Math 481: Homework 8

1. Let M be a manifold with boundary. Prove that its interior, $\mathrm{int}(M)$, inherits the structure of a manifold.

2. Consider $\mathbb{R}^n_+$ with the (standard) orientation determined by the orienting atlas $\mathscr{A} = \{(\mathrm{int}(\mathbb{R}^n_+), \mathrm{Id})\}$ for $\mathrm{int}(\mathbb{R}^n_+)$. Also observe that $\partial\mathbb{R}^n_+$ can be identified with $\mathbb{R}^{n-1}$ in an obvious way.

   Prove that the orientation on $\partial\mathbb{R}^n_+$ inherited from $\mathbb{R}^n_+$ agrees with the standard orientation on $\partial\mathbb{R}^n_+ \cong \mathbb{R}^{n-1}$ if and only if n is even.

3. Let M be a compact manifold with boundary, and let N be a compact manifold without boundary.

   (a) Show that $M \times N$ is a compact manifold with boundary and that

      i. $\partial(M \times N) = \partial M \times N$,

      ii. $\mathrm{int}(M \times N) = \mathrm{int}(M) \times N$.

   (b) Show that if M and N are both orientatble, then $M \times N$ is orientable.

4. Consider the following 2-form on $\mathbb{R}^3$:

$$\omega = x^1\, dx^2 \wedge dx^3 - x^2\, dx^1 \wedge dx^3 + x^3\, dx^1 \wedge dx^2.$$

   Let $i : S^2 \to \mathbb{R}^3$ be the inclusion.

   Compute the integral $\int_{S^2} i^*\omega$ up to sign—the sign depends on the orientation you choose, and you may use any choice.

   Hint: Use the appropriate 2-cube γ_1 from Lecture 26.