tilde{\psi}\psi} \mathrm{sgn}(\tilde{\psi}\circ\psi)\,(\tilde{\psi}\circ\psi * W)$$

$$= \mathrm{sgn}(\hat{\psi})\,\mathrm{Alt}(W)$$

<u>Example</u> $\mathcal{T}_{2,0}(\mathbb{R}^2) = \mathrm{Span}\{\sigma^1\otimes\sigma^1,\ \sigma^1\otimes\sigma^2,\ \sigma^2\otimes\sigma^1,\ \sigma^2\otimes\sigma^2\}$

$S_2 = \{\mathrm{Id},\ (12)\}$

$\mathrm{sgn}(\mathrm{Id}) = (-1)^0 = 1 \qquad \mathrm{sgn}((12)) = (-1)^1 = 1$

$$\mathrm{Alt}(\sigma^1\otimes\sigma^2) = \frac{1}{2}\left(\mathrm{Id}\cdot\sigma^1\otimes\sigma^2 - (12)\cdot\sigma^1\otimes\sigma^2\right)$$

$$= \frac{1}{2}\left(\sigma^1\otimes\sigma^2 - \sigma^2\otimes\sigma^1\right)$$

Check

$$(12)\circ \mathrm{Alt}(\sigma^1\otimes\sigma^2) = \frac{1}{2}(\sigma^2\otimes\sigma^1 - \sigma^1\otimes\sigma^2) = (-1)\,\mathrm{Alt}(\sigma^1\otimes\sigma^2)$$

✓

Similarly

$$\mathrm{Alt}(\sigma^2\otimes\sigma^1) = \frac{1}{2}(\sigma^2\otimes\sigma^1 - \sigma^1\otimes\sigma^2)$$

$$\mathrm{Alt}(\sigma^1\otimes\sigma^1) = 0 = \mathrm{Alt}(\sigma^2\otimes\sigma^2)$$

In terms of the basis $\{\sigma^i\otimes\sigma^j\}$ for $\mathcal{T}_{2,0}(\mathbb{R}^2)$

$$\left[\mathrm{Alt}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

viewed as map from $\mathcal{T}_{2,0}(\mathbb{R}^2) \longrightarrow \mathcal{T}_{2,0}(\mathbb{R}^2)$.

Back to manifolds

A k-form α on M locally looks like

$$\alpha(x) = \sum a_{t_1\ldots t_k}(x)\; dx^{t_1}\wedge\cdots\wedge dx^{t_k}$$

$$1 \le t_1 < t_2 \cdots < t_k \le n$$

In our new (alternating) basis there are no conditions on the $a_{t_1 \dots t_k}(x)$.

Let $\Lambda^k(M)$ be the space of k-forms on M.

$$\Lambda^0(M) = C^\infty(M) \qquad , \dots , \quad \Lambda^n(M)$$

$$\Lambda^1(M) = \mathcal{T}_{1,0}(M)$$

$$\Lambda^k(M) \ni \alpha(x) = \sum a_{t_1 \dots t_k}\, dx^{t_1} \wedge \dots \wedge dx^{t_k}.$$

$$\Lambda^n(M) \ni \alpha(f) = f(x)\, dx^1 \wedge \dots \wedge dx^n$$

$$\Lambda^{\wedge}(M) \ni 0 \in \mathcal{T}_{k+1}(M)$$

The wedge product extends to $\Lambda^k(M)$

$$\wedge : \Lambda^l(M) \times \Lambda^m(M) \longrightarrow \Lambda^{l+m}(M)$$

$$(\alpha , \omega) \longmapsto \alpha \wedge \omega$$

$$(\alpha \wedge \omega)(p) = \underset{\Lambda^l(T_pM)}{\underset{\cap}{\alpha(p)}} \wedge \underset{\Lambda^m(T_pM)}{\underset{\cap}{\omega(p)}}$$

Ex $\quad f, h : \mathbb{R}^n \longrightarrow \mathbb{R}$

$\quad df, dh \in \Lambda^1(\mathbb{R}^n)$

$$\Lambda^2(\mathbb{R}^n) \ni df \wedge dh = \left(\sum_i \frac{\partial f}{\partial x^i} dx^i \right) \wedge \left(\sum_j \frac{\partial h}{\partial x^j} dx^j \right)$$

$$= \sum_{i,j} \frac{\partial f}{\partial x^i} \frac{\partial h}{\partial x^j} \, dx^i \wedge dx^j$$

$$= \sum_{i \neq j} \frac{\partial f}{\partial x^i} \frac{\partial h}{\partial x^j} \, dx^i \wedge dx^j$$

$$= \sum_{i<j} \frac{\partial f}{\partial x^i} \frac{\partial h}{\partial x^j} \, dx^i \wedge dx^j + \sum_{i>j} \frac{\partial f}{\partial x^i} \frac{\partial h}{\partial x^j} \, dx^i \wedge dx^j$$

$$= \sum_{i \neq j} \left(\frac{\partial f}{\partial x^i} \frac{\partial h}{\partial x^j} - \frac{\partial f}{\partial x^j} \frac{\partial h}{\partial x^i} \right) dx^i \wedge dx^j$$

Note $\quad$ the "map" $d : \Lambda^0(M) \longrightarrow \Lambda^1(M)$

$$f \longmapsto df$$

Thm $\quad$ For $k = 0, 1, \ldots$ there is a unique map

$$d : \Lambda^k(M) \longrightarrow \Lambda^{k+1}(M) \quad \text{s.t.}$$

i) $\quad d(f) = df$

ii) $d(\alpha \wedge \omega) = d\alpha \wedge \omega + (-1)^{\ell} \alpha \wedge d\omega$. $\quad (\alpha \in \Lambda^{\ell}(M))$

iii) $d \circ d = 0$

$d \equiv$ the exterior derivative on M.

We define d in local coordinates

$$\alpha(x) = \sum a_{t_1 \cdots t_k}(x)\, dx^{t_1} \wedge \cdots \wedge dx^{t_k}.$$

$$(d\alpha)(x) = \sum d(a_{t_1 \cdots t_k}(x)) \wedge dx^{t_1} \wedge \cdots \wedge dx^{t_k}.$$

Ex $\quad \alpha = f(x^1, x^2, x^3)\, dx^1 \wedge dx^2 \in \Lambda^2(\mathbb{R}^3)$

$$\begin{aligned}
d\alpha &= df \wedge dx^1 \wedge dx^2 \\
&= \left(\frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \frac{\partial f}{\partial x^3} dx^3 \right) \wedge dx^1 \wedge dx^2 \\
&= \frac{\partial f}{\partial x^1} dx^1 \wedge dx^1 \wedge dx^2 + \frac{\partial f}{\partial x^2} dx^2 \wedge dx^1 \wedge dx^2 + \frac{\partial f}{\partial x^3} dx^3 \wedge dx^1 \wedge dx^2 \\
&= \frac{\partial f}{\partial x^3} dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}$$

Let's verify properties (i), (ii) and (iii).

(i) $\quad d(f) = df \quad \checkmark$

ii) $\quad \alpha = a(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_\ell}$

$\omega = b(x)\, dx^{j_1} \wedge \dots \wedge dx^{j_m}$

$$\alpha \wedge \omega = a(x) b(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_m}$$

$$d(\alpha \wedge \omega) = d(a(x) b(x)) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_m}$$

$$= (a\, db + b\, da) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_m}$$

$$= da \wedge dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge b\, dx^{j_1} \wedge \dots \wedge dx^{j_m} +$$

$$(-1)^\ell a\, dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge db \wedge dx^{j_1} \wedge \dots \wedge dx^{j_m}$$

$$= d\alpha \wedge \omega + (-1)^\ell \alpha \wedge d\omega$$