- Let $F : M \longrightarrow N$ be smooth.
- The derivative $F_* : T_pM \longrightarrow T_{F(p)}N$ is (meant to be) a linear approximation of the nonlinear map F.

Today: some nonlinear analogues to some familiar linear algebra results.

Recollections from Linear Algebra

Let $T : V \longrightarrow W$ be a linear map.

The range of T is $R(T) = \{ T(v) \in W \mid v \in V \}$

$R(T)$ is a subspace of W

$\dim(R(T))$ is also called the rank of T.

ex. $T(v) = 0_W \ \forall v \iff$ rank of T is 0

The nullspace of T is $N(T) = \{ v \in V \mid T(v) = 0_W \}$

$N(T)$ is a subspace of V.

ex $T(v) = 0_W \ \forall v \iff N(T) = V$

<u>Thm</u> (Rank-Nullity Thm)

$$\dim(N(T)) + \dim(R(T)) = \dim(V)$$

Fact 1 If T is 1-1 and onto then $\dim(W) = \dim(V)$

Pf T 1-1 $\Rightarrow N(T) = \{0_V\} \Rightarrow \dim(N(T)) = 0$

T onto $\Rightarrow R(T) = W \Rightarrow \dim(R(T)) = \dim(W)$.

Def^n $F: M \longrightarrow N$ is a diffeomorphism if it is 1-1 and onto and F^{-1} is smooth.

Ex $F: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is smooth 1-1 and onto, but $F^{-1}(x) = x^{1/3}$ is not smooth.

FACT 1 If $F: M \to N$ is a diffeomorphism then (each) $F_*: T_pM \longrightarrow T_{F(p)}N$ is an isomorphism, hence $\dim(M) = \dim(N)$.

Pf $Id_M = F^{-1} \circ F$

$\Rightarrow (Id_M)_* = (F^{-1})_* \circ F_*$ by chain rule

$\Rightarrow Id_{T_pM} = (F^{-1})_* \circ F_*$

$\Rightarrow N(F_*) \subset N(Id_{T_pM}) = \{0 \in T_pM\}$.

$\Rightarrow F_*$ is 1-1

$Id_N = F \circ F^{-1}$

$\Rightarrow Id_{T_{F(p)}N} = F_* (F^{-1})_*$

$\Rightarrow R(F_*) \supset R(Id_{T_{F(p)}N}) = T_{F(p)}N$

Hence F_* is 1-1 and onto ✓

There is a much deeper and more important partial converse to FACT 1.

INVERSE FUNCTION THEOREM (IFT)

Let $F: M^n \to N^n$ be smooth. If $F_*: T_pM \to T_{F(p)}N$

then F is a local diffeomorphism near p:

$\exists$ an open nbhd $U \ni p$ such that $F(U)$ is open in N

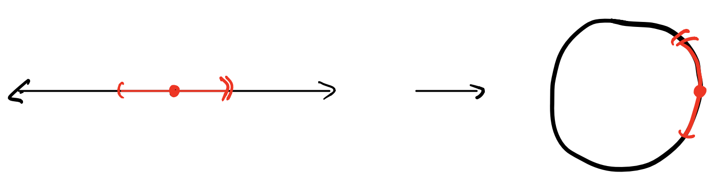
and $F: U \longrightarrow F(U)$ is a diffeomorphism.

Note the power of this result: you check a linear condition and get the existence of a nonlinear map F^{-1}.

<u>Ex</u> ("local" is necessary)

$$F: \mathbb{R} \longrightarrow S^1$$

$$t \longmapsto (\cos t, \sin t)$$



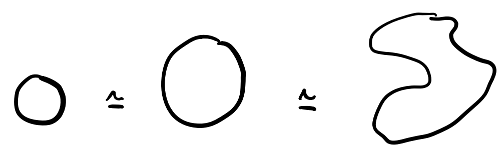
Can check that F_* is an isomorphism for all $x \in \mathbb{R}$.

But F is not 1-1 so not a diffeomorphism.

<u>Def$^{\underline{n}}$</u> M and N are diffeomorphic $(M \simeq N)$ if there is a diffeomorphism $F: M \to N$.

<u>Exercise</u> this defines an equivalence relation on manifolds.

<u>Ex</u> $\mathbb{R}^1 \simeq (0,1) \simeq (a,b) \quad \forall\, a<b$

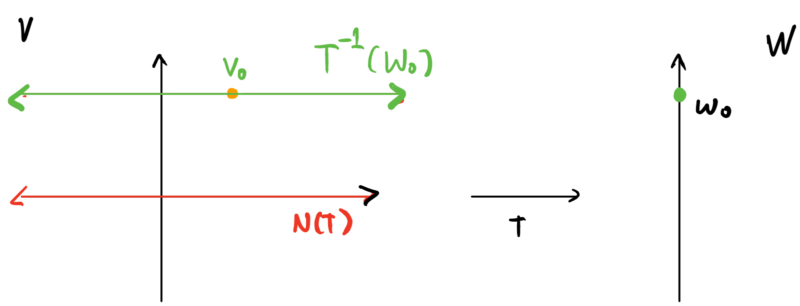
<u>Ex</u> $\mathbb{R}^1 \not\simeq S^1$, 

<u>Ex</u> $S^2 \not\simeq S^1 \times S^1 \not\simeq \mathbb{RP}^2$

<u>fact 2</u> Given $T: V \to W$ and $w_0 \in W$ consider

$T^{-1}(v_0) = \{ v \in V \mid T(v) = w_0 \}$. Given $v_0 \in T^{-1}(v_0)$

$$T^{-1}(w_0) = v_0 + N(T) = \{ v_0 + u \mid u \in N(T) \}$$



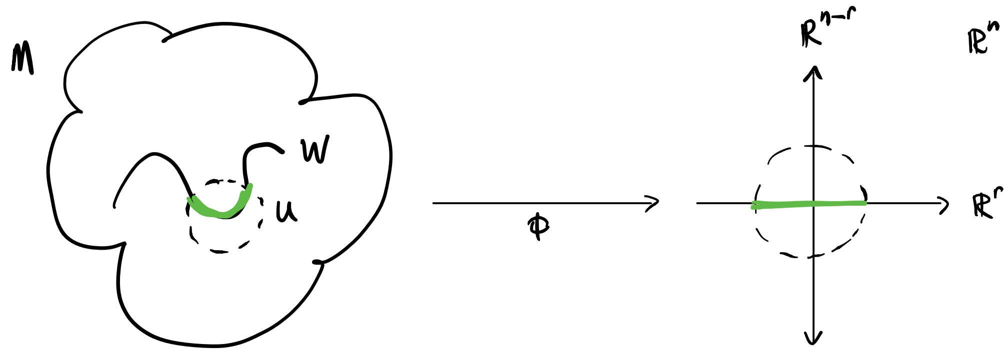
If nonempty, $T^{-1}(w_0)$ is a affine subspace of V.

Q. What can we say about $F^{-1}(q)$?

<u>Def^n</u> A subset W of M^n is an <u>embedded submanifold</u> of dimension r if for any $p \in W$, there is a chart (U, ϕ) of M such that

$$W \cap U = \phi^{-1}\left(\mathbb{R}^r \times 0_{\mathbb{R}^{n-r}}\right).$$

ie



" W looks like a subspace of $\mathbb{R}^n$ locally "

<u>Example</u> $T^{-1}(w_0)$ is an embedded submanifold of V of dimension equal to $\dim(N(T))$

Q. Given $F: M \to N$ and $q \in N$, is $F^{-1}(q)$ an embedded submanifold of M ?