Math 481: Homework 7

1. Consider the $(3,0)$-tensor $\sigma^1 \otimes \sigma^2 \otimes \sigma^3$ in $\mathcal{T}_{3,0}(\mathbb{R}^3)$. Compute $\mathrm{Alt}(\sigma^1 \otimes \sigma^2 \otimes \sigma^3)$.

   Hint: It is useful to have the following notation for the permutation group on three elements:

   $$S_3 = \{\mathrm{Id}, (12), (23), (13), (123), (132)\},$$

   where, for instance, (12) is the transposition $1 \mapsto 2 \mapsto 1$, and (132) is the cyclic permutation $1 \mapsto 3 \mapsto 2 \mapsto 1$.

2. Consider the following differential forms on $\mathbb{R}^4$:

   - $\alpha = x^2\, dx^1 \wedge dx^3 \wedge dx^4 + x^1\, dx^2 \wedge dx^3 \wedge dx^1$,
   - $\lambda = x^1\, dx^1 + x^2\, dx^2 + x^3\, dx^3 + x^4\, dx^4$,
   - $\omega = x^1\, dx^2 \wedge dx^4 + x^2\, dx^1 \wedge dx^3$.

   Compute

   (a) $\alpha \wedge \lambda$,

   (b) $\omega \wedge \omega$,

   (c) $d\lambda$,

   (d) $d(\lambda \wedge \omega)$.

3. Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by

   $$F\left(x^1, x^2, x^3\right) = \left(x^2, x^1, e^{x^1 + x^2 + x^3}\right)$$

   and the two-form $\omega = dy^1 \wedge dy^2 + y^3\, dy^3 \wedge dy^1 \in \Lambda^2(\mathbb{R}^3)$.

   Compute

   (a) $F^*\omega$,

   (b) $d(F^*\omega)$,

   (c) $F^*(d\omega)$.

4. Find an orienting atlas for S^1. *Hint:* Fix our "standard" atlas by adjusting the chart maps.

5. Show that our standard atlas for $\mathbb{RP}^2$, $\{(U_i, \phi_i)\}_{i=1}^3$, is not orienting. *Bonus problem:* Prove that it cannot be fixed by adjusting the chart maps.

6. Bonus Problem. Let $\alpha = \{e_1, e_2, \ldots, e_n\}$ be an (ordered) basis of V. For $\psi \in S_n$, prove that the basis $\{e_{\psi(1)}, e_{\psi(2)}, \ldots, e_{\psi(n)}\}$ is equivalent to α (in the sense of orientation) if and only if $\mathrm{sgn}(\psi) = 1$.