$$\mathcal{T}_{r,s}(M) = \{ (p, W) \mid W \in \mathcal{T}_{r,s}(T_pM) \}$$

A (r,s)-tensor field on M is a map $T: M \to \mathcal{T}_{r,s}(M)$ such that $\pi \circ T = Id_M$.

$$T(p) = (p, \underbrace{W(p)}_{\in \mathcal{T}_{r,s}(T_pM)})$$

$$W(x) = \sum_{\substack{1 \le k_1, \ldots, k_s \le n \\ 1 \le t_1, \ldots, t_r \le m}} W^{k_1 \ldots k_s}_{t_1 \ldots t_r}(x) \, \partial/\partial x^{k_1} \otimes \cdots \otimes \partial/\partial x^{k_s} \otimes dx^{t_1} \otimes \cdots \otimes dx^{t_r}$$

where $x = \phi(p)$

Can reduce this to $W^{k_1 \cdots k_s}_{t_1 \cdots t_r}$

r times covariant, s times contravariant tensor field.

<u>Def^n</u> A Riemannian metric is a $(2,0)$-tensor field g which is symmetric and nondegenerate.

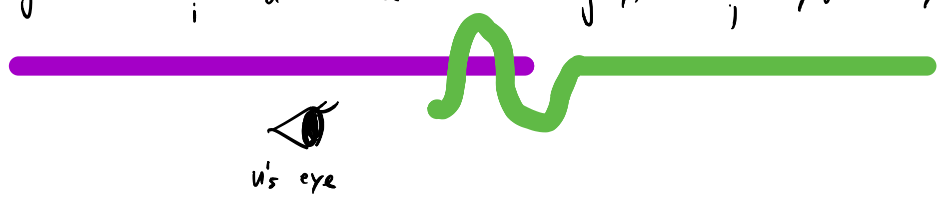
i.e. $(g_{ij}(x)) = (g_{ij}(x))^T$ and eigenvalues of $(g_{ij}(x))$ are positive.

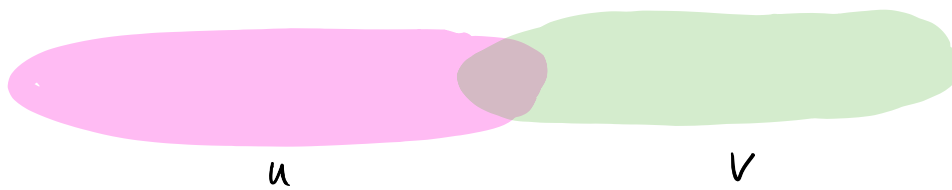
<u>Thm</u> Every manifold M admits a Riemannian metric

(and hence many others).

Idea: Given chart (U,ϕ) and (V,ψ) with $U \cap V \neq \emptyset$.

$$g^U(x) = \sum_i dx_U^i \otimes dx_U^i \qquad g^V(y) = \sum_j dy_V^j \otimes dy_V^j$$





We need to patch these together nicely.

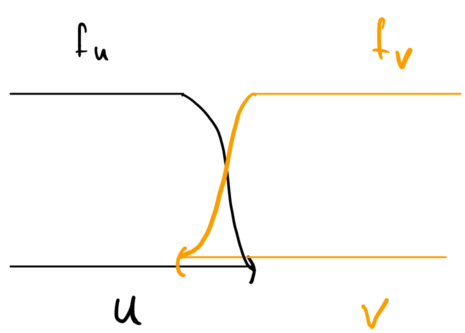
<u>Prop</u> There exist smooth functions $f_U, f_V : U \cup V \to \mathbb{R}$

s.t. i) $f_U(p) \geq 0$ and $f_U(p) = 0$ if $p \notin U$

ii) $f_V(p) \geq 0$ and $f_V(p) = 0$ if $p \notin V$

iii) $(f_U + f_V)(p) = 1 \quad \forall p \in U \cup V$.

i.e.



<u>Note</u> $f_u g^u + f_v g^v$ is a metric on $U \cup V$.

$$f_u(\phi(p))\, g^u(\phi(p)) + f_v(\psi(p))\, g^v(\psi(p))$$

By (i), (ii), (iii) f_u and f_v never vanish simultaneously.

Continuing in this way we get a g on $M = \bigcup_{\alpha \in A} U_\alpha$.

(Use countable base assumption for M here.)

Let g be a metric on M. Let's make use of it.

<u>Use 1</u> Translation

For every $p \in M$ g defines an isomorphism

$$I_g : T_pM \longrightarrow T_p^*M$$

$$V \longmapsto g(p)(V, \bullet) : T_pM \longrightarrow \mathbb{R}$$

$$W \longmapsto g(p)(V, W)$$

To verify that this is an isomorphism let's compute its matrix representative.

$$g(p)\left(\partial/\partial x^i, \bullet\right) = \sum_{j=1}^{n} a_{ji}\, dx^j$$

$$\Rightarrow g(p)\left(\partial/\partial x^i, \partial/\partial x^k\right) = \sum_{j=1}^{n} a_{ji}\, dx^j\left(\partial/\partial x^k\right)$$

$$\Rightarrow \quad g_{ik}(x) = a_{ki}$$

So
$$\left[\mathcal{I}_g \right]_{\{\frac{\partial}{\partial x^i}\}}^{\{dx^i\}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{pmatrix} = \begin{pmatrix} g_{11} & \cdots & g_{n1} \\ g_{12} & & \vdots \\ \vdots & & \vdots \\ g_{1n} & & g_{nn} \end{pmatrix}$$

$$= \left(g_{ij}(x) \right)^T$$

$$= \left(g_{ij}(x) \right) \quad \text{by symmetry.}$$

<u>Use It</u>
$$\mathcal{I}_g : TM \longrightarrow T^*M$$
$$(p, V) \longmapsto (p, g(p)(V, \cdot))$$

is a diffeomorphism.

This allows us to turn a vector field $X : M \to TM$ into a 1-form

$$\mathcal{I}_g \circ X : M \to T^*M.$$

We can also turn a 1-form $\alpha : M \to T^*M$ into a vector field $I_g^{-1} \circ \alpha : M \longrightarrow TM$.

<u>Example</u> $I_g^{-1} \circ df$ is the gradient vector field of f with respect to g, $\nabla_g f$.

The matrix representative of I_g^{-1} is $(g_{ij}(x))^{-1}$.

Convention $(g_{ij})^{-1} = (g^{ij})$

<u>Use 2</u> Measuring length of paths and distances b/w points

Fix g on M. (M, g) = Riemannian manifold.

<u>Def^n</u> The <u>norm of $V \in T_pM$</u> is

$$\|V\|_g = \sqrt{g(p)(V,V)}$$

<u>Def^n</u> The <u>length of a smooth curve $\gamma : [a,b] \to M$</u>

is $$L_g(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_g \, dt$$

<u>FACT</u> If γ and $\tilde{\gamma}$ are both 1-1 and have the same image, then $L_g(\gamma) = L_g(\tilde{\gamma})$