Def$^{\underline{n}}$ (provisional)

A <u>smooth manifold</u> is a set M equipped with a smooth atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$.

Let's deal with the "provisional"

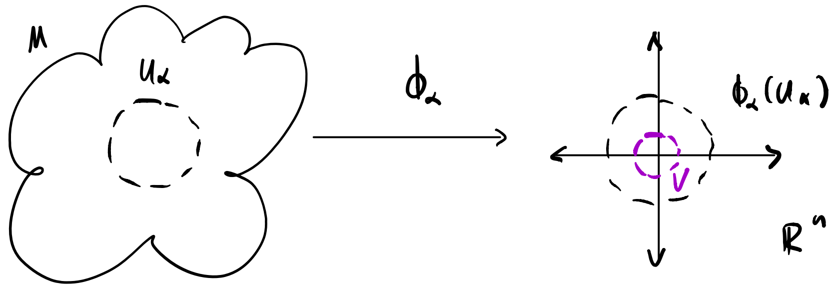
Def$^{\underline{n}}$ a chart (U, ϕ) is <u>compatible with $\mathcal{A}$</u> if it is compatible with each $(U_\alpha, \phi_\alpha) \in \mathcal{A}$.

Exercise If (U, ϕ) is compatible with $\mathcal{A}$, then $\mathcal{A} \cup \{(U, \phi)\}$ is an atlas on M.

Basic Problem It is easy to find (U, ϕ) compatible w/ $\mathcal{A}$ and we don't want to view $(M, \mathcal{A})$ and $(M, \mathcal{A} \cup \{(U, \phi)\})$ as different smooth manifolds.

Method 1 Restriction

Choose $(U_\alpha, \phi_\alpha) \in \mathcal{A}$ and $V \underset{\text{open}}{\subset} \phi_\alpha(U_\alpha)$



<u>Exercise</u> $\left(\phi_\alpha^{-1}(V), \phi_\alpha\right)$ is a chart compatible w/ $\mathcal{A}$.

<u>Method 2</u> Composition

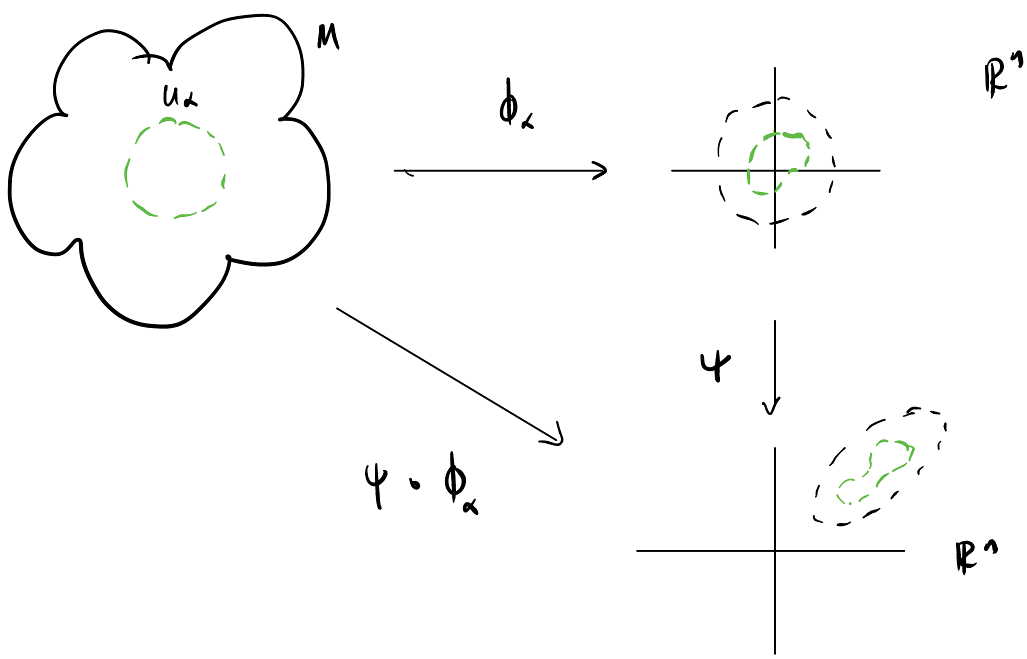
Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be smooth with smooth inverse.

<u>ex</u> If A is an invertible matrix and $c \in \mathbb{R}^n$

the map $\psi(x) = Ax + c$ is smooth as is its

inverse $\psi^{-1}(y) = A^{-1}(y - c)$

<u>Exercise</u> For $(U_\alpha, \phi_\alpha) \in \mathcal{A}$, $(U_\alpha, \psi \circ \phi_\alpha)$ is a chart compatible with $\mathcal{A}$.



Let's deal with the Basic Problem.

<u>Def$^{\underline{n}}$</u> two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are compatible ($\mathcal{A} \sim \mathcal{A}'$) if each chart of $\mathcal{A}$ is compatible with each chart of $\mathcal{A}'$.

<u>Def$^{\underline{n}}$</u> (?) A smooth manifold is a set M equipped with a equivalence class of smooth atlases, $[\mathcal{A}]$.

Now $(M, [\mathcal{A}]) = (M, [\mathcal{A} \cup (U,\phi)])$

There is still a sublte (topological) problem.

Given an atlas $\mathcal{A}$ for M add to it all charts compatible with $\mathcal{A}$ to get $\max(\mathcal{A})$.

- $\max(\mathcal{A})$ is an atlas on M
- $[\max(\mathcal{A})] \sim [\mathcal{A}]$.
- every chart compatible w/ $\max(\mathcal{A})$ is in $\max(\mathcal{A})$.
- $\mathcal{A} \sim \mathcal{A}' \iff \max(\mathcal{A}) = \max(\mathcal{A}')$.

Topology from an atlas.

Let $\mathcal{A}$ be an atlas on M. $(M, [\mathcal{A}])$ is smooth mfld.

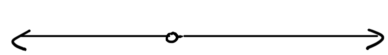
$W \subset M$ is <u>open</u> if for each $p \in W$ there is a $(U, \phi) \in \max(\mathcal{A})$ such that $p \in U \subset W$.

This collection of open sets of M defines a topology on M.

> <u>Def^{n}</u> A smooth manifold is a pair $(M, [\mathcal{A}])$ such that the (induced) topology on M is Hausdorff and has a countable base.

<u>Hausdorff</u> : For $p \neq q \in M$ there are open sets $U, V \subset M$ s.t. $p \in U$, $q \in V$ and $U \cap V = \emptyset$.

<u>Example</u> $M = \mathbb{R} \setminus \{0\} \cup \{p_1, p_2\}$



$\{p_1, p_2\}$

$U_1 = M \setminus \{p_2\}$ $\qquad$ $U_2 = M \setminus \{p_1\}$

$\phi_1 : U_1 \longrightarrow \mathbb{R}$ $\qquad$ $\phi_2 : U_2 \longrightarrow \mathbb{R}$

$x \longmapsto x$ $\qquad$ $x \longmapsto x$

$p_1 \longmapsto 0$ $\qquad$ $p_2 \longmapsto 0$

$\{(U_1, \phi_1), (U_2, \phi_2)\}$ is a smooth atlas on M

The topology on M is not Hausdorff.

Let $(M, [a])$ be a smooth manifold of dimension n.

Let $C^\infty(M) = \{$ smooth functions on $M \}$

$C^\infty(M)$ is a vector space. $\quad (f+g)(p) = f(p) + g(p)$

$(cf)(p) = c(f(p))$

Let $f : M \to \mathbb{R}$ be smooth.

Q. What is the derivative of f at $p \in M$?

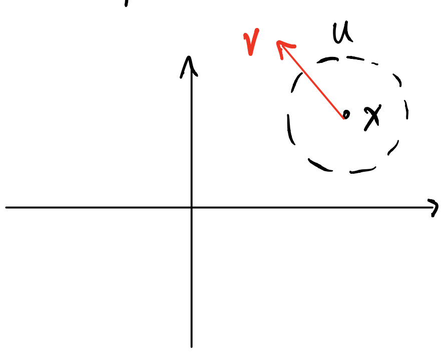
A. It is a linear map from $T_p M$ to $\mathbb{R}$.

Q. What is $T_p M$?

- By name, T_pM is the tangent space to M at p.
- T_pM is a vector space of dimension n.
- T_pM is a collection of linear maps from $C^\infty(M)$ to $\mathbb{R}$.
- Each chart (U, ϕ) with $p \in U$ determines a basis for T_pM.

Let's look at things in Example 0.

$M = U \underset{\text{open}}{\subset} \mathbb{R}^n \qquad \mathcal{A} = \{ (U, Id_U) \}$.



- A tangent vector to M at x is a vector v w/ base at x
- The space of all such vectors is a copy of $\mathbb{R}^n$ with origin at x. Hence a vector space of dimension n.

- Each v determines a map

$$D_v : C^\infty(M) \longrightarrow \mathbb{R}$$
$$f \longmapsto \left.\frac{d}{dt}\right|_{t=0} f(x+tv)$$

This map is linear.

$$D_v(f+cg) = \left.\frac{d}{dt}\right|_{t=0} (f+cg)(x+tv)$$
$$= \left.\frac{d}{dt}\right|_{t=0} \Big(f(x+tv) + c\,(g(x+tv)) \Big)$$
$$= D_v(f) + c\,D_v(g).$$

This map satisfies the product rule

$$D_v(fg) = D_v f \cdot g(x) + f(x)\, D_v g$$

<u>Fact</u> Any linear map satisfying this rule is equal to D_v for some v.

So $\quad v \longleftrightarrow D_v$

<u>Def$^{\underline{n}}$</u> For a smooth manifold $(M, [a])$, the tangent space to M at p is

$$T_p M = \left\{ L : C^\infty(M) \to \mathbb{R} \;\middle|\; \begin{array}{l} L \text{ is linear and} \\ L(fg) = L(f)g(p) + f(p)L(g) \end{array} \right\}$$