k-forms

A <u>k-form</u> on M is a $(k,0)$-tensor field on M

$\alpha : M \longrightarrow \mathcal{T}_{k,0}(M)$ which is <u>alternating</u>.

$$\alpha(p)(V_1, \dots, V_i, \dots, V_j, \dots, V_k) = -\alpha(p)(V_1, \dots, V_j, \dots, V_i, \dots, V_k)$$

Lets understand these in local coordinates.

<u>Example</u> let α be a 2-form on a two dimensional manifold M

Like any $(2,0)$-tensor field on M, α locally looks like

$$\alpha(x) = \sum_{1 \leq i,j \leq 2} a_{ij}(x)\, dx^i \otimes dx^j$$

where $a_{ij}(x) = \alpha(x)\left(\partial/\partial x^i, \partial/\partial x^j \right)$

The alternating condition implies $a_{ij}(x) = -a_{ji}(x)$.

So $a_{11} = a_{22} = 0$, $a_{12} = -a_{21}$ and

$$\alpha(x) = a_{12}(x)\left(dx^1 \otimes dx^2 - dx^2 \otimes dx^1 \right)$$

Note $dx^1 \otimes dx^2 - dx^2 \otimes dx^1$ is a local basis for 2-forms

Let's generalize this to k-forms on M^n.

Some more multi-linear algebra.

V w/ basis $\{e_1, \dots, e_n\}$

V^* w/ dual basis $\{\sigma^1, \dots, \sigma^n\}$

$\mathcal{T}_{k,0}(V)$ has basis $\{ \sigma^{t_1} \otimes \dots \otimes \sigma^{t_k} \mid 1 \le t_1, \dots, t_k \le n \}$

Define a product $\otimes : \mathcal{T}_{l,0}(V) \times \mathcal{T}_{m,0}(V) \longrightarrow \mathcal{T}_{l+m,0}(V)$

by $(W_1, W_2) \longmapsto W_1 \otimes W_2$ where

$$W_1 \otimes W_2 (V_1, \dots, V_{l+m}) = W_1(V_1, \dots, V_l)\, W_2(V_{l+1}, \dots, V_{l+m})$$

Then our basis for $\mathcal{T}_{k,0}(V)$ is built from $\{\sigma^1, \dots, \sigma^n\}$ and $\otimes$.

Let $\Lambda^k(V) = \{ W \in \mathcal{T}_{k,0}(V) \mid W \text{ is alternating} \}$

ex $\Lambda^1(V) = \mathcal{T}_{1,0}(V) = V^*$

ex $\det \in \Lambda^n(\mathbb{R}^n)$

ex $W : \underset{x}{\mathbb{R}^n} \times \underset{y}{\mathbb{R}^n} \longrightarrow \det\begin{pmatrix} x^1 & x^2 \\ y^1 & x^2 \end{pmatrix} \in \Lambda^2(\mathbb{R}^n)$

Exercise $\Lambda^k(V)$ is a subspace of $\mathcal{T}_{k,0}(V)$

Ex $\Lambda^2(\mathbb{R}^2)$ is a 1 dimensional subspace of $\mathcal{T}_{2,0}(\mathbb{R}^2)$ which has dimension $n^{r+s} = 4$.

Fact $\otimes$ does not preserve the alternating property

Goal : Build a basis for $\Lambda^k(V)$ from $\{\sigma^1, \dots, \sigma^n\}$ and a new (wedge) product , $\wedge$.

Guess $\sigma^1 \wedge \sigma^2 = \sigma^1 \otimes \sigma^2 - \sigma^2 \otimes \sigma^1$.

Thm There is a linear map $\mathrm{Alt} : \mathcal{T}_{k,0}(V) \longrightarrow \Lambda^k(V)$ such that

i) $\mathrm{Alt}(W) = W \iff W \in \Lambda^k(W)$ (projection)

ii) $\mathrm{Alt}(W_1 \otimes W_2 \otimes W_3) = \mathrm{Alt}(W_1 \otimes \mathrm{Alt}(W_2 \otimes W_3))$

ex $\mathrm{Alt}(\sigma^i \otimes \sigma^j) = \frac{1}{2}(\sigma^i \otimes \sigma^j - \sigma^j \otimes \sigma^i)$

We will define Alt later. Lets first use it to define

$$\wedge : \Lambda^l(V) \times \Lambda^m(V) \longrightarrow \Lambda^{l+m}(V)$$

$$(\alpha, \omega) \longmapsto \alpha \wedge \omega = \frac{(l+m)!}{l!\,m!} \mathrm{Alt}(\alpha \otimes \omega)$$

ex $\sigma^i \wedge \sigma^j = \frac{2!}{1!\,1!}\left(\frac{1}{2}(\sigma^i \otimes \sigma^j - \sigma^j \otimes \sigma^i)\right)$

$= \sigma^i \otimes \sigma^j - \sigma^j \otimes \sigma^i$

Facts 1) $(\alpha_1 + c\,\alpha_2) \wedge \omega = \alpha_1 \wedge \omega + c(\alpha_1 \wedge \omega)$

2) $\alpha \wedge \omega = (-1)^{lm}\, \omega \wedge \alpha$

3) $(\alpha \wedge \omega) \wedge \eta = \alpha \wedge (\omega \wedge \eta)$

Cor 1 If $\alpha \in \Lambda^k(V)$ for k-odd then

$\alpha \wedge \alpha = -\alpha \wedge \alpha = 0$

Cor 2 If $\alpha \in \Lambda^{2k}(V)$ then

$$\alpha \wedge \omega = \omega \wedge \alpha$$

These features are special to $\Lambda^k(V) \subset \mathcal{T}_{k,0}(V)$

$W_1 \otimes W_1 \neq 0$, if $W_1 \neq 0$

$\sigma^i \otimes \sigma^j \neq \sigma^j \otimes \sigma^i$ if $i \neq j$

<u>Fact</u> $\left\{ \sigma^{i_1} \wedge \sigma^{i_2} \wedge \dots \wedge \sigma^{i_k} \;\middle|\; 1 \leq i_1 < i_2 < \dots < i_k \leq n \right\}$

is a basis of $\Lambda^k(V)$.

No repealed indices by Cor 1. Can order indices by using Fact 2)

<u>Rmk</u> For $k > n$, $\Lambda^k(V)$ has dimension 0.

<u>Rmk</u> For $k \leq n$, $\Lambda^k(V)$ has dimension $\binom{n}{k} = \frac{n!}{(n-k)!\, k!}$

<u>Rmk</u> $\Lambda^n(V)$ has dimension 1. It is spanned by

$\sigma^1 \wedge \sigma^2 \wedge \dots \wedge \sigma^n$.

For $V = \mathbb{R}^n$ $\quad \sigma^1 \wedge \sigma^2 \wedge \dots \wedge \sigma^n (x_1, x_2, \dots, x_n)$

$$= \det \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \text{volume of parallelopiped spanned by } x_1, \dots, x_n$$

Now we need to define $\mathrm{Alt} : \mathcal{T}_k(V) \longrightarrow \Lambda_k(V)$

S_k = the symmetric group on k letters $\{1, 2, \dots, k\}$.

$\psi \in$

$\psi(\{1, 2, \dots, k\}) = \{\psi(1), \dots, \psi(k)\}$

ex $\quad \{1, 2, 3, \dots, k\} \overset{\psi = (12)}{\longmapsto} \{2, 1, 3, \dots, k\}$

(i, j) is called a transposition.

fact Each $\psi \in S_k$ can be expressed as a composition of transpositions

ex

$$\begin{tikzcd}
\{1,2,3\} \arrow[rr] \arrow[dr, "(13)"'] & & \{3,1,2\} \\
& \{3,2,1\} \arrow[ur, "(23)"'] &
\end{tikzcd}$$

fact The number of transpositions in the expressions for a fixed ψ all have the same parity.

$\mathrm{sign}(\psi) = (-1)^r$ is well-defined.

- S_k acts on $\mathcal{T}_{k,0}(V)$ as follows

$$(\psi \cdot W)(v_1, \dots, v_k) = W(v_{\psi(1)}, \dots, v_{\psi(k)})$$

ex $\quad (12)(\sigma^1 \otimes \sigma^2) = \sigma^2 \otimes \sigma^1$

- $\tilde{\psi}(\psi \cdot W) = (\tilde{\psi}\psi) \cdot W$

Now W alternating

$\Leftrightarrow \quad (ij) \cdot W = -W$ for every transposition

$\Leftrightarrow \quad \psi \cdot W = \text{sign}(\psi) W$ for every $\psi \in S_k$.

S_k identifies $\Lambda^k(V) \subset \mathcal{T}_{k,0}(V)$

Define

$$\begin{aligned} \text{Alt} : \mathcal{T}_{k,0}(V) &\longrightarrow \Lambda^k(V) \\ W &\longmapsto \frac{1}{k!} \sum_{\psi} \text{sgn}(\psi)\,(\psi \cdot W) \end{aligned}$$

<u>Example</u> $\quad \mathcal{T}_{2,0}(\mathbb{R}^2) = \text{Span}\{\sigma^1 \otimes \sigma^1, \sigma^1 \otimes \sigma^2, \sigma^2 \otimes \sigma^1, \sigma^2 \otimes \sigma^2\}$

$S_2 = \{\text{Id}, (12)\}$

$\text{sgn}(\text{Id}) = (-1)^0 = 1 \qquad \text{sgn}((12)) = (-1)^1 = 1$

$$\mathrm{Alt}(\sigma^1 \otimes \sigma^2) = \frac{1}{2}\left(\mathrm{Id} \cdot \sigma^1 \otimes \sigma^2 - (12) \cdot \sigma^1 \otimes \sigma^2 \right)$$

$$= \frac{1}{2}\left(\sigma^1 \otimes \sigma^2 - \sigma^2 \otimes \sigma^1 \right)$$

$$(12) \cdot \mathrm{Alt}(\sigma^1 \otimes \sigma^2) = \frac{1}{2}\left(\sigma^2 \otimes \sigma^1 - \sigma^1 \otimes \sigma^2 \right) = (-1)\, \mathrm{Alt}(\sigma^1 \otimes \sigma^2)$$

$$\mathrm{Alt}(\sigma^2 \otimes \sigma^1) = \frac{1}{2}\left(\sigma^2 \otimes \sigma^1 - \sigma^1 \otimes \sigma^2 \right)$$

$$\mathrm{Alt}(\sigma^1 \otimes \sigma^1) = 0 = \mathrm{Alt}(\sigma^2 \otimes \sigma^2)$$

In terms of the basis above

$$[\mathrm{Alt}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$