$$\mathcal{T}_{r,s}(M) = \{ (p, W) \mid W \in \mathcal{T}_{r,s}(T_pM) \}$$

A $(r,s)$-tensor field on $M$ is a map $T: M \to \mathcal{T}_{r,s}(M)$ such that $\pi \circ T = Id_M$.

$$T(p) = (p, W(p))$$

$$\in \mathcal{T}_{r,s}(T_pM).$$

In coordinates given by a chart $(U, \phi)$

$\left\{ \frac{\partial}{\partial x_u^i}\Big|_p \right\}$ is a basis for $T_pM$.

$\left\{ dx_u^j(p) \right\}$ is the dual basis for $T_p^*M$

and

$$W(p) = \sum_{\substack{1 \le k_1, \dots, k_s \le n \\ 1 \le t_1, \dots, t_r \le n}} W^{k_1 \dots k_s}_{t_1 \dots t_r}(p) \frac{\partial}{\partial x_u^{k_1}}\Big|_p \otimes \dots \otimes \frac{\partial}{\partial x_u^{k_s}}\Big|_p \otimes dx_u^{t_1}(p) \otimes \dots \otimes dx_u^{t_r}(p)$$

We will simplify this notation to

$$W(x) = \sum W^{k_1 \dots k_s}_{t_1 \dots t_r}(x) \frac{\partial}{\partial x^{k_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{k_s}} \otimes dx^{t_1} \otimes \dots \otimes dx^{t_r}$$

where $x = \phi(p)$.

<u>Rmk</u> given $f \in C^\infty(M)$ and an $(r,s)$-tensor field we can construct another $(r,s)$-tensor field $fW$.

$$fW^{k_1 \cdots k_s}_{t_1 \cdots t_r}(x) = f(x)\, W^{k_1 \cdots k_s}_{t_1 \cdots t_r}(x)$$

<u>Rmk</u> given two $(r,s)$-tensor fields $W, \tilde{W}$ there sum $W + \tilde{W}$ is the $(r,s)$-tensor field defined (locally) by.

$$(W + \tilde{W})^{k_1 \cdots k_s}_{t_1 \cdots t_r}(x) = \left( W^{k_1 \cdots k_s}_{t_1 \cdots t_r}(x) + \tilde{W}^{k_1 \cdots k_s}_{t_1 \cdots t_r}(x) \right)$$

Now a new kind of tensor field.

<u>Def$^{\underline{n}}$</u> A Riemannian metric on $M$ is a smooth $(2,0)$-tensor field $g : M \longrightarrow \mathcal{T}_{(2,0)}(M)$ such that

i) $\quad g(p)(X, Y) = g(p)(Y, X) \quad \forall\ X, Y \in T_pM$

(g is symmetric)

ii) $\quad g(p)(X, X) > 0$ for all $\bar{0} \neq X \in T_pM$

(g is positive definite)

Let's view (i) and (ii) in local coordinates.

In general local coordinates :

$$g(x) = \sum_{1 \le t_1, t_2 \le 2} g_{t_1 t_2}(x) \, dx^{t_1} \otimes dx^{t_2}$$

where $g_{t_1 t_2}(x) = g(x)\left( \partial/\partial x^{t_1}, \partial/\partial x^{t_2} \right)$

or $$g(x) = \sum_{1 \le i,j \le n} g_{ij}(x) \, dx^i \otimes dx^j$$

where $g_{ij}(x) = g(x)\left( \partial/\partial x^i, \partial/\partial x^j \right)$

- $g$ is locally defined by the $n \times n$ matrix function $(g_{ij}(x))$.

$g$ symmetric $\iff g_{ij}(x) = g_{ji}(x)$

$\iff (g_{ij}(x)) = (g_{ij}(x))^T$

Recall from linear algebra

$A = A^T \Rightarrow \exists \, B$ s.t. $B^T B = I_n$ and

$$B^T A B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad \lambda_i \in \mathbb{R}.$$

A has real eigenvalues and is diagonlizable.

$g$ symmetric and positive definite $\Leftrightarrow$ all eigenvalues
(i) + (ii) are positive.

**Note** $(g_{ij}(x))$ is invertible!

Example 0 $\quad M = \mathbb{R}^n \quad g(x) = \sum_{i=1}^{n} dx^i \otimes dx^i$

Here $\left( g_{ij}(x) \right) = \mathbb{I}_n$

This is the standard (Euclidean) metric (dot product) on $\mathbb{R}^n$.

Example 1 $\quad M = \{ (x^1, x^2) \in \mathbb{R}^2 \mid x^2 > 0 \} \overset{\text{open}}{\subset} \mathbb{R}^2$

$$g(x) = \frac{1}{(x^2)^2} \sum_{i=1}^{2} dx^i \otimes dx^i$$

$$\left( g_{ij}(x) \right) = \frac{1}{(x^2)^2} \mathbb{I}_2$$

This is called the hyperbolic (or Poincaré) metric.

Example 2

Let $g_{st}$ be the standard metric on $\mathbb{R}^n$

Let $q$ be a regular value of $F: \mathbb{R}^n \longrightarrow \mathbb{R}$

Recall $T_x(F^{-1}(q)) = N(F_*) \subset T_x\mathbb{R}^n$.

The restriction of $g_{st}$ to $T_x(F^{-1}(q))$ is a Riemannian metric on $F^{-1}(q)$

ex 2 $F: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$x \longmapsto \|x\|^2$

The restriction of $g_{st}$ to $F^{-1}(1) = S^2$ is called the round metric on $S^2$.

Example 3 If $N$ is an embedded submanifold of $M$ and $g$ is a metric on $M$, then it restricts to a metric on $N$.

Thm Every manifold $M$ admits a Riemannian metric.

This is VERY fortunate since metrics are our primary measuring tools.

This is not trivial. $g$ positive definite $\Rightarrow$ $g$ is nonvanishing.

Recall: $S^2$ doesn't admit nonvanishing vector fields.

In fact each manifold admits lots of metrics.

<u>Method 1</u> given $g$ and $f \in C^\infty(M)$ s.t. $f(p) > 0 \ \forall p \in M$

$fg$ is also a metric

$$(fg)_{ij}(x) = f(x)\, g_{ij}(x)$$

<u>Method 2</u> If $g$ and $\tilde{g}$ are metrics so is

$g + \tilde{g}$.

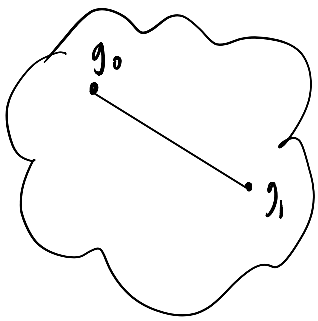
<u>Pf</u>

$$(g + \tilde{g})(X, X) = g(X,X) + \tilde{g}(X,X)$$

$$> 0 \quad \text{if } X \neq 0.$$

<u>Example</u> Any two metrics $g_0$ and $g_1$ are connected by a straight line of metrics.

$$g_t = (1-t)\, g_0 + t\, g_1$$




"The set of metrics in $\mathcal{T}_{2,0}(M)$ is convex."

Idea behind existence of a metric on $M$.

- Given $(U, \phi)$ define $g^U$ on $U$ by

$$(g^U_{ij}(x)) = \mathbb{I}_n.$$

- Given another chart $(V, \psi)$ with $U \cap V \neq \emptyset$ define $g^V$ by $(g^V_{ij}(y)) = \mathbb{I}_n$.

We need to patch these together nicely.

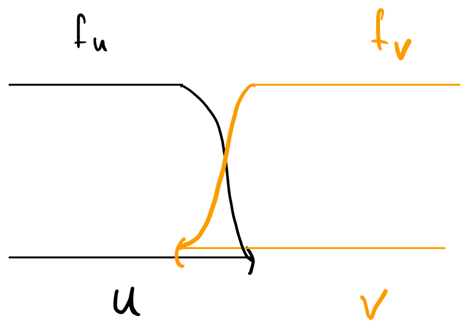
**Prop** There exist smooth functions $f_u, f_v : U \cup V \to \mathbb{R}$

s.t. i) $f_u(p) \geq 0$ and $f_u(p) = 0$ if $p \notin U$

ii) $f_v(p) \geq 0$ and $f_v(p) = 0$ if $p \notin V$

iii) $(f_u + f_v)(p) = 1 \quad \forall p \in U \cup V$.

i.e.




Now $f_u g^u + f_v g^v$ is a metric on $U \cup V$.

Continuing in this way we get a $g$ on $M$.

( Use countable base assumption for $M$ here. )