REGULAR VALUE THEOREM (RVT)

If q is a regular value of $F: M \longrightarrow N$

$\left(p \in F^{-1}(q) \Rightarrow F_*: T_pM \to T_{F(p)}N \text{ is onto (has rank} = n) \right)$

then $F^{-1}(q)$ is an embedded submanifold of dimension

equal to $\dim(M) - \dim(N)$.

SARD's THEOREM

Almost every $q \in N$ is a regular value of $F: M \to N$.

Example $\quad F: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$

$$(x^1, \dots, x^{n+1}) \longmapsto (x^1)^2 + \dots + (x^{n+1})^2$$

Claim every $c \neq 0$ is a regular value of F.

For all $x \in \mathbb{R}^{n+1}$

$$[F_*] = \left(\frac{\partial F}{\partial x^j}(x) \right) = (2x^1, \dots, 2x^{n+1})$$

- If $c < 0$, then $F^{-1}(c) = \emptyset$ and c regular by default.
- $F^{-1}(0) = \bar{0} \in \mathbb{R}^{n+1}$ and $[F_*] = (0, \dots, 0)$ has rank 0.

  So 0 is a critical value.

- For $c > 0$ we have

$$x \in F^{-1}(c) \iff (x^1)^2 + \cdots + (x^{n+1})^2 = c$$

$$\Rightarrow x^i \neq 0 \text{ for some } i$$

$$\Rightarrow [F_*] \text{ has rank } 1.$$

Hence each $c > 0$ is a regular value.

<u>Note</u> $F^{-1}(1) = S^n$.

<u>Lie Groups.</u>

Let $M_{n \times n} = \{ A \mid A \text{ is } n \times n \text{ matrix} \}$.

As a set $M_{n \times n}$ is in bijection with $\mathbb{R}^{n^2}$.

$$\text{List}(a_{ij}) = (a_{11}, a_{12}, \ldots, a_{n-1\,n}, a_{nn})$$

So, $M_{n \times n}$ can be given the structure of a smooth mfld.

(Has atlas with one chart $(M_{n \times n}, \text{List})$)

<u>Ex!</u> Consider $SL(n) = \{ A \in M_{n \times n} \mid \det(A) = 1 \}$

$SL(n)$ is called the special linear group.

Note $A, B \in SL(n) \Rightarrow AB \in SL(n)$ since $\det(AB) = \det(A)\det(B) = 1$

Also $A \in SL(n) \Rightarrow A^{-1} \in SL(n)$ since

$$AA^{-1} = I_n \Rightarrow \det(A)\det(A^{-1}) = 1$$

$SL(n)$ with usual matrix multiplication is a group

<u>Claim</u> $SL(n)$ is also a manifold. (Lie Group)

<u>Pf</u> Suffices to show 1 is a regular value of the map

$$F : M_{n\times n} \longrightarrow \mathbb{R}$$
$$A \longmapsto \det(A)$$

(i.e. $SL(n) = F^{-1}(1)$)

Let's prove this for the case $n=2$.

$$F \circ \mathrm{List}^{-1}(x^1, x^2, x^3, x^4) = \det\begin{pmatrix} x^1 & x^2 \\ x^3 & x^4 \end{pmatrix} = x^1x^4 - x^2x^3.$$

$$[F_*] = (x^4, -x^3, -x^2, x^1) \quad \text{at } A = \begin{pmatrix} x^1 & x^2 \\ x^3, & x^4 \end{pmatrix}$$

$$A = \begin{pmatrix} x^1 & x^2 \\ x^3 & x^4 \end{pmatrix} \in F^{-1}(1) \Rightarrow x^1x^4 - x^2x^3 = 1$$

$$\Rightarrow x^i \neq 0 \text{ for some } i$$

$\Rightarrow$ F_* has rank 1.

<u>Rmk 1</u> $SL(n)$ is a manifold of dimension n^2-1.

For example $SL(2) \simeq S^1 \times \mathbb{R}^2$

<u>Rmk 2</u> The group structure and manifold structure are compatible

For each $B \in SL(n)$ the (group) map

$$F_B : SL(n) \longrightarrow SL(n)$$
$$A \longmapsto BA$$

is a diffeomorphism.

smooth : each entry of BA is a lin comb of entries of A.

invertible: $(F_B)^{-1} = F_{B^{-1}}$, smooth for same reason.

<u>Ex 2</u> Consider $O(n) = \{ A \in M_{n \times n} \mid A^T A = \mathbb{I}_n \}$.

<u>Exercise</u> $O(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} \longleftrightarrow S^1$

$O(n)$ is called the orthogonal group.

$$A, B \in O(n) \Rightarrow (AB)^T(AB) = B^T A^T A B$$
$$= B^T \mathbb{I}_n B$$
$$= \mathbb{I}_n$$
$$\Rightarrow AB \in O(n)$$

<u>Exercise</u> Prove $A \in O(n)$ is invertible and $A^{-1} \in O(n)$.

<u>Claim</u> $O(n)$ is a manifold.

<u>Pf</u> (Idea) $O(n) = F^{-1}(\mathbb{I}_n)$ where

$$F: M_{n \times n} \longrightarrow M_{n \times n}$$
$$A \longmapsto A^T A$$

<u>Problem</u> $\mathbb{I}_n$ is <u>not</u> a regular value of this map!

(If it where, then $O(n)$ would be 0-dimensional)

- Observe that $(A^T A)^T = A^T A$.
- Change target of F to $Sym(n) = \{ B \in M_{n \times n} \mid B = B^T \}$

$Sym(n)$ is a vector space isomorphic to $\mathbb{R}^{(n + (n-1) + \cdots + 1)} = \mathbb{R}^{\frac{(n+1)n}{2}}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \longleftrightarrow (a_{11}, a_{12}, a_{13}, a_{22}\ a_{23}, a_{33})$$

$$\frac{(3+1)(3)}{2} = 6$$

<u>Claim</u> $\mathbb{I}_n$ is a regular value of the map

$$M_{n \times n} \longrightarrow \mathrm{Sym}(n)$$

$$A \longmapsto A^T A.$$

Hence $O(n)$ is a manifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n^2}{2} - \frac{n}{2}$.