Math 481: Homework 10

1. Prove that a connection ∇ is torsion free, that is,

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad \forall X, Y,$$

if and only if its Christoffel symbols satisfy

$$\Gamma^k_{ij} = \Gamma^k_{ji} \quad \forall i, j, k.$$

2. Any metric g on M can be used to measure the angle between two vectors $X, Y \in T_pM$:

$$\measuredangle_g(X, Y) = \arccos\left(g\left(\frac{X}{\|X\|_g}, \frac{Y}{\|Y\|_g}\right)\right).$$

(Think of the formula $X \cdot Y = \|X\| \|Y\| \cos\theta$.)

   (a) Consider the following two metrics on $\mathbb{H}$:
   - the standard metric $g_0 = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$,
   - the hyperbolic metric $g_1 = \frac{1}{(x^2)^2}\left(dx^1 \otimes dx^1 + dx^2 \otimes dx^2\right)$.

   Show that, for any $X, Y \in T_x\mathbb{H}$, we have

$$\measuredangle_{g_0}(X, Y) = \measuredangle_{g_1}(X, Y).$$

   (b) Construct a "triangle" in $\mathbb{H}$ whose three sides are parts of geodesics for the hyperbolic metric g_1 (as described in Lecture 36). Compute the sum of the interior angles of your triangle. *Hint:* You don't need to parametrize the sides of the triangle as geodesics to compute angles.

3. Derive the geodesic equations for the metric

$$g = \frac{1}{(x^2)^2}\left(dx^1 \otimes dx^1\right) + \frac{1}{(x^1)^2}\left(dx^2 \otimes dx^2\right)$$

on the manifold $E = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^1 > 0,\ x^2 > 0\}$.

4. Use the Flow Box Theorem (Lecture 34) to prove Cartan's Magic Formula

$$\mathscr{L}_X \alpha = d(\alpha(X, \cdot)) + (d\alpha)(X, \cdot)$$

in the case when $\alpha \in \Lambda^2(\mathbb{R}^n)$.