- A vector field on $\mathbb{R}^n$ is a map

$$X : \mathbb{R}^n \longrightarrow T\mathbb{R}^n = \{ (x, v_x) \mid v_x \in T_x M \} \quad \text{s.t.}$$

$$\pi \circ X = Id_{\mathbb{R}^n}$$

- $X(x) = (x, V(x))$

$$\begin{tikzcd}
\text{Vector Field on } \mathbb{R}^n \arrow[d, Leftrightarrow, "①"'] & & \text{1-parameter group of diffeomorphisms} \arrow[ll, Leftarrow, "④"'] \arrow[d, shift left] \\
\text{IVP's for } n \text{ first order ODE's} \arrow[rr, Leftrightarrow, "②", "\text{uniqueness} \quad \text{existence}"'] & & \text{Solutions} \arrow[u, dashed, shift left, "③"]
\end{tikzcd}$$

$$\begin{tikzcd}
V(x) = x^1 \frac{\partial}{\partial x^1} + 2x^2 \arrow[d, Leftrightarrow] & & \phi_t(x) = (x^1 e^t, x^2 + 2t) \arrow[ll, Leftarrow] \arrow[d, shift left] \\
\begin{cases} \gamma_x(0) = x \\ \frac{d}{dt}(\gamma_x^1) = \gamma_x^1(t) \\ \frac{d}{dt}(\gamma_x^2) = 2 \end{cases} \arrow[rr, Leftrightarrow] & & \gamma_x(t) = (x^1 e^t, x^2 + 2t) \arrow[u, dashed, shift left]
\end{tikzcd}$$

(2) Existence and Uniqueness Thm for IVP's

- for all x $\exists$ $b>0$ and a solution

  $\gamma_x : (-b,b) \to \mathbb{R}^n$ s.t. $\gamma(0) = x$.

- any two solutions agree on the intersection of their domains

- For all x there is an open nbhd $U_x \ni x$ and $\varepsilon > 0$

s.t.

$$\begin{aligned} \Phi : U_x \times (-\varepsilon, \varepsilon) &\longrightarrow \mathbb{R}^n \\ (y,t) &\longmapsto \gamma_y(t) \end{aligned}$$

is well-defined and smooth.

- The maps

$$\begin{aligned} \phi_t : U_x &\longrightarrow \mathbb{R}^n \\ y &\longmapsto \gamma_y(t) \end{aligned}$$

  all satisfy $\phi_0 = Id_{U_x}$ and the rule

$$\phi_t \circ \phi_s(y) = \phi_{s+t}(y)$$

whenever $t, s, s+t \in (-\varepsilon, \varepsilon)$

<u>Rmk</u> Note $\phi_t \circ \phi_{-t} = \phi_0 = Id$ so each ϕ_t is a diffeomorphism (from U_x to $\phi_t(U_x)$)

<u>Fact</u> Under mild assumption on V, every solution $\gamma_x(t)$ exists for all $t \in \mathbb{R}$.

$\Rightarrow$ $\Phi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$

$\Rightarrow$ Each $\phi_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeo of $\mathbb{R}^n$.

For example, this global existence holds if $\exists\, K > 0$ s.t.

$$\left| \frac{\partial V^i}{\partial x^j}(x) \right| \leq K \quad \forall\, i, j, x$$

③ In this case, the ϕ_t are examples of ...

<u>Def^n</u> A 1-parameter group of diffeomorphisms (flow) on M is a collection $\{\phi_t : M \to M\}_{t \in \mathbb{R}}$ of diffeomorphisms such that

$$\phi_0 = Id_M$$

$$\phi_s \circ \phi_t = \phi_{s+t}$$

Ex $\phi_t : S^1 \longrightarrow S^1$

$$(x^1, x^2) \longmapsto (\cos t\, x^1 - \sin t\, x^2, \sin t\, x^1 \ \cos t\, x^2)$$

$$= \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \right)^T$$

(4) $\Leftarrow$

Given a flow on M, one can define a $V(p) \in T_pM$ for any $p \in M$ by

$$V(p)(f) = \left.\frac{d}{dt}\right|_{t=0} f(\phi_t(p))$$

This completes the diagram for $M = \mathbb{R}^n$.

To complete it for general M we first need to define vector fields on M.

Def^n The <u>tangent bundle</u> of M is

$$TM = \{ (p, V) \mid V \in T_pM \}$$

Set $\pi : TM \longrightarrow M$
$(p, V) \longmapsto p.$

<u>Prop</u> TM inherits from M the structure of a smooth manifold of dimension $2n$. The map $\pi : TM \to M$ is smooth.

<u>Rmk</u> In general $TM \not\cong M \times \mathbb{R}^n$! So some of these are new manifolds.

<u>Pf of Proposition</u>

Let $\mathcal{A} = \{ (U_\alpha, \phi_\alpha) \}$ be an atlas on M

Given (U_α, ϕ_α) define

$$\hat{U}_\alpha = \pi^{-1}(U_\alpha) = \{ (p, V) \in TM \mid p \in U_\alpha \}$$

Note $\bigcup_\alpha \hat{U}_\alpha = TM$

Need $\hat{\phi}_\alpha : \hat{U}_\alpha \longrightarrow \mathbb{R}^{2n}$

For $(p, V) \in U_\alpha$ we have.

$$V = V_\alpha^1 \left.\frac{\partial}{\partial x^1_{U_\alpha}}\right|_p + \cdots + V_\alpha^n \left.\frac{\partial}{\partial x^n_{U_\alpha}}\right|_p.$$

Set $\hat{\phi}_\alpha(p, V) = \left(\phi_\alpha(p), (V^1_\alpha, \dots, V^n_\alpha)\right)$

ϕ_α 1-1 and $\left\{ \partial/\partial x^i_{u_\alpha} \right\}$ basis $\Rightarrow$ $\hat{\phi}_\alpha$ is 1-1

$\hat{\phi}_\alpha(U_\alpha) = \phi_\alpha(U_\alpha) \times \mathbb{R}^n \underset{\text{open}}{\subset} \mathbb{R}^n \times \mathbb{R}^n$

Hence $\hat{\phi}_\alpha$ is a chart on TM.

We still need to check compatibility.

$$\hat{\phi}_\alpha(\hat{U}_\alpha \cap \hat{U}_\beta) = \left\{ (\phi_\alpha(p), v) \,\middle|\, p \in U_\alpha \cap U_\beta , v \in \mathbb{R}^n \right\}$$

$$= \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

$$\underset{\text{open}}{\subset} \mathbb{R}^n \times \mathbb{R}^n$$

$$\hat{\phi}_\beta \circ \phi_\alpha^{-1} (x, v)$$

$$= \phi_\beta \left(\phi_\alpha^{-1}(x), \sum_i v^i \partial/\partial x^i_{u_\alpha} \right)$$

$$= \phi_\beta \left(\phi_\alpha^{-1}(x), \sum_i \sum_j v^i \frac{\partial x^j_{u_\beta}}{\partial x^i_{u_\alpha}} \partial / \partial x^j_{u_\beta} \right)$$

$$= \left(\phi_\beta \circ \phi_\alpha^{-1}(x) , \left(\sum_i v^i \frac{\partial x^1_{u_\beta}}{\partial x^i_{u_\alpha}} , \dots , \sum_i v^i \frac{\partial x^n_{u_\beta}}{\partial x^i_{u_\alpha}} \right) \right)$$

This is smooth since $\phi_\beta \circ \phi_\alpha^{-1}$ is smooth, as is

$$\frac{\partial x^j_{u_\beta}}{\partial x^i_{u_\alpha}} = \frac{\partial}{\partial x^i} \left(\pi_j \circ \phi_\beta \circ \phi_\alpha^{-1} \right)$$

<u>Def^n</u> A vector field on M is a map $X : M \longrightarrow TM$ such that $\pi \circ X = Id_M$.

What about the flow of X on M?