$X : M \longrightarrow TM$ s.t. $\pi \circ X = Id_M$

$X(p) = (p, V(p))$

If M is compact or $\operatorname{supp} X$ is compact the flow ϕ_t of X is global.

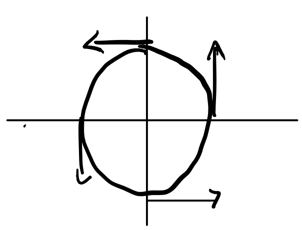
<u>Example</u> TS^1

- We constructed a nonvanishing vector field on S^1

$X(x) = (x, \dot{\gamma}(t))$ where $\gamma(t) = x$.

- Another version of the same vector field is

$$X(x) = \Big((x^1, x^2), \underbrace{-x^2 \frac{\partial}{\partial x^1}\Big|_x + x^1 \frac{\partial}{\partial x^2}\Big|_x}_{\frac{\partial}{\partial \theta}\big|_x \,=\, \dot{\gamma}(t).}\Big)$$



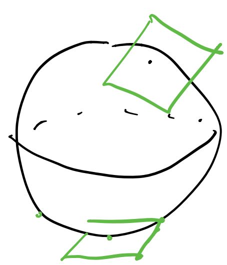
This yielded a diffeomorphism

$$F : S^1 \times \mathbb{R} \longrightarrow TS^1$$
$$(x, c) \longmapsto \left(x, c \frac{\partial}{\partial \theta}\Big|_x\right)$$

Fact $TM \simeq M \times \mathbb{R}^n$ if and only if there are n vector fields $X^1, \dots, X^n$ on M such that at each p, the collection $\{V^1(p), \dots, V^n(p)\}$ spans $T_p M$ where $X^i(p) = (p, V^i(p))$.

Example TS^2

Again we can picture this ...



Thm Every vector field X on S^2 vanishes at some pt.

i.e. $\exists\, p \in S^2$ such that $X(p) = (p, \bar{0} \in T_p S^2)$.

Corollary $TS^2 \not\cong S^2 \times \mathbb{R}^2$.

OK Let's take stock.

$$\mathbb{R}^m \rightsquigarrow M$$

$$F: \mathbb{R}^m \to \mathbb{R}^n \rightsquigarrow F: M \longrightarrow N$$

$$\left(\frac{\partial F^j}{\partial x^i}(x)\right) \rightsquigarrow F_*: T_pM \longrightarrow T_{F(p)}N$$

We can do optimization on M and solve IVP's.

Next we would like to integrate!

The things we integrate will be special tensor fields.

Aside: Dual Spaces

Let V be a finite dim'l vector space (over $\mathbb{R}$).

$$V^* = \{ \alpha : V \longrightarrow \mathbb{R} \mid \alpha \text{ is linear} \}$$

<u>Prop</u> V^* is a vector space w.r.t. the operations

$$(\alpha + c\beta)(v) = \alpha(v) + c(\beta(v)).$$

Moreover, $\dim(V^*) = \dim(V)$.

Let $\{e_1, \ldots, e_n\}$ be a basis for V

Define $\sigma^i \in V^*$ by $\sigma^i(e_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = \delta^i_j$

So $\sigma^i(v) = \sigma^i(a_1 e_1 + \cdots + a_n e_n)$

$$= a_1 \sigma^i(e_1) + \cdots + a_n \sigma^i(e_n)$$

$$= a_i$$

<u>Exercise</u> Prove that $\{\sigma^i\}_{i=1,\ldots,n}$ is a basis for V^*.

$\{\sigma^i\}$ is <u>the</u> basis dual to $\{e_j\}$

<u>Aside</u> The dot product on $V = \mathbb{R}^n$ defines an isomorphism $\mathbb{R}^n \longrightarrow (\mathbb{R}^n)^*$ where

$$x \longmapsto \sigma_x$$

$$\sigma_x(y) = x \cdot y.$$

Consider the vector space T_pM

$(T_pM)^* = T_p^*M$ the cotangent space to M at p.

• Each $f \in C^\infty(M)$ determines an element $df(p) \in T_p^*M$ (for each $p \in M$) via the formula

$$df(p)\big(\underset{\substack{\cap \\ T_pM}}{X}\big) = X(f).$$

$df(p) : T_pM \longrightarrow \mathbb{R}$ is the <u>differential of f</u> at p.

<u>Example</u> 0

On $M = \mathbb{R}^n$ we have the functions

$$\begin{aligned} x^i : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto x^i \end{aligned}$$ (we called this π_i previously)

$$dx^i(x) : T_x\mathbb{R}^n \longrightarrow \mathbb{R}$$

<u>Claim</u>

$\{dx^i(x)\}$ is the basis of $T_x^*\mathbb{R}^n$ dual to $\left\{ \frac{\partial}{\partial x^i}\Big|_x \right\}$.

$$dx^i(x)\left(\frac{\partial}{\partial x^j}\Big|_x \right) = \frac{\partial}{\partial x^j}\Big|_x (x^i) = \delta^i_j$$

Example 1 Let (U, ϕ) be a chart.

Consider $$x_U^i : U \longrightarrow \mathbb{R}$$
$$p \longmapsto x^i(\phi(p))$$

$dx_U^i(p) \in T_p^* M$ is defined by.

$$dx_U^i(p)(V) = V(x_U^i)$$

$\{dx_U^i(p)\}$ is basis dual to $\left\{ \frac{\partial}{\partial x_U^j}\Big|_p \right\}$.

$$dx_U^i(p)\left(\frac{\partial}{\partial x_U^j}\Big|_p \right) = \frac{\partial}{\partial x_U^j}\Big|_p (x_U^i)$$

$$= \frac{\partial}{\partial x^j}\Big|_{\phi(p)} (x^i \circ \phi \circ \phi^{-1})$$

$$= \frac{\partial}{\partial x^j}\Big|_{\phi(p)} (x^i)$$

$$= \delta_j^i$$

Change of coordinates for differentials

$$dx^i_V = \sum_j \frac{\partial x^i_V}{\partial x^j_u} dx^j_u$$

Claim Given $f \in C^\infty(M)$, (U, ϕ) a chart and $p \in U$

$$df(p) = \sum_i \frac{\partial f}{\partial x^i_u}(p)\, dx^i_u(p)$$

Cor $df(p) = \bar{0} \in T^*_p M \Leftrightarrow p$ is a critical point of f

In Calculus one calls the differential of f

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n$$

but doesn't say where it lives

Def$^{\underline{n}}$ The cotangent bundle of M is the set

$$T^*M = \{ (p, \alpha) \mid p \in M, \ \alpha \in T^*_p M \}$$

Thm T^*M inherits from M the structure of a smooth manifold of dimension $2(\dim(M))$. The map $\pi : T^*M \longrightarrow M$ is smooth.

$$(p, \alpha) \longmapsto p$$