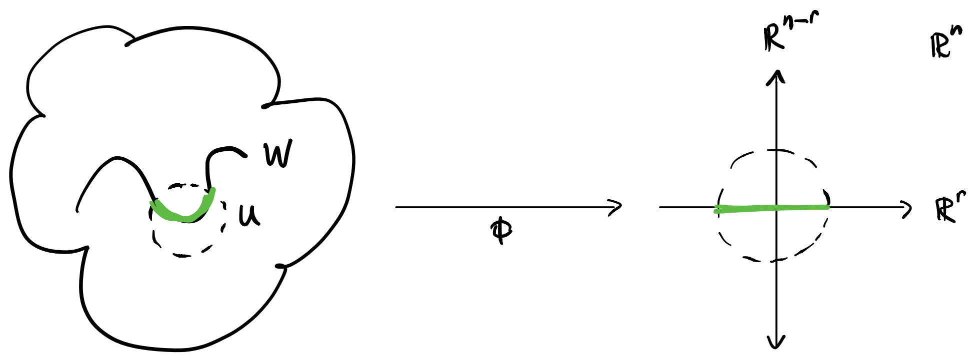
<u>Def^n</u> A subset W of M^n is an embedded submanifold of dimension r if for any $p \in W$, there is a chart (U, ϕ) of M such that

$$\phi(W \cap U) = \phi(U) \cap (\mathbb{R}^r \times \{0_{n-r}\})$$

ie



" W looks like a subspace of $\mathbb{R}^n$ locally "

<u>Prop</u> An embedded submanifold $W \subset M$ of dim r inherits the structure of a manifold of dim r.

<u>Pf</u> (Sketch)

- Fix an atlas $\mathcal{A}$ on M.
- For all $p \in W$ there is a chart (U_p, ϕ_p) of $\max(\mathcal{A})$ as in the definition above.
- Let $\pi_r : \mathbb{R}^n \longrightarrow \mathbb{R}^r$

$$(x^1, \ldots, x^r, \ldots, x^n) \longmapsto (x^1, \ldots, x^r)$$

- Define $V_p = U_p \cap W$ and $\psi_p = \pi_r \circ \Phi_p|_{V_p}$

- Exercise 1. (V_p, ψ_p) is a chart on W.

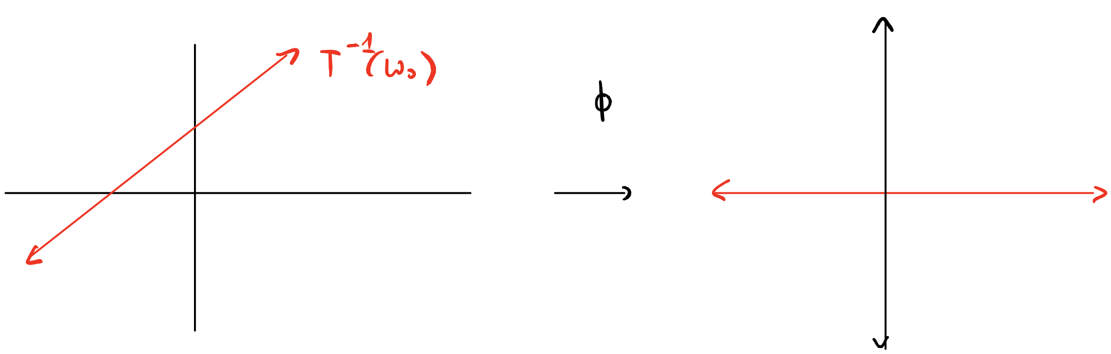
- Exercise 2 $\{(V_p, \psi_p)\}_{p \in W}$ is a smooth atlas on W.

Q. Given a smooth map $F: M \longrightarrow N$ and a $q \in N$ when is $F^{-1}(q)$ an embedded submfld of M?

<u>Ex 0</u> (In the linear case: Always!)

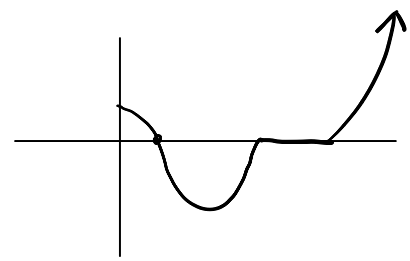
For $T: \mathbb{R}^n \to \mathbb{R}^m$ and $w_0 \in \mathbb{R}^m$

$T^{-1}(w_0)$ is an embedded submanifold. (affine subspace).



Ex 1 (Not always.)

Consider $F: \mathbb{R} \to \mathbb{R}$ with graph



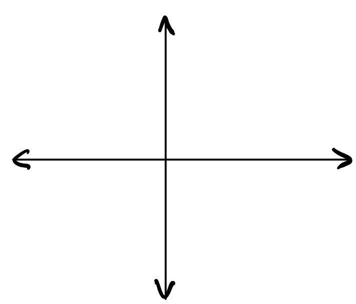
$F^{-1}(0) = \{1\} \cup [3,5]$

Ex 2 (Not always, another way)

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x^1, x^2) \longmapsto x^1 x^2$$

$$F^{-1}(0) = \{ (x^1, x^2) \mid x^1 = 0 \text{ or } x^2 = 0 \}$$

$=$



The actual answer is ... "ALMOST ALWAYS!"

Rmk By Prop, this will be an easy way to construct manifolds.

Def$^{\underline{n}}$ $p \in M$ is a regular point of $F: M^m \longrightarrow N^n$ if $F_*: T_pM \longrightarrow T_{F(p)}N$ is onto (has rank $= n$). Otherwise p is a critical point.

<u>Ex</u> $F : \mathbb{R}^m \longrightarrow \mathbb{R}$

$$[F_*] = \left(\frac{\partial F}{\partial x^i}(x)\right) = \left(\frac{\partial F}{\partial x^1}(x) \quad \dots, \quad \frac{\partial F}{\partial x^n}(x)\right)$$

x regular $\Leftrightarrow$ F_* has rank 1 $\Leftrightarrow$ $\frac{\partial F}{\partial x^i}(x) \neq 0$ for some i

x critical $\Leftrightarrow$ $\frac{\partial F}{\partial x^i}(x) = 0$ for all i

Ex $F : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$$[F_*] = \left(\frac{\partial F^j}{\partial x^i}(x)\right)$$

x is regular $\Leftrightarrow$ rank of $F_* = n \leq m$

x is critical $\Leftrightarrow$ rank of $F_* < n$

<u>Note</u> $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has rank $1 < 2$ but no zeros.

<u>Def^n</u> $q \in N$ is a <u>regular value</u> of $F : M \to N$ if each $p \in F^{-1}(q)$ is a regular point.

(Need $F(p) = q \Rightarrow [F_*]$ full rank ?

REGULAR VALUE THEOREM (RVT)

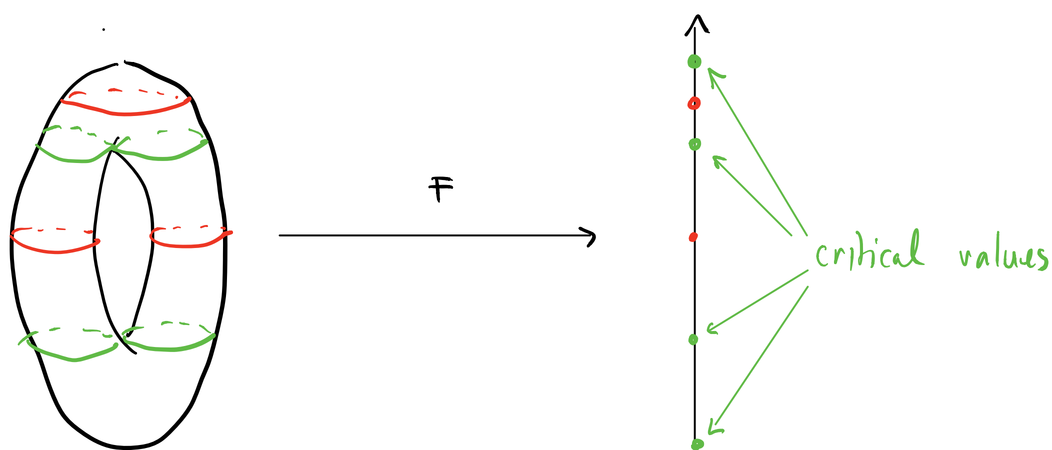
If q is a regular value of $F: M \longrightarrow N$, then $F^{-1}(q)$ is an embedded submanifold of dimension equal to $\dim(M) - \dim(N)$.

Rmk $\quad T_p F^{-1}(q) = N(F_*) \subset T_p M.$

SARD's THEOREM

Almost every $q \in N$ is a regular value of $F: M \to N$.

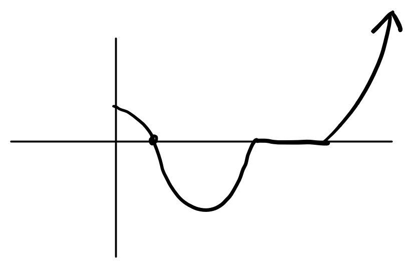
Here is a famous picture illustrating these ideas



$$S^1 \times S^1 \simeq M \subset \mathbb{R}^3 \ni x \longmapsto x^3 \in \mathbb{R}$$

We can also observe them in our previous examples.

Ex 1: Consider again the map $F: \mathbb{R} \to \mathbb{R}$ with graph



For any $\varepsilon \neq 0$ the set $F^{-1}(\varepsilon)$ is a finite collection of disjoint points, and hence a manifold of dimension 0.

Ex 2:
$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x^1, x^2) \longmapsto x^1 x^2$$

Choose $\varepsilon > 0$. Then $F^{-1}(\varepsilon)$ is an embedded submfld.

$$[F_*] = \left[\frac{\partial F}{\partial x^i}\right] = [x^2, x^1]$$

This has rank $= 1$ for $(x^1, x^2) \in F^{-1}(\varepsilon)$ since

$(x^1, x^2) \in F^{-1}(\varepsilon) \Rightarrow x^1 x^2 = \varepsilon \Rightarrow x^1 \neq 0$ and $x^2 \neq 0$