Riemannian metrics as measuring tools.

Let g be a Riemannian metric on M

$$\| \ \|_g : T_pM \longrightarrow \mathbb{R}$$
$$V \longmapsto \sqrt{g(p)(V,V)}$$

Let $\gamma : [a,b] \longrightarrow M$ be a smooth curve.

$$L_g(\gamma) = \int_a^b \| \dot{\gamma}(t) \|_g \, dt.$$

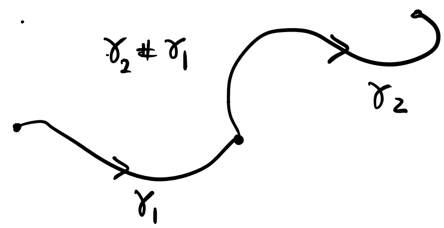
<u>Def^{n}</u> γ is <u>regular</u> if $\dot{\gamma}(t) \neq \bar{0} \in T_{\gamma(t)}M$
for at $t \in [a,b]$.

<u>Fact</u> If two regular curves $\gamma, \tilde{\gamma}$ have the same image then $L_g(\gamma) = L_g(\tilde{\gamma})$.

<u>Def^{n}</u> $\gamma : [a,b] \longrightarrow M$ is <u>piecewise smooth</u> if it is continuous and there are numbers $a < t_1 < t_2 < \dots < t_k < b$ such that γ is smooth on the domains $(a,t_1), (t_1,t_2), \dots, (t_k, b)$.

<u>ex</u> If $\gamma_1, \gamma_2 : [0,1] \longrightarrow M$ are smooth and $\gamma_1(1) = \gamma_2(0)$ then their concatination is the piecewise smooth curve

$$\gamma_2 \# \gamma_1 = \begin{cases} \gamma_1(2t) & t \in [0, 1/2] \\ \gamma_2(2t-1) & t \in [1/2, 1] \end{cases}$$



Exercise $L_g(\gamma)$ makes sense for piecewise smooth curves.

Exercise $L_g(\gamma_2 \# \gamma_1) = L_g(\gamma_1) + L_g(\gamma_2)$.

Def^n The distance between $p, q \in M$ (w.r.t. g) is

$$d_g(p,q) = \underset{\substack{\gamma : [0,1] \to M \\ \gamma \text{ piecewise smooth} \\ \gamma(0) = p,\ \gamma(1) = q}}{\text{infimum}} \{ L_g(\gamma) \}$$

Could restrict to regular curves, same number.

Recall

infimum $\{S\}$ = greatest lower bound of S

ex infimum $\{ \frac{1}{n} \mid n \in \mathbb{N} \} = 0 \notin \{ \frac{1}{n} \mid n \in \mathbb{N} \}$.

facts: $S_1 \subset S_2 \Rightarrow$ infimum $S_1 \geq$ infimum S_2

infimum$(S_1 + S_2)$ = infimum S_1 + infimum S_2.

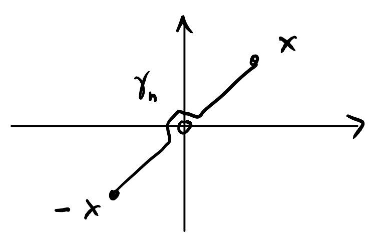
<u>Example 0</u> $M = \mathbb{R}^n$ $g = \sum_{i=1}^{n} dx^i \otimes dx^i$

$d_g(x,y) = \|y-x\|. = L_g(t \mapsto x + (1-t)y)$

<u>Example 1</u> $M = \mathbb{R}^n \setminus \{\bar{0}\}$ $g = \sum_{i=1}^{n} dx^i \otimes dx^i$

$d_g(x,-x) = \|x+x\| = 2\|x\|$

$\neq L_g(\gamma)$ for any γ in M from x to $-x$



<u>Thm</u> 1) $d_g(p,q) = d_g(q,p)$

2) $d_g(p,r) \leq d_g(p,q) + d_g(q,r)$

3) $d_g(p,q) > 0$ if $p \neq q$.

<u>Pf</u>

1)

Observation 1 :

$$\begin{array}{ccc} \gamma : [0,1] \longrightarrow M & \xrightarrow{\gamma(t) \mapsto \gamma(1-t)} & \eta : [0,1] \longrightarrow M \\ \gamma(0) = p \quad , \gamma(1) = q & \xleftarrow[\eta(1-t) \leftarrow \eta(t)]{} & \eta(0) = q \quad , \eta(1) = p \end{array}$$

Observation 2 : $L_g(\gamma(t)) = L_g(\gamma(1-t))$

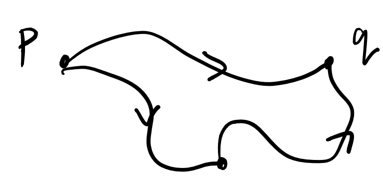
$$d_g(p,q) = \inf_{\substack{\gamma \\ p \rightsquigarrow q}} \{ L_g(\gamma) \}$$

$$= \inf_{\substack{\eta \\ q \rightsquigarrow p}} \{ L_g(\eta) \} \quad \text{(since same set of numbers)}$$

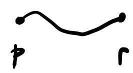
$$= d_g(q,p)$$

2)

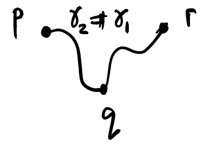
Observation : The set of paths from p to r contains the set of concatenated paths from p to r with matching ends at q.



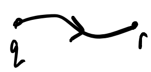
$$d_g(p,r) = \inf_{\gamma} \{ L_g(\gamma) \}$$



$$\leq \inf_{\gamma_2 \# \gamma_1} \{ L_g(\gamma_2 \# \gamma_1) \}$$



$$= \inf_{\gamma_1, \gamma_2} \{ L_g(\gamma_1) + L_g(\gamma_2) \}$$



$$= \inf_{\gamma_1} \{ L_g(\gamma_1) \} + \inf_{\gamma_2} \{ L_g(\gamma_2) \}$$

$$= d_g(p,q) + d_g(q,r)$$

3) Nondegeneracy is a bit more subtle, but crucial

Idea if $p, q \in U$ for a chart (U, ϕ), then

$$d_g(p,q) \geq C \, \| \phi(p) - \phi(q) \| \quad \text{for some } C(U,\phi).$$

k-forms.

Def^n A 0-form on M is a function $f \in C^\infty(M)$.

Def^n A 1-form on M is a covector field $\alpha : M \to T^*M$ $(\pi \circ \alpha = Id_M)$.

Def^n For $k \geq 2$ a k-form on M is a $(k,0)$-tensor field

$$\alpha : M \longrightarrow \mathcal{T}_{k,0}(M)$$

$$p \longmapsto (p, \alpha(p))$$

such that for all $p \in M$, all $V_1, \dots, V_k \in T_pM$ and all $i < j$ we have

$$\alpha(p)(V_1, \dots, V_i, \dots, V_j \dots V_k) = -\alpha(p)(V_1, \dots, V_j, \dots, V_i, \dots, V_k).$$

Such a tensor is said to be alternating.

- k-forms are the correct objects to integrate over k-dimensional domains
- the set of k-forms, over all k, has a rich algebraic structure which captures alot of topology.

- to understand k-forms we must understand what they look like in local coordinates.