M^n a smooth manifold. It's tangent bundle is:

$$TM = \{ (p, V) \mid p \in M , V \in T_pM \}$$

$$\pi : TM \longrightarrow M$$

<u>Rmk</u> TM is a "vector bundle" over M.

<u>Prop</u> TM inherits structure of $2n$-mfld from M.
The map π is smooth.

<u>Pf</u>

chart (U_α, ϕ_α) on M $\rightsquigarrow$ chart $(\hat{U}_\alpha, \hat{\phi}_\alpha)$ on TM.

$$\hat{U}_\alpha = \pi^{-1}(U_\alpha) = \{ (p, V) \in TM \mid p \in U_\alpha \}$$

$$\hat{\phi}_\alpha(p, V) = \left(\phi_\alpha(p), (V^1, \dots, V^n) \right) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

where $V = \sum_i V^i \frac{\partial}{\partial x^i_{U_\alpha}}$

Need Compatibility.

$$\hat{\phi}_\beta \circ (\hat{\phi}_\alpha)^{-1} \left(x , (V^1, \dots, V^n) \right)$$

$$= \hat{\phi}_\beta \left(\phi_\alpha^{-1}(x) \, , \, \sum_i V^i \frac{\partial}{\partial x^i_{u_\alpha}} \right)$$

$$= \hat{\phi}_\beta \left(\phi_\alpha^{-1}(x) \, , \, \sum_i V^i \sum_j \frac{\partial x^j_{u_\beta}}{\partial x^i_{u_\alpha}} \frac{\partial}{\partial x^j_{u_\beta}} \right)$$

$$= \hat{\phi}_\beta \left(\phi_\alpha^{-1}(x) \, , \, \sum_j \left(\sum_i V^i \frac{\partial x^j_{u_\beta}}{\partial x^i_{u_\alpha}} \right) \frac{\partial}{\partial x^j_{u_\beta}} \right)$$

$$= \left(\phi_\beta \circ \phi_\alpha^{-1}(x) \, , \, \left(\sum_i V^i \frac{\partial x^1_{u_\beta}}{\partial x^i_{u_\alpha}} , \ldots , \sum_i V^i \frac{\partial x^n_{u_\beta}}{\partial x^i_{u_\alpha}} \right) \right).$$

smooth since ϕ_β, ϕ_α compatible

linear combination of the V^i with coeff that are function of x

We just need these function to be smooth.

$$\frac{\partial x^j_{u_\beta}}{\partial x^i_{u_\alpha}}(x) = \frac{\partial}{\partial x^i} \left(x^j_{u_\beta} \circ \phi_\alpha^{-1} \right) = \frac{\partial}{\partial x^i} \left(\pi_j \circ \phi_\beta \circ \phi_\alpha^{-1} \right)$$

smooth.

Def^{n} A vector field on M is a map $X: M \longrightarrow TM$ such that $\pi \circ X = Id_M$.

Rmk A vector field is a section of TM (tensor field).

What about the flow of X on M?

Def^{n} $U \subset M$ is compact if for every collection of open subsets $\{V_\alpha\}_{\alpha \in A}$ with $\bigcup_{\alpha \in A} V_\alpha \supset U$, there is a finite subset of the V_α which also cover U.

Ex $[a,b] \subset \mathbb{R}$ is compact.

Ex $(a,b] \subset \mathbb{R}$ is not compact.

Ex S^n, $\mathbb{RP}^n$, $O(n)$ are all compact.

Fact 1 If M is compact, then any vector field X on M determines a global flow ϕ_t on M.

closure

Fact 2 Let $\operatorname{supp}(X) = \overline{\{ p \in M \mid X(p) \neq (p, \underset{T_pM}{\overline{0}}) \}}$

If $\operatorname{supp}(X)$ is compact then X determines a global flow.

Example On any M we have the zero vector field

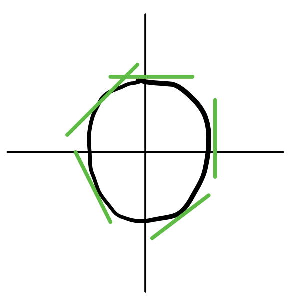
$$\begin{aligned} Z_M : M &\longrightarrow TM \\ p &\longmapsto (p, \underset{T_pM}{\overline{0}}) \end{aligned}$$

$\operatorname{supp}(Z_M) = \emptyset$ which is compact by default.

The flow of Z_M is $\phi_t = Id_M$.

Example TS^1

We have a "picture" of this



Consider the map
$$\begin{aligned} \gamma : \mathbb{R} &\longrightarrow S^1 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

The restriction $\gamma : [0, 2\pi) \to S^1$ is 1-1 and onto

Define $\dot{\gamma}(t) \in T_{\gamma(t)} S^1$ by $\dot{\gamma}(t)(f) = \left.\frac{d}{ds}\right|_{s=0} f(\gamma(t+s))$

$X : S^1 \longrightarrow TS^1$ is smooth vector field on S^1.
$x = \gamma(t) \longmapsto (x, \dot{\gamma}(t))$

The flow of X is $\phi_t(x^1, x^2) = \left(\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \right)^T$

- For each $t \in [0, 2\pi)$, $\dot{\gamma}(t)$ spans $T_{\gamma(t)} S^1$

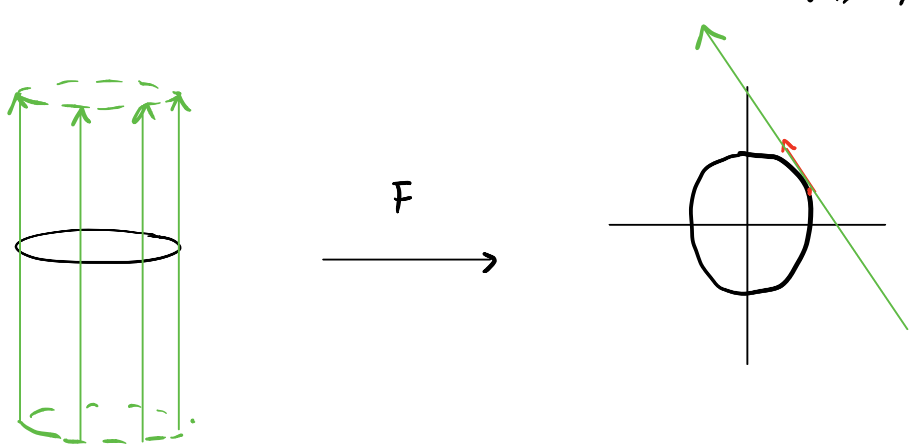
(X is nonvanishing)

The map $F : S^1 \times \mathbb{R} \longrightarrow TS^1$

$(x, c) \longmapsto (x, c\dot{\gamma}(t))$

for $t \in [0, 2\pi)$ s.t. $\gamma(t) = x$

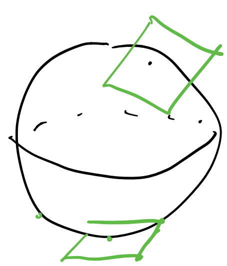
is a diffeomorphism.



Fact $TM \cong M \times \mathbb{R}^n$ if and only if there are n vector fields $X^1, \dots, X^n$ on M such that at each p, the collection $\{V^1(p), \dots, V^n(p)\}$ spans $T_p M$ where $X^i(p) = (p, V^i(p))$.

Example TS^2

Again we can picture this ...



Thm Every vector field X on S^2 vanishes at some pt.

i.e. $\exists\ p \in S^2$ such that $X(p) = (p, \bar{0} \in T_p S^2)$.

Corollary $TS^2 \not\cong S^2 \times \mathbb{R}^2$.