Def$^{\underline{n}}$ (provisional)

A smooth manifold is a set M equipped with a smooth atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$.

Ex $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ admits an atlas.

Ex So does $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$.

EXAMPLE

$$\mathbb{RP}^2 = \{\text{all lines through origin in } \mathbb{R}^3\}$$

$$= \{x \in \mathbb{R}^3 \setminus \{\text{origin}\}\} / \sim$$

where $x \sim y \iff x = \lambda y$ for $\lambda \neq 0$.

$$= \{[x^1, x^2, x^3]\} \quad \text{homogeneous coords.}$$

Here $[x^1, x^2, x^3]$ stands for the equivalence class of $x = (x^1, x^2, x^3)$. So $[x^1, x^2, x^3] = [\lambda x^1, \lambda x^2, \lambda x^3]$ for any $\lambda \neq 0$.

<u>Note</u> $\mathbb{RP}^2$ is comprised of subsets of $\mathbb{R}^3$ but does not live in $\mathbb{R}^3$ itself.

Let's construct a smooth atlas on $\mathbb{RP}^2$.

Set $U_1 = \{ [x^1, x^2, x^3] \mid x^1 \neq 0 \}$.

$=$ set of lines not in $x^2 x^3$-plane

Define $\phi_1 : U_1 \to \mathbb{R}^2$ by.

by $\phi_1([x^1, x^2, x^3]) = \left(\frac{x^2}{x^1}, \frac{x^3}{x^1} \right)$.

This is well defined since

$$\phi_1([\lambda x^1, \lambda x^2, \lambda x^3]) = \left(\frac{\lambda x^2}{\lambda x^1}, \frac{\lambda x^3}{\lambda x^1} \right) = \left(\frac{x^2}{x^1}, \frac{x^3}{x^1} \right)$$

Check (ϕ_1, U_1) is a chart.

- $\phi_1(U_1) = \mathbb{R}^2$ which is open. ✓

- $\phi_1([x^1, x^2, x^3]) = \phi_1([y^1, y^2, y^3])$

$\Leftrightarrow$ $\left(\frac{x^2}{x^1}, \frac{x^3}{x^1} \right) = \left(\frac{y^2}{y^1}, \frac{y^3}{y^1} \right)$

$\Leftrightarrow$ $x^2 = \frac{x^1}{y^1} y^2$ and $x^3 = \frac{x^1}{y^1} y^3$

$$\Leftrightarrow \quad (x^1, x^2, x^3) = \frac{x^1}{y^1}(y^1, y^2, y^3)$$

$$\Rightarrow \quad [x^1, x^2, x^3] = [y^1, y^2, y^3]$$

<u>Note</u> $\quad \phi_1^{-1}(u,v) = [1, u, v]$

Define (U_2, ϕ_2) and (U_3, ϕ_3) analogously

<u>Claim</u> $\quad \mathcal{A} = \{(U_i, \phi_i)\}_{i=1,2,3}$ is a smooth atlas on $\mathbb{RP}^2$.

<u>Pf</u> $\quad [x^1, x^2, x^3] \in \mathbb{RP}^2 \Rightarrow x^i \neq 0$ for some i

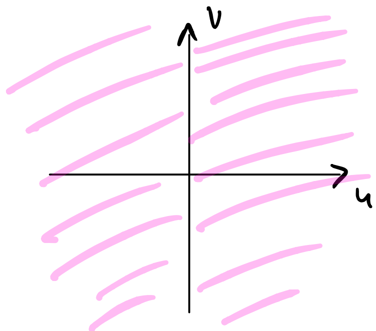
$$\Rightarrow [x^1, x^2, x^3] \in U_i$$

Consider compatibility of ϕ_1 and ϕ_2.

$$\phi_2(U_1 \cap U_2) = \phi_2\left(\{[x^1, x^2, x^3] \mid x^1 \neq 0, x^2 \neq 0\}\right)$$

$$= \left\{\left(\frac{x^1}{x^2}, \frac{x^3}{x^2}\right) \,\middle|\, x^1 \neq 0, x^2 \neq 0\right\}$$

$$= \{(u,v) \mid u \neq 0\}$$



<u>Exercise</u> this is open.

Now $\phi_1 \circ \phi_2^{-1}(u,v) = \phi_1([u,1,v])$

$$= \left(\frac{1}{u}, \frac{v}{u}\right)$$

This is smooth on its domain $(u \neq 0)$.

Consider the function $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}$

$$[x^1, x^2, x^3] \longmapsto \frac{(x^1)^2}{(x^1)^2 + (x^2)^2 + (x^3)^2}$$

Note this function is well-defined since

$$f([\lambda x^1, \lambda x^2, \lambda x^3]) = f([x^1, x^2, x^3])$$

Determine if f is smooth.

Need to check if $f \circ \phi_i^{-1} : \phi_i(U_i) \longrightarrow \mathbb{R}$ are smooth.

$$\phi_1([x^1, x^2, x^3]) = \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}\right)$$

$$\phi_1^{-1}(u,v) = [1, u, v]$$

$$f \circ \phi_1^{-1}(u,v) = \frac{1}{1+u^2+v^2}$$ which is smooth on $\mathbb{R}^2$

$$f \circ \phi_2^{-1}(u,v) = \frac{u^2}{u^2+1+v^2} = f \circ \phi_3^{-1}(u,v)$$

These are also smooth. Hence $f : \mathbb{RP}^2 \longrightarrow \mathbb{R}$ is smooth.

Let's deal with "provisional" part of definition of manifold

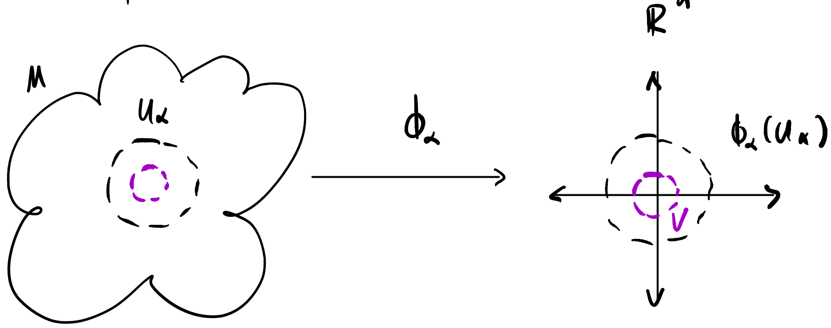
Let $\mathcal{A}$ be an atlas on M.

<u>Def^{n}</u> a chart (U, ϕ) is <u>compatible with $\mathcal{A}$</u> if it is compatible with each $(U_\alpha, \phi_\alpha) \in \mathcal{A}$.

<u>Exercise</u> If (U, ϕ) is compatible with $\mathcal{A}$, then $\mathcal{A} \cup \{(U, \phi)\}$ is an atlas on M.

It is easy to produce charts (U, ϕ) compatible with $\mathcal{A}$

<u>Example 1</u> (Restriction) Given $(U_\alpha, \phi_\alpha) \in \mathcal{A}$ choose $\underset{\text{open}}{V} \subset \phi_\alpha(U_\alpha)$.



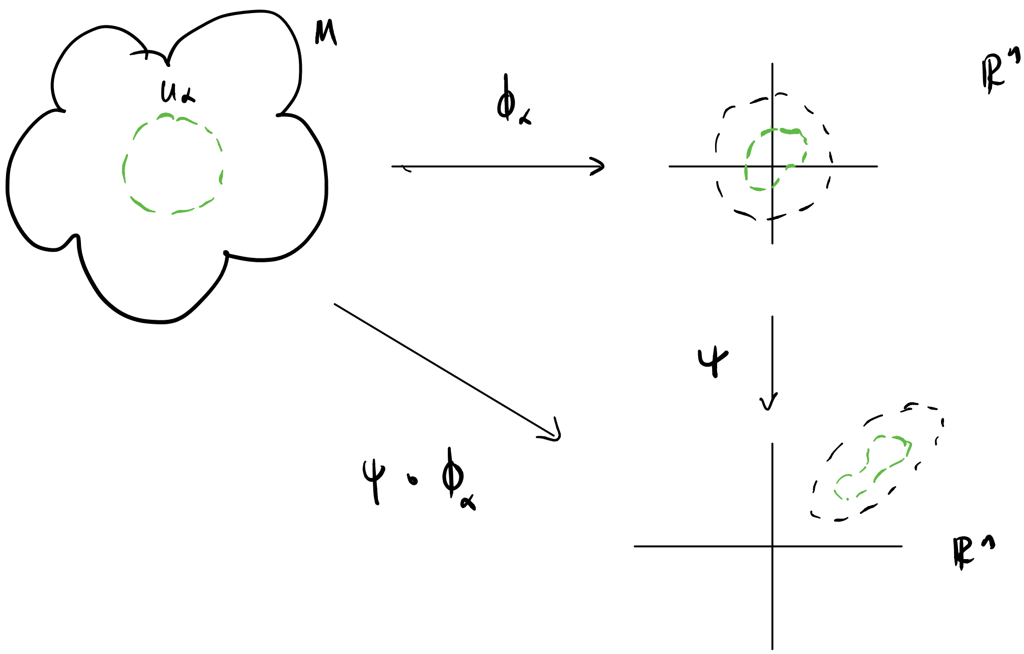
<u>Exercise</u> $(\phi_\alpha^{-1}(V), \phi_\alpha)$ is a chart compatible w/ $\mathcal{A}$.

<u>Example 2</u> Composition

Let $\Psi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be smooth with a smooth inverse.
$\Psi^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

ex. $\Psi x = Ax + c$ where A is invertible $n \times n$ matrix.
and $c \in \mathbb{R}$

Then for $(U_\alpha, \phi_\alpha) \in \mathcal{A}$, $(U_\alpha, \psi \circ \phi_\alpha)$ is a chart compatible with $\mathcal{A}$.



We <u>DO NOT</u> want to view $(M, \mathcal{A})$ and $(M, \mathcal{A} \cup \{(U, \phi)\})$ as different smooth manifolds.

For example $f : M \to \mathbb{R}$ is smooth wrt $\mathcal{A}$ iff it is smooth w.r.t. $\mathcal{A} \cup \{(U, \phi)\}$.

<u>Def^n</u> two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are compatible $(\mathcal{A} \sim \mathcal{A}')$ if each chart of $\mathcal{A}$ is compatible with each chart of $\mathcal{A}'$.

<u>Next try</u> A smooth manifold is a set M and an equivalence class of smooth atlases $[\mathcal{a}]$ on M.

$$(M, [\mathcal{a}])$$

Now $(M, [\mathcal{a}]) = (M, [\mathcal{a} \cup \{(U,\phi)\}])$

Almost ... there is one other matter to take care of.

- Given $\mathcal{a}$, add to it ALL charts compatible to $\mathcal{a}$ to get $\max(\mathcal{a})$. Note $[\mathcal{a}] = [\max(\mathcal{a})]$.

<u>Fact</u> If $\mathcal{a} \sim \mathcal{a}'$, then $\max(\mathcal{a}) = \max(\mathcal{a}')$.

- Given $(M, [\mathcal{a}])$ define a subset $W \subset M$ to be open if for any $p \in W$ there is a $(U, \phi) \in \max(\mathcal{a})$ such that $p \in U \subset W$ and $\phi(U)$ is open.

- This collection of open subsets of M defines a topology. We need it to have two basic properties.

Def^{n} A smooth manifold is a pair $(M, [\mathcal{a}])$ such that the induced topology on M is Hausdorff and has a countable base.

Ex Our atlas $\mathcal{a} = \{ (U_i^{\pm}, \phi_i^{\pm}) \}$ on S^n defines the smooth manifold $(S^n, [\mathcal{a}])$

Remark We will not need to consider these topological restrictions again

Hausdorff: For $p \neq q \in M$ there are open sets $U, V \subset M$ s.t. $p \in U$, $q \in V$ and $U \cap V = \emptyset$.