Def$^{\underline{n}}$ A smooth atlas on the set M is a collection of charts $\mathcal{A} = \{ (U_\alpha, \phi_\alpha) \}_{\alpha \in A}$ such that

I) $\bigcup_{\alpha \in A} U_\alpha = M$

II) Any two charts in $\mathcal{A}$ are compatible.

$$\left(\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha (U_\alpha \cap U_\beta) \to \mathbb{R}^n \quad \text{smooth} \right)$$

(Assumed that $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ for fixed n .)

Def$^{\underline{n}}$ (provisional)

A smooth manifold is a set M equipped with a smooth atlas $\mathcal{A}$.

Rmk $\dim (M, \mathcal{A}) = n$

Rmk Not all sets admit smooth atlases!

$\underline{\text{Def}^{\underline{n}}}$ Given $(M, \mathcal{A})$, a function $f : M \to \mathbb{R}$ is smooth near p if for some $(U_\alpha, \phi_\alpha) \in \mathcal{A}$ with $p \in U_\alpha$ the map $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}$ is smooth near $\phi_\alpha(p)$.

Examples of "smooth manifolds".

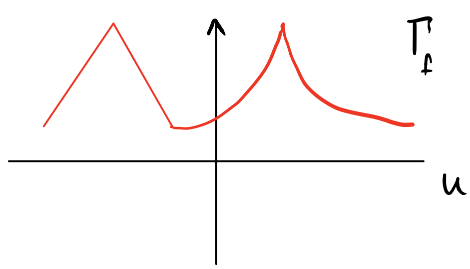
<u>Ex 0</u> $U \subset \mathbb{R}^n$ open

let
$$\begin{aligned} Id_U : U &\longrightarrow \mathbb{R}^n \\ x &\longmapsto x \end{aligned}$$

Then $\mathcal{A} = \{ (U, Id_U) \}$ is a smooth atlas

<u>Ex 1</u> For $f : U \overset{\text{open}}{\subset} \mathbb{R}^n \longrightarrow \mathbb{R}$ continuous, define.

$$\Gamma_f = \{ (x, f(x)) \mid x \in U \} \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}.$$



Let $\pi : \Gamma_f \longrightarrow \mathbb{R}^n$

$(x, f(x)) \longmapsto x$

<u>Claim</u> $\mathcal{A} = \{ (\Gamma_f, \pi) \}$ is a smooth atlas.

<u>Pf</u> I) $\Gamma_f = \Gamma_f$ $\quad (\bigcup_\alpha U_\alpha = M)$

II) $\pi \circ \pi^{-1} : \pi(\Gamma_f) \longrightarrow \mathbb{R}^n$

is the map Id_U which is smooth.

<u>Ex 2</u> $M = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \} = S^2$

Recall our previous chart

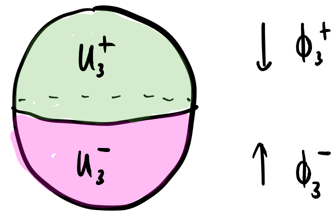
$U = \{ x \in S^2 \mid x^3 > 0 \}$ $\quad \phi_U(x) = (x^1, x^2)$

Rename this (U_3^+, ϕ_3^+).

Define $U_3^- = \{ x \in S^2 \mid x_3 < 0 \}$

and $\phi_3^- : U_3^- \longrightarrow \mathbb{R}^2$

$(x^1, x^2, x^3) \longmapsto (x^1, x^2)$



Define $(U_1^{\pm}, \phi_1^{\pm})$ and $(U_2^{\pm}, \phi_2^{\pm})$ analagously

<u>Claim</u> $\mathcal{A} = \{(U_i^{\pm}, \phi_i^{\pm})\}_{i=1,2,3}$ is a smooth atlas on S^2

<u>Pf</u>

Each $(U_i^{\pm}, \phi_i^{\pm})$ is a chart.

- Need $U_i^{\pm}$ to cover S^2.

$x \in S^2 \Leftrightarrow \|x\| = 1$

$\Rightarrow x_i \neq 0$ for some i

$\Rightarrow x \in U_i^+$ or U_i^-

- Need compatibility

For ϕ_i^+ and ϕ_i^- we have $U_i^+ \cap U_i^- = \emptyset$

Need to check compatibility of $\phi_i^{\pm}, \phi_j^{\pm}$ for $i \neq j$

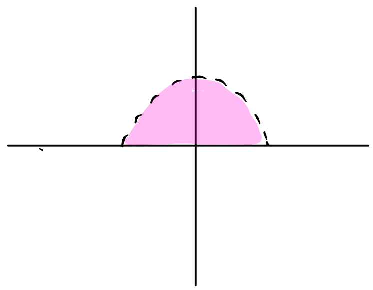
Consider ϕ_1^+ and ϕ_2^-.

Need $\phi_2(U_1^+ \cap U^-)$ to be open.

$$\phi_2(U_1^+ \cap U_2^-) = \phi_2\left(\{x \in S^2 \mid x_1 > 0,\ x_2 < 0\}\right)$$

$$= \{(x_1, x_3) \in B_1^2((0,0)) \mid x_1 > 0,\ \cancel{x_2 < 0}\}$$

$$= \{(u,v) \in B_1^2((0,0)) \mid u > 0\}$$



Exercise: this is an open subset of $\mathbb{R}^2$.

Need $\phi_1^+ \circ (\phi_2^-)^{-1}$ to be smooth on $\phi_2^-(U_1^+ \cap U_2^-)$.

$$\phi_1^+ \circ (\phi_2^-)^{-1}(u,v) = \phi_1^+\left(u, -\sqrt{1-u^2-v^2}, v\right)$$

$$= \left(-\sqrt{1-u^2-v^2}, v\right)$$

This is smooth on the domain of $\phi_1^+ \circ (\phi_2^-)^{-1}$, where $u^2+v^2 < 1$.

<u>Exercise</u> For any $n \geq 1$ set

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

Define a smooth atlas $\mathcal{A} = \{(U_i^{\pm}, \phi_i^{\pm})\}_{i=1,\dots,n+1}$

<u>Example</u> (Products)

Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ be a smooth atlas on M and

$\mathcal{B} = \{(V_\beta, \psi_\beta)\}_{\beta \in B}$ be a smooth atlas on N

On $M \times N = \{ (p,q) \mid p \in M, q \in N \}$ we can define

$$\mathcal{A} \times \mathcal{B} = \left\{ \left(U_\alpha \times V_\beta \,,\, \phi_\alpha \times \psi_\beta \right) \right\}_{(\alpha,\beta) \in A \times B}$$

where $\phi_\alpha \times \psi_\beta \,(p,q) = (\phi_\alpha(p), \psi_\beta(q)) \in \mathbb{R}^m \times \mathbb{R}^n$

<u>Exercise</u> $\mathcal{A} \times \mathcal{B}$ is a smooth atlas on $M \times N$.

eg. $S^1 \times S^1$, $S^2 \times S^1$, $S^1 \times S^2$, $(S^1 \times S^1) \times S^1$, ...

can all be equipped with smooth atlases.

EXAMPLE

$\mathbb{RP}^2 = \{$all lines through origin in $\mathbb{R}^3\}$

$= \{ x \in \mathbb{R}^3 \setminus \{\text{origin}\} \} / \sim$

where $x \sim y \iff x = \lambda y$ for $\lambda \neq 0$.

$= \{ [x^1, x^2, x^3] \}$ homogeneous coords.

Here $[x^1, x^2, x^3]$ stands for the equivalence class of $x = (x^1, x^2, x^3)$. So $[x^1, x^2, x^3] = [\lambda x^1, \lambda x^2, \lambda x^3]$ for any $\lambda \neq 0$.

<u>Note</u> $\mathbb{RP}^2$ is comprised of subsets of $\mathbb{R}^3$ but does not live in $\mathbb{R}^3$ itself.

Let's construct a smooth atlas on $\mathbb{RP}^2$.

Set $U_1 = \{ [x^1, x^2, x^3] \mid x^1 \neq 0 \}$.

$=$ set of lines not in $x^2 x^3$-plane

Define $\phi_1 : U_1 \to \mathbb{R}^2$ by.

by $\phi_1([x^1, x^2, x^3]) = \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}\right)$.

This is well defined since

$$\phi_1([\lambda x^1, \lambda x^2, \lambda x^3]) = \left(\frac{\lambda x^2}{\lambda x^1}, \frac{\lambda x^3}{\lambda x^1}\right) = \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}\right).$$

Check (ϕ_1, U_1) is a chart.

- $\phi_1(U_1) = \mathbb{R}^2$ which is open.
- $\phi_1([x^1, x^2, x^3]) = \phi_1([y^1, y^2, y^3])$

$$\Leftrightarrow \quad \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}\right) = \left(\frac{y^2}{y^1}, \frac{y^3}{y^1}\right)$$

$$\Leftrightarrow \quad x^2 = \frac{x^1}{y^1} y^2 \quad \text{and} \quad x^3 = \frac{x^1}{y^1} y^3$$

$$\Leftrightarrow \quad (x^1, x^2, x^3) = \frac{x^1}{y^1} (y^1, y^2, y^3)$$

$$\Rightarrow \quad x \sim y.$$

<u>Note</u> $\phi_1^{-1}(u, v) = [1, u, v]$

Define (U_2, ϕ_2) and (U_3, ϕ_3) analogously

Claim $\mathcal{A} = \{(U_i, \phi_i)\}_{i=1,2,3}$ is a smooth atlas on $\mathbb{RP}^2$.

Pf $[x^1, x^2, x^3] \in \mathbb{RP}^2 \Rightarrow x^i \neq 0$ for some i

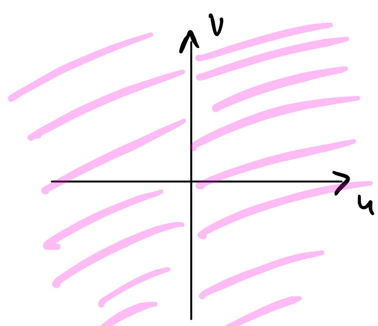
$\Rightarrow [x^1, x^2, x^3] \in U_i$

Consider compatibility of ϕ_1 and ϕ_2.

$$\phi_2(U_1 \cap U_2) = \phi_2\left(\{[x^1, x^2, x^3] \mid x^1 \neq 0, x^2 \neq 0\}\right)$$

$$= \left\{\left(\frac{x^1}{x^2}, \frac{x^3}{x^2}\right) \mid x^1 \neq 0, x^2 \neq 0\right\}$$

$$= \{(u, v) \mid u \neq 0\}$$



Exercise this is open.

Now $\phi_1 \circ \phi_2^{-1}(u, v) = \phi_1([u, 1, v])$

$$= \left(\frac{1}{u} , \frac{v}{u} \right)$$

This is smooth on its domain $(u \neq 0)$. $\checkmark$