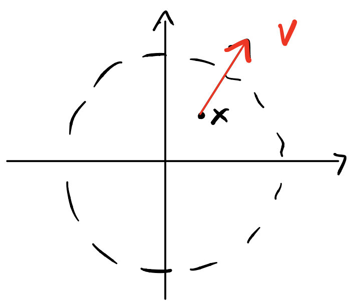
First Goal: Smooth manifolds ✓

M, $(M, [a])$ Hausdorff and countable base.

Next Goal: define T_pM, the tangent space to M at p.

<u>Example 0</u> $M = U \underset{\text{open}}{\subset} \mathbb{R}^n$.



$v \in \mathbb{R}^n_x$

Each v determines a map

$$D_v : C^\infty(M) \longrightarrow \mathbb{R}$$

$$f \longmapsto \left.\frac{d}{dt}\right|_{t=0} \Big(\underbrace{f(x + tv)}_{\text{function of } t} \Big)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)\, v^i$$

- D_v is a linear map.

Exercise: $D_v(f+g) = D_v(f) + D_v(g)$

$$D_v(cf) = c\,(D_v f).$$

- D_v satisfies the Liebnitz (product) rule at x.

$$D_v(fg) = D_v(f)\, g(x) + f(x)\, D_v(g)$$

note the x

<u>Pf</u>

$$D_v(fg) = \frac{d}{dt}\Big|_{t=0} (fg)(x+tv)$$

$$= \frac{d}{dt}\Big|_{t=0} \big(f(x+tv)\, g(x+tv)\big)$$

$$= \left(\frac{d}{dt}\Big|_{t=0} f(x+tv)\right) g(x) + f(x)\left(\frac{d}{dt}\Big|_{t=0} g(x+tv)\right)$$

$$= D_v f\; g(x) + f(x)\, D_v g.$$

Let $T_x U = \left\{ L : C^\infty(U) \to \mathbb{R} \;\middle|\; \begin{array}{l} L \text{ is linear and} \\ \text{satisfies Liebnitz rule at } x \end{array} \right\}$

<u>Fact</u> The map $\mathcal{C} : \mathbb{R}^n_x \longrightarrow T_x U$ is a bijection!

$$v \longmapsto D_v$$

(You will prove this in Q2 of HW 2.)

Def^{n}

$$T_p M = \left\{ L : C^\infty(M) \to \mathbb{R} \;\middle|\; \begin{array}{l} L \text{ linear and satisfies} \\ \text{Liebnitz rule at } p \end{array} \right\}.$$

Q. How do we get our hands on this (vector) space?

A. Curves!

Def^{n} A smooth curve in M is a map $\gamma : (a,b) \to M$ such that for any chart (U,ϕ) the map $\phi \circ \gamma : (a,b) \to \mathbb{R}^n$ is smooth.

Let γ be a smooth curve in M such that $\gamma(0) = p$.

Define $\dot{\gamma}(0) \in T_p M$ by

$$\dot{\gamma}(0)(f) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)).$$

Now a chart (U,ϕ) with $p \in M$ give us a basis for $T_p M$.

(U,ϕ) determines n curves through p.

$$\gamma^i(t) = \phi^{-1}\left(\phi(p) + (0,0,..,t,0..,0) \right)$$

Set. $\left.\frac{\partial}{\partial x_u^i}\right|_p = \dot{\gamma}^i(0).$

Note $\left.\frac{\partial}{\partial x_u^i}\right|_p f = \dot{\gamma}^i(0)\ (f)$

$$= \left.\frac{d}{dt}\right|_{t=0} f(\gamma^i(t))$$

$$= \left.\frac{d}{dt}\right|_{t=0} f \circ \phi^{-1}\left(\phi(p) + (0,0,\dots t, 0\dots, 0)\right)$$

$$= \left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} (f \circ \phi^{-1})$$

Note $\left\{ \left.\frac{\partial}{\partial x_u^i}\right|_p \right\}$ is a basis of $T_p M$

Suppose (V, ψ) is another chart w/ $p \in V$

How do we change from $\left\{ \left.\frac{\partial}{\partial x_u^i}\right|_p \right\}$ to $\left\{ \left.\frac{\partial}{\partial x_V^j}\right|_p \right\}$?

$$\left.\frac{\partial}{\partial x_u^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} f \circ \phi^{-1}(x)$$

$$= \left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} (f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})\ (x)$$

Recall the multivariable Chain Rule

$$\frac{\partial}{\partial x^i}\Big|_x \Big(h(\underbrace{y^1(x), \dots, y^n(x)}_{y(x)}) \Big) = \sum_{j=1}^{n} \frac{\partial h}{\partial y^j}(y(x)) \frac{\partial y^j}{\partial x^i}(x)$$

For $h = f \circ \psi^{-1}$, $y(x) = \psi \circ \phi^{-1}(x)$ and $x = \phi(p)$

$$\frac{\partial}{\partial x^i}\Big|_p f = \sum_{j=1}^{n} \frac{\partial}{\partial y^j}(f \circ \psi^{-1})(\psi(p)) \frac{\partial y^j}{\partial x^i}(\phi(p))$$

Now $\frac{\partial}{\partial y^j}(f \circ \psi^{-1})(\psi(p)) = \frac{\partial}{\partial x_v^j}\Big|_p f$

and $y^j = \pi_j \, \psi \circ \phi^{-1}(x) = x_v^j(\phi^{-1}(x))$

$$\Rightarrow \quad \frac{\partial y^j}{\partial x^i}(\phi(p)) = \frac{\partial}{\partial x_u^i}\Big|_p x_v^j$$

So $$\frac{\partial}{\partial x_u^i}\Big|_p f = \sum_{j=1}^{n} \frac{\partial x^j_v}{\partial x^i_u}(p) \frac{\partial}{\partial x_v^j}\Big|_p f$$

Or, in terms of linear algebra

$$\begin{pmatrix} \partial/\partial x_u^1 \\ \vdots \\ \partial/\partial x_u^n \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_v^j}{\partial x_u^i} \end{pmatrix} \begin{pmatrix} \partial/\partial x_v^1 \\ \vdots \\ \partial/\partial x_v^n \end{pmatrix}$$