$T_p^* M = (T_p M)^* = \{ \alpha : T_p M \to \mathbb{R} \mid \alpha \text{ is linear} \}$

$df(p) \in T_p^* M$ the differential of f at p

$df(p)(V) = V(f)$

Given a chart (U, ϕ)

$\{ dx_u^i(p) \}$ is basis of $T_p^* M$ dual to $\left\{ \frac{\partial}{\partial x_u^i}\Big|_p \right\}$

In particular $df(p) = \sum_i \frac{\partial f}{\partial x_u^i}(p) \, dx_u^i(p)$

$T^* M = \{ (p, \alpha) \mid \alpha \in T_p^* M \}$

<u>Thm</u> $T^* M$ inherits from M the structure of a smooth manifold of dimension $2(\dim(M))$. The map $\pi : T^* M \longrightarrow M$ is smooth.

$$(p, \alpha) \longmapsto p$$

<u>Def$^{\underline{n}}$</u> A 1-form (covector field) on M is a map $\nu : M \longrightarrow T^* M$ such that $\pi \circ \nu = Id_M$.

So $\nu(p) = (p, \alpha(p))$

<u>Example</u> Given $f \in C^\infty(M)$, the map

$$df : M \longrightarrow T^*M$$
$$p \longmapsto (p, df(p))$$

is a 1-form on M.

<u>Example</u> Recall the nonvanishing vector field on S^1

$$x \longmapsto \partial/\partial\theta\Big|_x = -x^2 \frac{\partial}{\partial x^1}\Big|_x + x^1 \frac{\partial}{\partial x^2}\Big|_x$$

Define the 1-form $d\theta$ on S^1 by $x \longmapsto (x, d\theta(x))$

where $d\theta(x)\left(\frac{\partial}{\partial\theta}\Big|_x\right) = 1.$

<u>Claim</u> this is not df for any $f \in C^\infty(S^1)$

<u>Pf</u>

Every $f \in C^\infty(S^1)$ attains its maximum value at some $x_{max} \in S^1$. (since S^1 is compact.)

$df(x_{max}) = \bar{0} \in T^*_{x_{max}} S^1$

Hence every df vanishes somewhere.

$d\theta(x) \neq \bar{0} \in T_x S^1$. for all $x \in S^1$.

Hence $d\theta \neq df$ for any f.

<u>Line Integrals.</u>

<u>Def^{n}</u> Given a 1-form $\nu(p) = (p, \alpha(p))$ on M and a smooth curve $\gamma : [a,b] \longrightarrow M$ we define

$$\int_\gamma \alpha = \int_a^b \alpha(\gamma(t))(\dot{\gamma}(t))\, dt$$

<u>Lemma</u> $\displaystyle\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$

<u>Pf</u>

$$\int_\gamma df = \int_a^b df(\gamma(t))(\dot{\gamma}(t))\, dt$$

$$= \int_a^b \dot{\gamma}(t)(f)\, dt$$

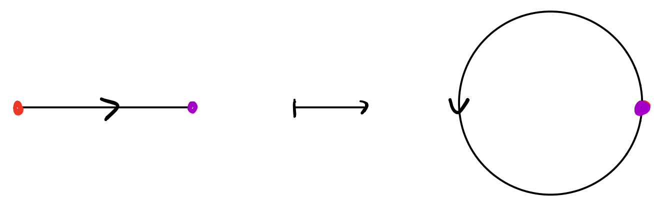
$$= \int_a^b \frac{d}{dt}(f \circ \gamma)\, dt$$

$$= (f \circ \gamma)(b) - (f \circ \gamma)(a) \quad \text{by FTC.}$$

$$= f(\gamma(b)) - f(\gamma(a)).$$

<u>Example</u> $\quad \gamma : [0, 2\pi] \longrightarrow S^1$

$$t \longmapsto (\cos t, \sin t)$$



For any $f \in C^\infty(S^1)$ we have

$$\int_\gamma df = f(\gamma(2\pi)) - f(\gamma(0)) = 0$$

But
$$\int_\gamma d\theta = \int_0^{2\pi} d\theta(\gamma(t))\, \dot\gamma(t)\, dt$$

$$= \int_0^{2\pi} d\theta(\gamma(t)) \left.\frac{\partial}{\partial\theta}\right|_{\gamma(t)} dt$$

$$= \int_0^{2\pi} 1 \; dt$$

$$= 2\pi .$$

For more interesting integrals (and measuring tools) we need ..

Tensors

Let V be a vector space of dimension n.

Let r, s be nonnegative integers.

<u>Def$^{\underline{n}}$</u> An (r,s)-tensor on V is a map

$$W : \underbrace{V^* \times V^* \times \cdots \times V^*}_{s} \times \underbrace{V \times V \times \cdots \times V}_{r} \longrightarrow \mathbb{R}$$

which is linear in each factor. (multi-linear)

i.e. $W(\alpha_1 + c\tilde{\alpha}_1, \alpha_2, \ldots) = W(\alpha_1, \alpha_2, \ldots) + c(\tilde{\alpha}_1, \alpha_2, \ldots)$

<u>Remark</u> As soon as there is more than one factor $(r+s>1)$
multi linear $\neq$ linear.

<u>Ex</u> The map

$$L : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto x+y$$

is linear but not multi-linear since

$$L(x, cy) \neq c\, L(x,y)$$
$$\| \qquad\qquad\qquad \|$$
$$x + cy \qquad\qquad c\,(x+y)$$

<u>Ex</u> The map

$$W : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto xy$$

is multilinear but not linear.

ex Each $\alpha \in V^*$ is a $(1,0)$-tensor on V.

$$\begin{aligned} \alpha : V &\longrightarrow \mathbb{R} \\ x &\longmapsto \alpha(x) \end{aligned}$$

ex Each $x \in V$ is a $(0,1)$-tensor on V

$$\begin{aligned} x : V^* &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto \alpha(x) \end{aligned}$$

ex The dot product
$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x^1y^1 + x^2y^2 + \dots + x^ny^n \end{aligned}$$
is a $(2,0)$-tensor on $\mathbb{R}^n$.

ex The map
$$\begin{aligned} \det : \mathbb{R}^n \times \dots \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x_1, x_2, \dots, x_n) &\mapsto \det \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \end{aligned}$$
is a $(n,0)$-tensor on $\mathbb{R}^n$.

ex A general $(3,0)$-tensor W on $\mathbb{R}^2$ has the form.

$$W : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x, y, z) \longmapsto \sum_{i,j,k=1}^{2} W_{ijk}\, x^i y^j z^k$$

<u>Note</u> W is determined by the $2\times 2\times 2$ array of real numbers W_{ijk}

Let $\mathcal{T}_{r,s}(V)$ be the set of (r,s)-tensors on V

<u>Thm</u> $\mathcal{T}_{r,s}(V)$ is a vector space of dimension n^{r+s}.

Let's build a basis for $\mathcal{T}_{r,s}(V)$

Let's start with its domain, $\underbrace{V^* \times \cdots \times V^*}_{s} \times \underbrace{V \times \cdots \times V}_{r}$

Let $\{e_1, \ldots, e_n\}$ be a basis for V

Let $\{\sigma^1, \ldots, \sigma^n\}$ be the dual basis for V^*

Then

$$\left\{ (\sigma^{i_1}, \ldots, \sigma^{i_s}, e_{j_1}, \ldots, e_{j_r}) \;\middle|\; 1 \le i_1, \ldots, i_s \le n \,,\; 1 \le j_1, \ldots, j_r \le n \right\}$$

is a basis for $\underbrace{V^* \times \cdots \times V^*}_{s} \times \underbrace{V \times \cdots \times V}_{r}$

Now we construct our basis for $\mathcal{T}_{r,s}(V)$ to be dual to our basis of $V^* \times \cdots \times V^* \times V \times \cdots \times V$.

Given numbers $1 \le k_1, \ldots, k_s \le n$ and $1 \le t_1, \ldots, t_r \le n$ let $e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes \sigma^{t_1} \otimes \cdots \otimes \sigma^{t_r}$ be the map on $V^* \times \cdots \times V^* \times V \times \cdots \times V$ defined by

$$e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes \sigma^{t_1} \otimes \cdots \otimes \sigma^{t_r} (\alpha_1, \ldots, \alpha_s, v_1, \ldots, v_r)$$

$$= \alpha_1(e_{k_1}) \cdots \alpha_s(e_{k_s}) \, \sigma^{t_1}(v_1) \cdots \sigma^{t_r}(v_r)$$

This map is multilinear.

$\otimes$ = tensor product

Notice

$$e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes \sigma^{t_1} \otimes \cdots \otimes \sigma^{t_r} (\sigma^{i_1}, \ldots, \sigma^{i_s}, e_{j_1}, \ldots, e_{j_r})$$

$$= \sigma^{i_1}(e_{k_1}) \cdots \sigma^{i_s}(e_{k_s}) \, \sigma^{t_1}(e_{j_1}) \cdots \sigma^{t_r}(e_{j_r})$$

$$= \begin{cases} 1 & \text{if } i_\ell = k_\ell \text{ for all } \ell \text{ and } i_m = j_m \text{ for all } m \\ 0 & \text{otherwise} \end{cases}$$

<u>Thm</u> $\{ e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes \sigma^{t_1} \otimes \cdots \otimes \sigma^{t_r} \}$

is a basis for $\mathcal{T}_{r,s}(V)$

Idea $W \in \mathcal{T}_{r,s}(V)$ can be written uniquely in the form

$$W = \sum_{\substack{k_1, \ldots, k_s \\ t_1, \ldots, t_r}} W^{k_1 \cdots k_s}_{t_1 \cdots t_r} \, e_{k_1} \otimes \cdots \otimes e_{k_s} \otimes \sigma^{t_1} \otimes \cdots \otimes \sigma^{t_r}$$

where $W^{k_1 \cdots k_s}_{t_1 \cdots t_r} = W(\sigma^{k_1}, \ldots, \sigma^{k_s}, e_{t_1}, \ldots, e_{t_r})$

<u>Example</u> $\bullet : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$(x, y) \longmapsto x^1 y^1 + \cdots + x^n y^n$

A basis of $\mathcal{T}_{2,0}(\mathbb{R}^n)$ is $\{ \sigma^{t_1} \otimes \sigma^{t_2} \mid 1 \leq t_1, t_2 \leq n \}$

$= \{ \sigma^i \otimes \sigma^j \mid 1 \leq i, j \leq n \}$

$W_{i,j} = W(e_i, e_j) = \delta^i_j$

So $\bullet = \sum_{1 \le i,j \le n} \delta^i_j \, \sigma^i \otimes \sigma^j$

$$= \sum_{i=1}^{n} \sigma^i \otimes \sigma^i$$