

Math 481: Homework 6

1. Let W be a $(2, 2)$ -tensor field on a two-dimensional manifold M . Suppose that, in a certain local coordinate system given by a chart (U, ϕ) on M , we have

$$\begin{aligned} W_{11}^{12}(x) &= (x^1)^2 + 1, \\ W_{22}^{21}(x) &= 3x^2 - 1, \end{aligned}$$

and all other components $W_{t_1 t_2}^{k_1 k_2}$ are equal to 0. Evaluate $W(x)$ on

$$\left(x^1 dx^1 + dx^2, 3dx^1, \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, 2\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} \right).$$

2. Consider the manifold $M = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^2 > 0\}$ equipped with the Riemannian metric

$$g(x) = \frac{1}{(x^2)^2} (dx^1 \otimes dx^1 + dx^2 \otimes dx^2).$$

- (a) Sketch the following three curves in M

- $\gamma(t) = (1 - 2t, 1), \quad t \in [0, 1].$
- $\sigma(t) = (\sqrt{2} \cos t, \sqrt{2} \sin t), \quad t \in [\pi/4, 3\pi/4].$
- $\eta(t) = \begin{cases} (1, 1 + t), & t \in [0, 1], \\ (1 - 2(t - 1), 2), & t \in [1, 2], \\ (-1, 2 - (t - 2)), & t \in [2, 3]. \end{cases}$

- (b) Compute the lengths of γ , σ , and η with respect to g . Which curve is the shortest? Which is the longest?

3. Let V be a finite-dimensional vector space. Prove that $\Lambda^k(V)$, the subset of alternating $(k, 0)$ -tensors, is a subspace of $\mathcal{T}_{k,0}(V)$.

4. Bonus Problem.

- (a) Prove that $\text{GL}(n) = \{A \in M_{n \times n} \mid \det(A) \neq 0\}$ is an open subset of $M_{n \times n}$ and hence a manifold.
 (b) Prove that the map

$$\begin{aligned} \text{GL}(n) &\rightarrow \text{GL}(n) \\ A &\mapsto A^{-1} \end{aligned}$$

is well-defined and smooth.