Let $F : M^m \longrightarrow N^n$ be smooth.

The derivative of $F$ at $p \in M$ is the map

$$F_* : T_p M \longrightarrow T_{F(p)} N$$

defined by $(F_*(V))(f) = V(f \circ F)$

<u>Claim 1</u> $F_*$ is linear.

Need $F_*(V + cW) = F_*(V) + c\,F_*(W)$.

$$\begin{aligned}(F_*(V+cW))(f) &= (V+cW)(f \circ F) \\ &= V(f\circ F) + cW(f \circ F) \\ &= F_*(V) f + c(F_*(W) f) \\ &= (F_*(V) + c\,F_*(W))(f).\end{aligned}$$

To see derivatives we need to compute a matrix representative of $F_*$ with respect to bases given by charts on $M$ and $N$.

<u>Aside</u>

- Any linear map $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is of the form $T(x) = Ax$ for some $m \times n$ matrix $A$

ex $T\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} ax^1 + bx^2 \\ cx^1 + dx^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$

- Consider vector space $V$ and $W$ of dimensions $m$ and $n$.

  let $T: V \longrightarrow W$ be a linear map.

  To get a matrix rep of $T$ we need to make two choices

- a basis for $V$, $\alpha = \{v_1, \dots, v_m\}$ and

  a basis for $W$, $\beta = \{w_1, \dots, w_n\}$.

Then $v = a_1 v_1 + \dots + a_m v_m \longrightarrow [v]_\alpha = (a_1, \dots, a_m)^T \in \mathbb{R}^m$

$w = b_1 w_1 + \dots + b_n w_n \longrightarrow [w]_\beta = (b_1, \dots, b_n)^T \in \mathbb{R}^n$

and $T$ has an $n \times m$ matrix representative $[T]_\beta^\alpha$

defined by $[Tv]_\beta = [T]_\alpha^\beta [v]_\alpha \quad \forall \ v \in V.$

To compute $[T]_\beta^\alpha$ we find $c_{ij}$ such that

- $Tv_j = c_{1j} w_1 + \dots + c_{nj} w_n$

Then $[T]_\alpha^\beta = (c_{ij})$

($Tv_j$ determines $j^{th}$ column of $[T]_\alpha^\beta$)

Back to the derivative of $F: M \to N$ at $p$

$$F_*: T_pM \longrightarrow T_{F(p)}N$$

Choose $(U, \phi)$ with $p \in U$ and $(V, \psi)$ with $F(p) \in V$.

- $F_{UV} = \psi \circ F \circ \phi^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$$x \longmapsto (F_{UV}^1(x), \dots, F_{UV}^n(x))$$

- $\alpha = \left\{ \frac{\partial}{\partial x_U^i}\Big|_p \right\}$ basis of $T_pM$

- $\beta = \left\{ \frac{\partial}{\partial y_V^j}\Big|_{F(p)} \right\}$ basis of $T_pN$

<u>Lemma</u> $[F_*]_\alpha^\beta = \left( \frac{\partial F_{UV}^j}{\partial x^i}(\phi(p)) \right)$

<u>Pf</u> Need $F_*\left(\frac{\partial}{\partial x_U^i}\right) = \sum_j \frac{\partial F_{UV}^j}{\partial x^i}(\phi(p)) \frac{\partial}{\partial y_V^j}$

$$\left[F_*\left(\frac{\partial}{\partial x_u^i}\right)\right](f) = \frac{\partial}{\partial x_u^i}\left(f \circ F\right)$$

$$= \frac{\partial}{\partial x^i}\left(f \circ F \circ \phi^{-1}\right)$$

$$= \frac{\partial}{\partial x^i}\left(f \circ \psi^{-1} \circ \psi \circ F \circ \phi^{-1}\right)$$

$$= \frac{\partial}{\partial x^i}\left(f \circ \psi^{-1} \circ F_{uv}\right)$$

$$= \sum_j \frac{\partial}{\partial y^j}\left(f \circ \psi^{-1}\right) \frac{\partial}{\partial x^i}\left(F_{uv}^j\right) \qquad \text{chain rule}$$

$$= \sum_j \frac{\partial}{\partial x^i}\left(F_{uv}^j\right) \frac{\partial}{\partial y^j}\left(f \circ \psi^{-1}\right)$$

$$= \left[\sum_j \frac{\partial F_{uv}^j}{\partial x^i} \frac{\partial}{\partial y_v^j}\right](f)$$

<u>Example 0</u>

$$F : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$(x^1, \dots, x^m) \longmapsto (F^1(x^1, \dots, x^m), \dots, F^n(x^1, \dots, x^m))$$

$$(U, \phi) = (\mathbb{R}^m, Id_{\mathbb{R}^m}) \qquad (V, \psi) = (\mathbb{R}^n, Id_{\mathbb{R}^n})$$

$$F_{uv} = F$$

$$\left[F_*\right]^{\beta}_{\alpha} = \left( \frac{\partial F^i}{\partial x^j}(x) \right) \quad \text{the Jacobian of } F$$

<u>Example</u> $\quad F: S^2 \longrightarrow \mathbb{R}P^2$

$$(x^1, x^2, x^3) \longmapsto [x^2, 2x^1, 3x^3]$$

Find a matrix representative of $F_*$ for $p = (1, 0, 0)$.

$p \in U_1^+ \qquad F(p) = [0, 2, 0] \in U_2$

$$F_{U_1^+ U_2}(x^1, x^2) = \phi_2 \circ F \circ (\phi_1^+)^{-1}(x^1, x^2)$$

$$= \phi_2 \, F\left( \left( \sqrt{1-(x^1)^2-(x^2)^2}, \, x^1, \, x^2 \right) \right)$$

$$= \phi_2 \left[ x^1, \, 2\sqrt{1-(x^1)^2-(x^2)^2}, \, 3x^2 \right]$$

$$= \left( \frac{x^1}{2\sqrt{1-(x^1)^2-(x^2)^2}}, \, \frac{3x^2}{2\sqrt{1-(x^1)^2-(x^2)^2}} \right)$$

$$\phi_1^+(p) = \phi_1^+(1, 0, 0) = (0, 0)$$

$$\left( \frac{\partial F^j_{U_1^+ U_2}}{\partial x^i}(0,0) \right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3/2 \end{pmatrix}$$

- To apply this to $V \in T_p S^2$ one needs to write $V$

as $V = V^1 \, \partial/\partial x^1_{u_1,t} + V^2 \, \partial/\partial x^2_{u_1,t}$.

The result $F_*(V) = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/2 \end{pmatrix} \begin{pmatrix} V^1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} V_1 \\ \frac{3}{2} V_2 \end{pmatrix}$

should be interpreted as $\frac{1}{2} V_1 \, \frac{\partial}{\partial x^1_{u_2}} + \frac{3}{2} \, \frac{\partial}{\partial x^2_{u_2}}$.

<u>Fact</u> If $F : M^m \longrightarrow L^\ell$ and $G : L^\ell \longrightarrow N^n$ are smooth maps then

$$\underset{\text{at } p}{(G \circ F)_*} = \underset{\text{at } F(p)}{G_*} \circ \underset{\text{at } p}{F_*}$$

In terms of matrix representations $[G_* \circ F_*] = \underset{\text{matrix multiplication}}{\overset{n\times \ell}{[G_*]}\overset{\ell \times m}{[F_*]}}$

<u>Pf</u>

$$\left[ (G_* \circ F_*)(V) \right] f$$

$$= \left[ G_* \left( F_*(V) \right) \right] f$$

$$= F_*(V) \, ( f \circ G )$$

$$= V \, ( f \circ G \circ F )$$

$$= \left[ (G \circ F)_{*}(V) \right] f \qquad \checkmark$$