Math 481: Homework 9

1. Consider the following vector fields on $\mathbb{R}^2$:

   - $X(x) = x^1 \frac{\partial}{\partial x^1} + 3\frac{\partial}{\partial x^2}$,
   - $Y(x) = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$.

   (a) Compute the Lie bracket $[X, Y]$.

   (b) Compute ϕ_t^X and the Lie derivative

   $$\mathscr{L}_X Y = \lim_{t\to 0} \frac{\left(\phi_{-t}^X\right)_* Y(\phi_t^X(x)) - Y(x)}{t}$$

   (c) For the 1-form $\alpha(x) = x^1\, dx^2$ compute

   $$\mathscr{L}_X \alpha = \lim_{t\to 0} \frac{\left(\left(\phi_t^X\right)^* \alpha\right)(x) - \alpha(x)}{t}$$

   (d) Compute $d(\alpha(X)) + d\alpha(X, \cdot)$.

2. Consider a general 2-form on $\mathbb{R}^3$

   $$\omega = A\left(x^1, x^2, x^3\right) dx^1 \wedge dx^2 + B\left(x^1, x^2, x^3\right) dx^1 \wedge dx^3 + C\left(x^1, x^2, x^3\right) dx^2 \wedge dx^3.$$

   (a) Write down explicit conditions on A, B, and C that are equivalent to $d\omega = 0$.

   (b) Write down explicit conditions on A, B, and C that are equivalent to the condition that $\omega = d\alpha$ for some 1-form $\alpha = F\, dx^1 + G\, dx^2 + H\, dx^3$.

   (c) Show that your conditions from (b) imply those from (a).

3. Bonus Problem. Consider the twice punctured plane

   $$M = \mathbb{R}^2 \setminus \{(-2, 0), (2, 0)\}.$$

   Construct two 1-forms α, β on M that satisfy all of the following conditions:

   (a) $d\alpha = d\beta = 0$, and

   (b) α and β are not exact, and

   (c) $\alpha \neq c\beta$ for any $c \in \mathbb{R}$, and

   (d) $\alpha - \beta$ is not exact.