V a vector space of dimension n

- An (r,s) - tensor on V is a multi-linear map

$$W : V^* \times \cdots \times V^* \times V \times \cdots \times V \longrightarrow \mathbb{R}.$$

Let $\mathcal{T}_{r,s}(V)$ be the set of such maps

<u>ex</u> $\mathcal{T}_{1,0}(V) = V^*$

$$\alpha : V \longrightarrow \mathbb{R}$$
$$v \longmapsto \alpha(v)$$

<u>ex</u> $\mathcal{T}_{0,1}(V) \supset V$ in fact $\mathcal{T}_{0,1}(V) = V$

$$v : V^* \longrightarrow \mathbb{R}$$
$$\alpha \longmapsto \alpha(v)$$

<u>ex</u> $\bullet : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \quad \in \mathcal{T}_{2,0}(\mathbb{R}^n)$

$$(x, y) \longmapsto x_1 y_1 + \cdots + x_n y_n$$

ex $\quad \det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \in \mathcal{T}_{n,0}(\mathbb{R}^n)$

$$(x_1, \ldots, x_n) \longmapsto \det \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Thm $\quad \mathcal{T}_{r,s}(V)$ is a vector space of dimension n^{r+s}.

Example $\quad$ A general $W \in \mathcal{T}_{3,0}(\mathbb{R}^2)$ has the form

$$W(x,y,z) = \sum_{i,j,k=1}^{2} W_{ijk} \, x^i y^j z^k$$

- W is determined by the $2 \times 2 \times 2$ array W_{ijk}.
- The "basis" functions for $\mathcal{T}_{3,0}(\mathbb{R}^2)$ are

$$(x, y, z) \longmapsto x^i y^j z^k.$$

To prove the Thm we will build a basis for $\mathcal{T}_{r,s}(V)$

We start with the domain.

$\{e_1, \ldots, e_n\}$ a basis for V

$\{\sigma^1, \ldots, \sigma^n\}$ the (dual) basis for V^*

Consider the following subset of $V^* \times \cdots \times V^* \times V \times \cdots \times V$.

$$\left\{ (\sigma^{i_1}, \ldots, \sigma^{i_s}, e_{j_1}, \ldots, e_{j_r}) \;\middle|\; 1 \le i_1, \ldots, i_s \le n \;,\; 1 \le j_1, \ldots, j_r \le n \right\}$$

<u>FACT</u> By multilinearity, every $W \in \mathcal{T}_{r,s}(V)$ is determined uniquely by its values on this set

Now we build a basis for $\mathcal{T}_{r,s}(V)$ "dual" to this set

Given numbers $1 \le k_1, \ldots, k_s \le n$ and $1 \le t_1, \ldots, t_r \le n$

let $e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes \sigma^{t_1} \otimes \cdots \otimes \sigma^{t_r}$ be the map

on $V^* \times \cdots \times V^* \times V \times \cdots \times V$ defined by

$$e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes \sigma^{t_1} \otimes \cdots \otimes \sigma^{t_r} (\alpha_1, \ldots, \alpha_s, v_1, \ldots, v_r)$$

$$= \alpha_1(e_{k_1}) \cdots \alpha_s(e_{k_s}) \, \sigma^{t_1}(v_1) \cdots \sigma^{t_r}(v_r)$$

This map is multilinear.

$\otimes$ = tensor product

<u>Remark 1</u>

$$e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes \sigma^{t_1} \otimes \cdots \otimes \sigma^{t_r} (\sigma^{i_1}, \ldots, \sigma^{i_s}, e_{j_1}, \ldots, e_{j_r})$$

$$= \sigma^{i_1}(e_{k_1}) \cdots \sigma^{i_s}(e_{k_s}) \, \sigma^{t_1}(e_{j_1}) \cdots \sigma^{t_r}(e_{j_r})$$

$$= \begin{cases} 1 & \text{if } i_\ell = k_\ell \text{ for all } \ell \text{ and } i_m = j_m \text{ for all } m \\ 0 & \text{otherwise} \end{cases}$$

<u>Remark 2</u> There are $\underbrace{n \cdot n \cdots n}_{s} \cdot \underbrace{n \cdot n \cdots n}_{r} = n^{r+s}$

of these maps

These form a basis of $\mathcal{T}_{r,s}(V)$

Every $W \in \mathcal{T}_{r,s}(V)$ can be expressed uniquely in the form

$$W = \sum_{\substack{k_1, \ldots, k_s \\ t_1, \ldots, t_r}} W^{k_1 \cdots k_s}_{t_1 \cdots t_r} \, e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes \sigma^{t_1} \otimes \cdots \otimes \sigma^{t_r}$$

where $W^{k_1 \cdots k_s}_{t_1 \cdots t_r} = W(\sigma^{k_1}, \ldots, \sigma^{k_s}, e_{t_1}, \ldots, e_{t_r})$

Example $\quad \bullet : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$(x, y) \longmapsto x^1y^1 + \cdots + x^ny^n$$

A basis of $\mathcal{T}_{2,0}(\mathbb{R}^n)$ is $\{\sigma^{t_1} \otimes \sigma^{t_2} \mid 1 \leq t_1, t_2 \leq n\}$

$$= \{\sigma^i \otimes \sigma^j \mid 1 \leq i, j \leq n\}$$

$$W_{i,j} = W(e_i, e_j) = \delta^i_j$$

So $\quad \bullet = \sum_{1 \leq i,j \leq n} \delta^i_j \, \sigma^i \otimes \sigma^j$

$$= \sum_{i=1}^{n} \sigma^i \otimes \sigma^i$$

<u>Example</u> $\quad \det : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(x, y) \longmapsto \begin{vmatrix} x^1 & x^2 \\ y^1 & y^2 \end{vmatrix} = x^1y^2 - x^2y^1$$

$$\det = \sigma^1 \otimes \sigma^2 - \sigma^2 \otimes \sigma^1$$

Now let's put all this structure on manifolds

let $\mathcal{T}_{r,s}(M) = \{ (p, W) \mid p \in M, \; W \in \mathcal{T}_{r,s}(T_pM) \}$

<u>Thm</u> $\mathcal{T}_{r,s}(M)$ inherits from M the structure of a smooth manifold of dimension $n + n^{r+s}$.

The map $\pi : \mathcal{T}_{r,s}(M) \longrightarrow M$ is smooth.

<u>Def$^{\underline{n}}$</u> A smooth (r,s)-tensor field on M is a smooth map $T : M \longrightarrow \mathcal{T}_{r,s}(M)$ s.t. $\pi \circ T = Id_M$.

i.e. $T(p) = (p, W(p))$

In local coordinates given by a chart (U, ϕ)

$$W(p) = \sum_{\substack{1 \le k_1, \dots, k_s \le n \\ 1 \le t_1, \dots, t_r \le n}} W^{k_1 \dots k_s}_{t_1 \dots t_r}(p) \left.\frac{\partial}{\partial x_u^{k_1}}\right|_p \otimes \cdots \otimes \left.\frac{\partial}{\partial x_u^{k_s}}\right|_p \otimes dx_u^{t_1}(p) \otimes \cdots \otimes dx_u^{t_r}(p)$$

Henceforth we will simplify this to

$$W(x) = \sum W^{k_1 \cdots k_s}_{t_1 \cdots t_r}(x) \; \partial/\partial x^{k_1} \otimes \cdots \otimes \partial/\partial x^{k_s} \otimes dx^{t_1} \otimes \cdots \otimes dx^{t_r}$$

where $x = \phi(p)$.