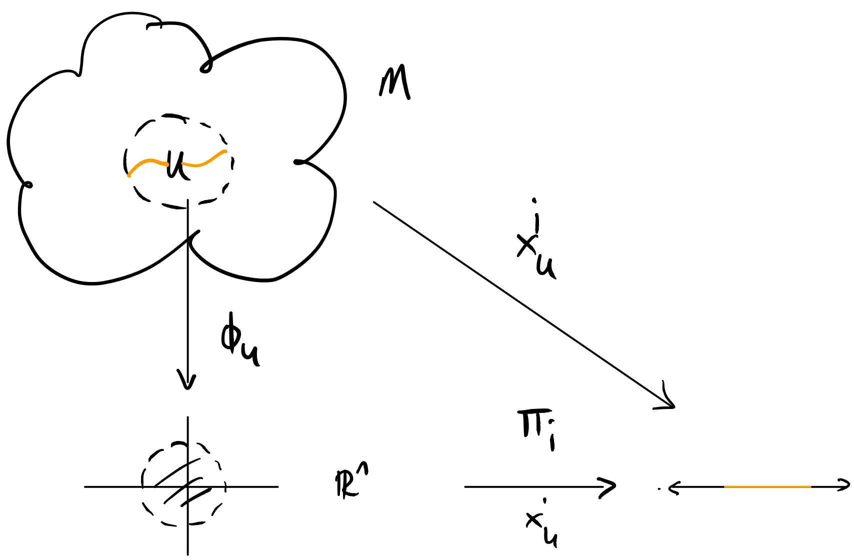
Let M be a set.

eg $\quad M = \{ \text{possible configurations of a robot} \}$

Q. When can define what it means for $f : M \to \mathbb{R}$ to be continuous or smooth?

<u>Def$^{\underline{n}}$</u> A <u>coordinate chart</u> on M is a subset $U \subset M$ and a map $\phi_U : U \longrightarrow \mathbb{R}^n$ such that $\phi_U(U) \subset \mathbb{R}^n$ is open and ϕ_U is 1-1.



<u>Rmk</u> Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be projection $\pi_i(x^1, .., x^n) = x^i$

Given (U, ϕ_U) we get n local coordinate functions

$$x^i_U : U \longrightarrow \mathbb{R}$$
$$p \longmapsto \pi_i \circ \phi_U(p)$$

Ex $\quad M = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \}$

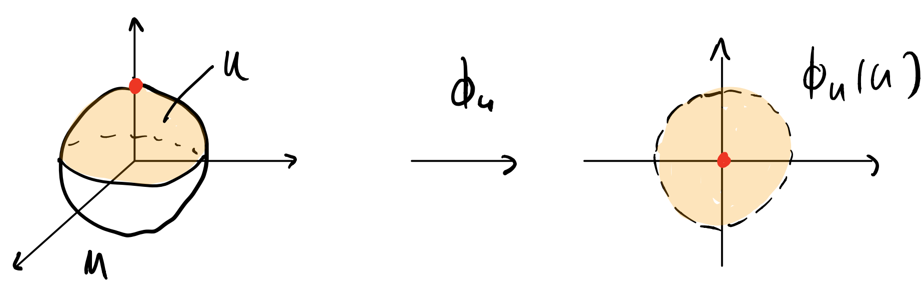
$= \{ (x^1, x^2, x^3) \mid \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = 1 \}$

$=$ unit sphere, S^2

Let $\mathcal{U} = \{ x \in S^2 \mid x^3 > 0 \}$ (Northern hemisphere)

and $\phi_u : \mathcal{U} \longrightarrow \mathbb{R}^2$

$(x^1, x^2, x^3) \longrightarrow (x^1, x^2)$



Claim ϕ_u is a coordinate chart

$\phi_u(\mathcal{U}) = \{ (x^1, x^2) \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1, \; x^3 > 0 \}$

$= \{ (x^1, x^2) \mid (x^1)^2 + (x^2)^2 < 1 \}$

$= B_1^2((0,0))$ which is open.

Need ϕ_u to be 1-1.

Suppose $x, y \in \mathcal{U}$ and $\phi_u(x) = \phi_u(y)$.

$\Rightarrow \quad x = \left(x^1, x^2, \sqrt{1-(x^1)^2-(x^2)^2}\right), \quad y = \left(y^1, y^2, \sqrt{1-y_1^2-y_2^2}\right)$

and $(x^1, x^2) = (y^1, y^2)$.

$\Rightarrow \quad x = y$.

Note $\quad \phi_u^{-1}(u, v) = \left(u, v, \sqrt{1-u^2-v^2}\right)$

<u>Attempt 1</u> $\quad f : M \longrightarrow \mathbb{R}$ is smooth near $p \in M$

if there is a chart (U, ϕ_U) with $p \in U$

such that $\quad f \circ \phi_U^{-1} : \phi_U(U) \longrightarrow \mathbb{R}$

is smooth.

<u>Problems</u>

P1) Need a chart around (each) $p \in M$.

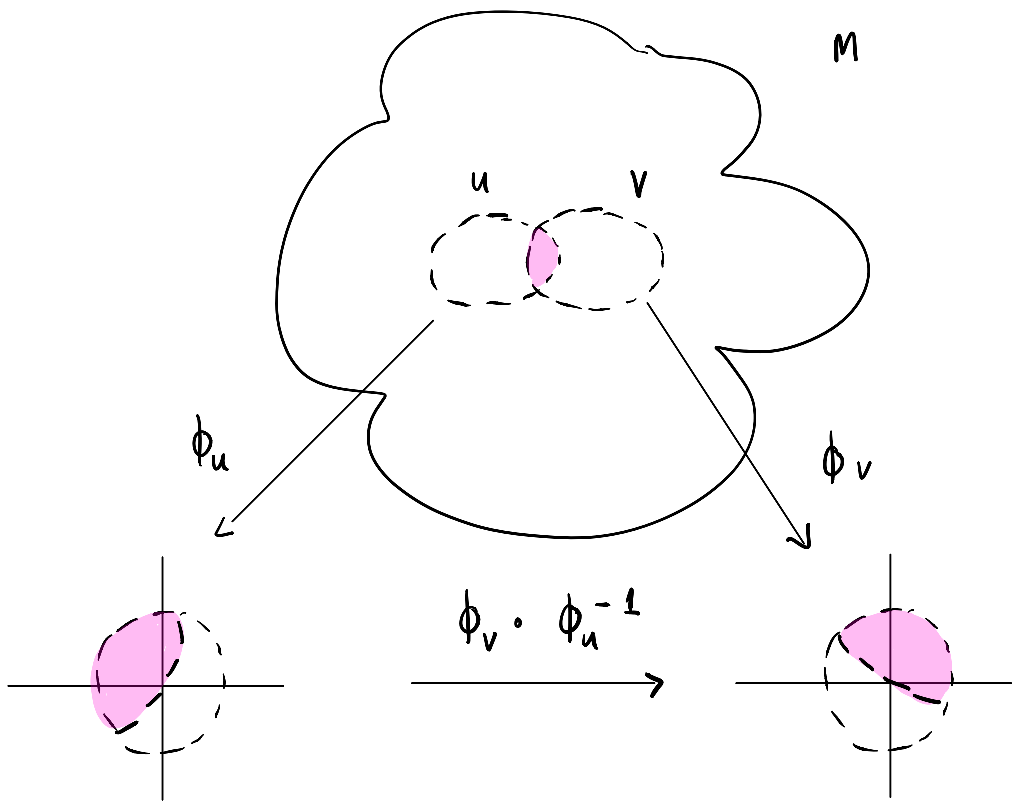
P2) Smoothness near p should be independent of choice of this chart.

<u>Def^n</u> Two charts (U, ϕ_U) and (V, ϕ_V) on M

are (smoothly) <u>compatible</u> if either $U \cap V = \emptyset$

or if $\quad \phi_U(U \cap V) \subset \mathbb{R}^n$ is open and

$\phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \longrightarrow \mathbb{R}^n$ is smooth.



Def$^{\underline{n}}$ A smooth atlas on M is a collection of charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ such that

I) $\bigcup_{\alpha \in A} U_\alpha = M$

II) Any two charts in $\mathcal{A}$ are compatible

"Def$^{\underline{n}}$" a smooth manifold (of dimension n) is a set M equipped a smooth atlas $\mathcal{A}$ (where each

$\phi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$).

$(M, \mathcal{A})$.

<u>Def$^{\underline{n}}$</u> A function $f : M \longrightarrow \mathbb{R}$ is <u>smooth near p</u> (w.r.t. $\mathcal{A}$) if for any $(U_\alpha, \phi_\alpha) \in \mathcal{A}$ with $p \in U_\alpha$ the function

$$f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \longrightarrow \mathbb{R}$$

is smooth.

~~P1~~ $p \in U_\alpha$ for some $\alpha \in A$ by (I).

~~P2~~ Suppose $p \in U_\alpha \cap U_\beta$ and

$f \circ \phi_\alpha^{-1}$ is smooth. We need

$f \circ \phi_\beta^{-1}$ to also be smooth.

$$f \circ \phi_\beta^{-1} = f \circ \left(\phi_\alpha^{-1} \circ \phi_\alpha\right) \circ \phi_\beta^{-1}$$

$$= \underset{\substack{\uparrow \\ \text{smooth by} \\ \text{assumption}}}{\left(f \circ \phi_\alpha^{-1}\right)} \circ \underset{\substack{\uparrow \\ \text{smooth by} \\ \text{II}}}{\left(\phi_\alpha \circ \phi_\beta^{-1}\right)}$$

<u>Exercise</u> The composition of smooth maps is smooth.

This follows from chain rule and properties of continuity.