481 Vector and Tensor Analysis

Lecture 1

Recall some of the Wonders of Calculus.

<u>Optimization</u>

Given $f: \underset{\substack{\Downarrow \\ x = (x^1, \dots, x^n)}}{\mathbb{R}^n} \longrightarrow \mathbb{R}$, find $\max(f)$.

Suppose f is continuous and has continuous partial derivatives

x_0 is a maximum (or minimum) value of f only if

$$\frac{\partial f}{\partial x^i}(x_0) = 0 \quad \text{for } i = 1, \dots, n.$$

(n equations in n-unknowns (nonlinear))

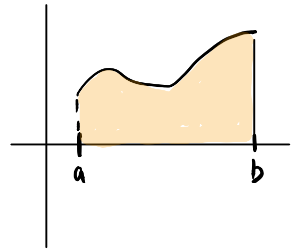
- We can look for maxima by starting at a point y_0 and moving in the direction

$$\nabla f(y_0) = \left(\frac{\partial f}{\partial x^1}(y_0), \dots, \frac{\partial f}{\partial x^n}(y_0) \right)$$

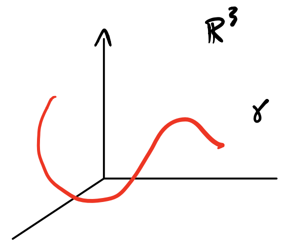
Basis of Newton's method and all Greedy Algorithms.

With integral we can also define/compute geometric quantities

0) $$\int_a^b f(x)\,dx =$$

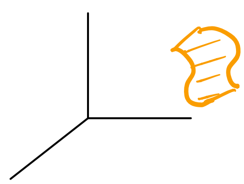


1) $$\gamma : [0,1] \longrightarrow \mathbb{R}^3$$



$$\int_0^1 \|\dot{\gamma}(t)\|\,dt = \text{length of } \gamma$$

2) $$u : [0,1] \times [0,1] \longrightarrow \mathbb{R}^3$$



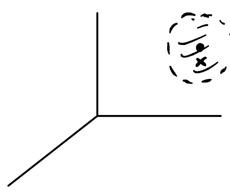
$$\int_0^1\int_0^1 \left\| \frac{\partial u}{\partial s} \times \frac{\partial u}{\partial t} \right\| ds\,dt = \text{surface area of } u.$$

...

Task 1. Generalize from $\mathbb{R}^n$ to smooth manifolds.

Some background definitions

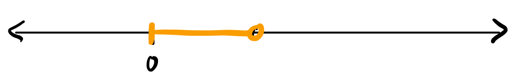
<u>Def$^{\underline{n}}$</u> $B_\varepsilon^n(x) = \{ y \in \mathbb{R}^n \mid \|y-x\| < \varepsilon \}$



<u>Def$^{\underline{n}}$</u> $U \subset \mathbb{R}^n$ is <u>open</u> if for each $x \in U$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

<u>ex</u> Each $B_\varepsilon(x)$ is open.

<u>ex</u> $[0,1) \subset \mathbb{R}^1$ is not open.



<u>Def$^{\underline{n}}$</u> $f : \overset{\text{open}}{U \subset \mathbb{R}^n} \longrightarrow \mathbb{R}$ is <u>continuous</u> (C^0) if

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \text{for all } x_0 \in U$$

<u>Def$^{\underline{n}}$</u> $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is <u>differentiable</u> (C^1) if

f is continuous and each $\frac{\partial f}{\partial x^i} : U \longrightarrow \mathbb{R}$

$$\frac{\partial f}{\partial x^i}(y) = \lim_{\varepsilon \to 0} \frac{f(y + (0,..,\varepsilon,..,0)) - f(y)}{\varepsilon}$$

exist and is continuous

<u>Def$^{\underline{n}}$</u> $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is <u>smooth</u> (C^∞) if

$$\frac{\partial^k f}{(\partial x^1)^{k_1} \cdots (\partial x^n)^{k_n}} : U \longrightarrow \mathbb{R}$$

exists and is continuous for all $k = 1, 2,$ and

$k_1 + \cdots + k_n = k$.

<u>Def$^{\underline{n}}$</u>

$$F : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$$
$$(x^1,..,x^n) \longmapsto (f_1(x^1,..,x^n), \ldots, f_k(x^1,..,x^n))$$

is <u>smooth</u> if each f_i is smooth.

UNLESS OTHERWISE STATED ALL MAPS ARE SMOOTH.

<u>ex</u> $f(x) = \sqrt{1-x^2}$ is not smooth on $\mathbb{R}$ since

$f'(x) = -x(1-x^2)^{-1/2}$ is not continuous at $x = \pm 1$.

$f(x)$ is smooth on $U = (-1, 1)$.

ex If f, g are smooth on $U \subset \mathbb{R}^n$

f/g is smooth on $U \setminus \{x \in U \mid g(x) = 0\}$.

Let M be a set.

eg $M = \{$ configurations of a robot $\}$

The set of functions any set M makes sense.

What about set of continuous (smooth) functions on M?

Need subsets of M to look like open subsets of $\mathbb{R}^n$.

Def$^{\underline{n}}$ A coordinate chart on M is a subset $U \subset M$ and a map $\phi_U : U \longrightarrow \mathbb{R}^n$ such that ϕ_U is 1-1 and $\phi_U(U) \subset \mathbb{R}^n$ is open.

Recall $\phi : U \to V$ is 1-1 if

$$\phi(x) = \phi(y) \Rightarrow x = y$$

$$(x \neq y \Rightarrow \phi(x) \neq \phi(y))$$

+ If ϕ is 1-1, then $\phi^{-1} : \phi(U) \longrightarrow U$, $\phi(x) \longmapsto x$ is well-defined