Last time we used the RVT to construct manifolds

$$SL(n) = \det^{-1}(1) \quad \text{where} \quad \det : M_{n\times n} \longrightarrow \mathbb{R}$$
$$A \longmapsto \det(A)$$

$$O(n) = \mathrm{Sym}^{-1}(\mathbb{I}_n) \quad \text{where} \quad \mathrm{Sym} : M_{n\times n} \longrightarrow \mathrm{Sym}(n)$$
$$A \longmapsto A^T A$$

$$SO(n) = \{ A \in O(n) \mid \det(A) = 1 \} \underset{\text{open}}{\subset} O(n)$$

These manifolds are also groups $\Rightarrow$ Lie Groups.

<u>Correction</u> $\quad SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\} \subsetneq O(2)$

<u>ex</u> $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2) \setminus SO(2)$

Some other nice facts

$$SL(n) \cong SO(n) \times \mathbb{R}^{\frac{(n+2)(n-1)}{2}} \qquad \left(SL(2) \cong S^1 \times \mathbb{R}^2 \right)$$

$$SO(3) \cong \mathbb{RP}^3$$

Rest of the manifolds are new, to us.

Vector Fields, ODE's and their solutions.

<u>Def^{n} 1</u> A vector field on $\mathbb{R}^n$ is a rule which assigns to each $x \in \mathbb{R}^n$ a vector $V(x) \in T_x\mathbb{R}^n$

Each $T_x\mathbb{R}^n$ has basis $\left\{ \partial/\partial x^i \big|_x \right\}$.

So $V(x) = V^1(x) \, \partial/\partial x^1 \big|_x + \cdots + V^n(x) \, \partial/\partial x^n \big|_x$.

and V is encoded by the n functions $V^1(x), \ldots, V^n(x)$

Note: V is not a function (yet).

Problem: the target of V, $T_x\mathbb{R}^n$, changes with x.

Let $T\mathbb{R}^n = \left\{ (x, v_x) \;\middle|\; v_x \in T_x\mathbb{R}^n \right\}$

This is the <u>tangent bundle to $\mathbb{R}^n$</u>.

As a set, it is in bijection with $\mathbb{R}^n \times \mathbb{R}^n$

$$(x, v_x) \longleftrightarrow (x, v)$$

This works ($\longleftrightarrow$) <u>only</u> because we can unambiguously identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ since $\mathbb{R}^n$ has a global chart.

$$V_x = V^1 \, \partial/\partial x^1 + \cdots + V^n \, \partial/\partial x^n \longleftrightarrow (V^1, \ldots, V^n) = V$$

$\cap$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\cap$

$T_x\mathbb{R}^n$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\mathbb{R}^n$

- $T\mathbb{R}^n$ is a smooth manifold of dimension $2n$.
- Let $\pi : T\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

$$(x, v_x) \longmapsto x$$

<u>Def</u>

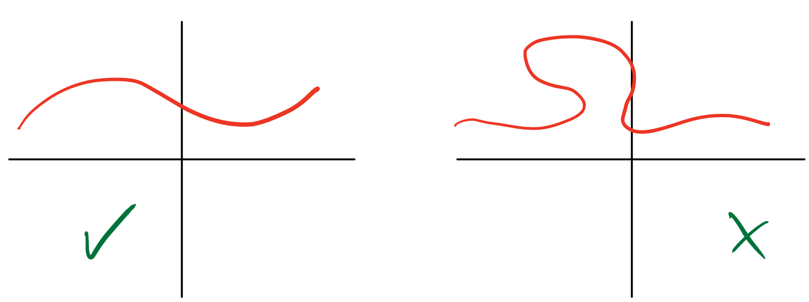
Def$^{\underline{n}}$ A <u>vector field</u> on $\mathbb{R}^n$ is a map

$$X : \mathbb{R}^n \longrightarrow T\mathbb{R}^n \quad \text{s.t.} \quad \pi \circ X = Id_{\mathbb{R}^n}.$$

The condition $\pi \circ X = Id_{\mathbb{R}^n}$ implies that

$$X(x) = (\underline{x}, V(x))$$

$\uparrow$ rule from previous definition.



There is a related set of ideas which we will later extend to manifolds

$$
\begin{tikzcd}
\text{Vector Field on } \mathbb{R}^n \arrow[d, Leftrightarrow, "1"] & & \text{1-parameter group of diffeomorphisms} \arrow[ll, Leftarrow, "4"'] \arrow[d, shift left, "3"] \\
\text{IVP's for } n \text{ first order ODE's} \arrow[rr, Leftrightarrow, "2", "\text{uniqueness} \qquad \text{existence}"'] & & \text{Solutions} \arrow[u, dashed, shift left]
\end{tikzcd}
$$

1 To the vector field $X(x) = (x, V(x))$ with

$$V(x) = V^1(x) \frac{\partial}{\partial x^1}\Big|_x + \cdots + V^n(x) \frac{\partial}{\partial x^n}\Big|_x$$

we assorciate the following Initial Value Problems.

For each x, find a map

$$\gamma_x(t) = (\gamma_x^1(t), \ldots, \gamma_x^n(t))$$

such that $\gamma_x(0) = x$ and

$$\left.\begin{aligned} \frac{d}{dt}\left(\gamma_x^1(t)\right) &= V^1(\gamma_x(t)) \\ &\vdots \\ \frac{d}{dt}\left(\gamma_x^2(t)\right) &= V^2(\gamma_x(t)) \end{aligned}\right\} \; n \text{ first order ODE's}$$

<u>Example</u> $V(x^1,x^2) = x^1 \frac{\partial}{\partial x^1}\Big|_x + 2\frac{\partial}{\partial x^2}\Big|_x$

The initial condition is $\gamma_x(0) = x$

$(\gamma_x^1(0) = x^1 \ , \ \gamma_x^2(0) = x^2)$

The system of ODE's is

$$\frac{d}{dt}(\gamma_x^1(t)) = V^1(\gamma_x(t)) = \gamma_x^1(t)$$

$$\frac{d}{dt}(\gamma_x^2(t)) = V^2(\gamma_x(t)) = 2$$

Note we can solve <u>this</u> IVP

$$\gamma_x^1(t) = \gamma_x^1(0)e^t = x^1 e^t$$

$$\gamma_x^2(t) = 2t + x^2.$$

and this solution works for all $t \in \mathbb{R}$.

For a general IVP of this type we have

② <u>Existence and Uniqueness for solutions of IVP's.</u>

Let $X(x) = (x, V(x))$ be a smooth vector field on $\mathbb{R}^n$

- For each $x \in \mathbb{R}$ there is a $b>0$ and a smooth curve $\gamma_x : (-b, b) \longrightarrow \mathbb{R}^n$ such that $\gamma_x(0) = x$ and $\gamma_x^{i\,\prime}(t) = V^i(\gamma_x(t))$ for $i = 1, \ldots, n$.
(Existence)

- Any two such maps agree on the intersection of their domains. (Uniqueness.)

- There is an open nbhd $U_x \subset \mathbb{R}^n$ and an $\varepsilon > 0$ such that the map

$$\begin{aligned} \Phi : U_x \times (-\varepsilon, \varepsilon) &\longrightarrow \mathbb{R}^n \\ (y, t) &\longmapsto \gamma_y(t) \end{aligned}$$

is well defined and smooth. $\left(\gamma_x(t) \text{ depends smoothly on } x \right)$

- The maps

$$\begin{aligned} \phi_t : U_p \subset \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ y &\longmapsto \gamma_y(t) \end{aligned}$$

are smooth and satisfy

$$\phi_t \circ \phi_s (y) = \phi_{t+s}(y)$$

whenever $s, t, s+t \in (-\varepsilon, \varepsilon)$